Supplementary Material

for "Threshold Spatial Panel Regression with Fixed Effects"

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The `Supplementary Material` contains detail proofs of Lemmas B.1−B.4. Lemma C.1 and Lemma C.2 are first established to help prove the main results.

Lemma C.1. *There is a* $c_1 < \infty$ *such that for* $\underline{\gamma} \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \bar{\gamma}$ *and* $1 \leqslant r \leqslant 4$,

$$\begin{aligned} &(i)\ \ \mathrm{E}h_{it}^r(\gamma_1,\gamma_2) \leqslant c_1|\gamma_2-\gamma_1|, \quad (ii)\ \mathrm{E}f_{it}^r(\gamma_1,\gamma_2) \leqslant c_1|\gamma_2-\gamma_1|, \\ &(iii)\ \mathrm{E}k_{it}^r(\gamma_1,\gamma_2) \leqslant c_1|\gamma_2-\gamma_1|, \quad (iv)\ \mathrm{E}l_{it}^r(\gamma_1,\gamma_2) \leqslant c_1|\gamma_2-\gamma_1|. \end{aligned}$$

where, $h_{it}(\gamma_1,\gamma_2) = \|h_{it}\||d_{it}(\gamma_2,\gamma_1)|$, $\quad f_{it}(\gamma_1,\gamma_2) = \|h_{it}v_{it}\||d_{it}(\gamma_2,\gamma_1)|$,
$k_{it}(\gamma_1,\gamma_2) = |v_{it}^2-\sigma_0^2||g_{ii,t}||d_{it}(\gamma_2,\gamma_1)|$, $\quad l_{it}(\gamma_1,\gamma_2) = |v_{it}|\sum_{j\neq i}^n |g_{ij,t}||v_{jt}||d_{it}(\gamma_2,\gamma_1)|$.

Proof: We only show (i), as the others can be shown similarly. We have

$$\mathrm{E}[Zd_{it}(\gamma)] = \mathrm{E}[\mathrm{E}(Z|q_{it})d_{it}(\gamma)] = \int_{-\infty}^{\gamma} \mathrm{E}(Z|q_{it})dF(q_{it})$$

for any random variable Z, where $F(\cdot)$ denotes the CDF of q_{it}. Hence, $\frac{d}{d\gamma}\mathrm{E}[Zd_{it}(\gamma)] = \mathrm{E}(Z|q_{it} = \gamma)f(\gamma)$. Thus by the Jensen inequality and Assumption B(iii), one has

$$\frac{d}{d\gamma}\mathrm{E}(\|h_{it}\|^r d_{it}(\gamma)) = \ \mathrm{E}(\|h_{it}\|^r|q_{jt}=\gamma)f(\gamma) \leqslant [\mathrm{E}(\|h_{it}\|^4|q_{it}=\gamma)]^{r/4}f(\gamma) \leqslant c^{1+r/4},$$

by Assumption B (iii). Since $d_{jt}(\gamma_2) - d_{jt}(\gamma_1)$ equals either zero or one,

$$\mathrm{E}[\|h_{it}\|^r|d_{it}(\gamma_2)-d_{jt}(\gamma_1)|] = \mathrm{E}[\|h_{it}\|^r d_{it}(\gamma_2)] - \mathrm{E}[\|h_{it}\|^r d_{it}(\gamma_1)] \leqslant c_1|\gamma_2-\gamma_1|,$$

for some $c_1 < \infty$, by a first-order Taylor series expansion, establishing (i). Assume this c_1 is large enough so that results (ii)-(iv) also hold.

Lemma C.2. *There is a* $c_2 < \infty$ *such that for all* $\underline{\gamma} \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \bar{\gamma}$,

$$\mathrm{E}\Big|\tfrac{1}{\sqrt{nT}}\textstyle\sum_{i=1}^n\sum_{t=1}^T[h_{it}^2(\gamma_1,\gamma_2)-\mathrm{E}h_{it}^2(\gamma_1,\gamma_2)]\Big|^2 \leqslant c_2|\gamma_2-\gamma_1|, \tag{C.1}$$

$$\mathrm{E}\Big|\tfrac{1}{\sqrt{nT}}\textstyle\sum_{i=1}^n\sum_{t=1}^T[f_{it}^2(\gamma_1,\gamma_2)-\mathrm{E}f_{it}^2(\gamma_1,\gamma_2)]\Big|^2 \leqslant c_2|\gamma_2-\gamma_1|, \tag{C.2}$$

$$\mathrm{E}\Big|\tfrac{1}{\sqrt{nT}}\textstyle\sum_{i=1}^n\sum_{t=1}^T[k_{it}^2(\gamma_1,\gamma_2)-\mathrm{E}k_{it}^2(\gamma_1,\gamma_2)]\Big|^2 \leqslant c_2|\gamma_2-\gamma_1|, \tag{C.3}$$

$$\mathrm{E}\Big|\tfrac{1}{\sqrt{nT}}\textstyle\sum_{i=1}^n\sum_{t=1}^T[l_{it}^2(\gamma_1,\gamma_2)-\mathrm{E}l_{it}^2(\gamma_1,\gamma_2)]\Big|^2 \leqslant c_2|\gamma_2-\gamma_1|. \tag{C.4}$$

Proof: We only show (C.4) when $r=2$, as the proofs of the others are similar and less difficult, using Lemma C.1. As $l_{it}(\gamma_1,\gamma_2)$ are independent across t, we have

$$\begin{aligned}
&\mathrm{E}\Big|\tfrac{1}{\sqrt{nT}}\textstyle\sum_{i=1}^n\sum_{t=1}^T[l_{it}^2(\gamma_1,\gamma_2)-\mathrm{E}l_{it}^2(\gamma_1,\gamma_2)]\Big|^2\\
=&\tfrac{1}{nT}\textstyle\sum_{t=1}^T\mathrm{E}\Big|\sum_{i=1}^n[l_{it}^2(\gamma_1,\gamma_2)-\mathrm{E}l_{it}^2(\gamma_1,\gamma_2)]\Big|^2\\
=&\tfrac{1}{nT}\textstyle\sum_{t=1}^T\sum_{i=1}^n\sum_{j=1}^n\{\mathrm{E}[l_{it}^2(\gamma_1,\gamma_2)l_{jt}^2(\gamma_1,\gamma_2)]-\mathrm{E}l_{it}^2(\gamma_1,\gamma_2)\mathrm{E}l_{jt}^2(\gamma_1,\gamma_2)\}\\
=&\tfrac{1}{nT}\textstyle\sum_{t=1}^T\sum_{i=1}^n\{\mathrm{E}l_{it}^4(\gamma_1,\gamma_2)-[\mathrm{E}l_{it}^2(\gamma_1,\gamma_2)]^2\}\\
&+\tfrac{1}{nT}\textstyle\sum_{t=1}^T\sum_{i=1}^n\sum_{j\neq i}^n\{\mathrm{E}[l_{it}^2(\gamma_1,\gamma_2)l_{jt}^2(\gamma_1,\gamma_2)]-\mathrm{E}l_{it}^2(\gamma_1,\gamma_2)\mathrm{E}l_{jt}^2(\gamma_1,\gamma_2)\}\\
\equiv& I_1(\gamma_1,\gamma_2)+I_2(\gamma_1,\gamma_2).
\end{aligned}$$

It is easy to verify that $I_1(\gamma_1,\gamma_2) \leqslant \frac{2}{nT}\sum_{t=1}^T\sum_{i=1}^n \mathrm{E}[l_{it}^4(\gamma_1,\gamma_2)] \leqslant 2c_1|\gamma_2-\gamma_1|$. Further,

$$\begin{aligned}
&I_2(\gamma_1,\gamma_2)=\tfrac{l_0^4}{nT}\textstyle\sum_{t=1}^T\sum_{i=1}^n\sum_{j\neq i}^n\sum_{l\neq i}^n\sum_{k\neq i}^n\sum_{m\neq j}^n\sum_{p\neq j}^n\Big\{\mathrm{E}(|g_{il,t}||g_{ik,t}||g_{jm,t}||g_{jp,t}|)\\
&\mathrm{E}|d_{it}(\gamma_2,\gamma_1)|\mathrm{E}|d_{jt}(\gamma_2,\gamma_1)|\big[\mathrm{E}(|v_{it}^2||v_{lt}^2||v_{kt}^2||v_{jt}^2||v_{mt}^2||v_{pt}^2|)-\mathrm{E}(|v_{it}^2||v_{lt}^2||v_{kt}^2|)\mathrm{E}(|v_{jt}^2||v_{mt}^2||v_{pt}^2|)\big]\Big\}.
\end{aligned}$$

Consider the term with the highest order in error term, i.e., $l=k=m=p$, as the analyses of other terms are similar and less difficult. This term equals to

$$\begin{aligned}
&\tfrac{l_0^4}{nT}\textstyle\sum_{t=1}^T\sum_{i=1}^n\sum_{j\neq i}^n\sum_{l\neq i,j}^n\mathrm{E}(|g_{il,t}|^2|g_{jl,t}|^2)\mathrm{E}(|d_{it}(\gamma_2,\gamma_1)||d_{jt}(\gamma_2,\gamma_1)|)\mathrm{E}|v_{it}^2|\mathrm{E}|v_{jt}^2|[\mathrm{E}|v_{lt}^8|-(\mathrm{E}|v_{lt}^4|)^2]\\
\leqslant&\tfrac{l_0^4}{nT}\textstyle\sum_{t=1}^T\sum_{i=1}^n\mathrm{E}[(\sum_{l=1}^n|g_{il,t}|^2)(\sum_{j=1}^n|g_{jl,t}|^2)]\mathrm{E}|d_{it}(\gamma_2,\gamma_1)|\mathrm{E}|v_{it}^2|\mathrm{E}|v_{jt}^2|\mathrm{E}|v_{lt}^8| \leqslant c|\gamma_2-\gamma_1|,
\end{aligned}$$

for some $c<\infty$, as $\mathrm{E}(|d_{it}(\gamma_2,\gamma_1)||d_{jt}(\gamma_2,\gamma_1)|) \leqslant \mathrm{E}^{\frac{1}{2}}|d_{it}(\gamma_2,\gamma_1)|\mathrm{E}^{\frac{1}{2}}|d_{jt}(\gamma_2,\gamma_1)| = \mathrm{E}|d_{it}(\gamma_2,\gamma_1)| \leqslant c_1|\gamma_2-\gamma_1|$ based on (i) of Lemma C.1. Let c be large enough, and hence we can similarly show all the other non-zero terms in $I_2(\gamma_1,\gamma_2)$ are also bounded by $c|\gamma_2-\gamma_1|$. Thus, the desired result follows.

Proof of Lemma B.1: Firstly, we define $J_{1,nT}(\gamma) = \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} H_t' d_t(\gamma) V_t$ and $J_{2,nT}(\gamma) = \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} [V_t' d_t(\gamma) G_t V_t - \sigma_0^2 \mathtt{tr}(d_t(\gamma) G_t)]$. As the analysis of $\mathcal{J}_{s,nT}(\gamma)$ is tedious but follows the similar arguments to that of $J_{s,nT}(\gamma)$ for $s = 1, 2$, we show the uniform convergences of $J_{s,nT}(\gamma)$ instead. Lemma C.1 implies that $\mathrm{E}[\|h_{it}\|^4 d_{it}(\gamma)] < \infty$ for each γ. Meanwhile, it is easy to see that $\{d_t(\gamma) G_t\}$ are matrices with bounded row and column sum norms by Lemma A.1. Hence, $J_{1,nT}(\gamma)$ and $J_{2,nT}(\gamma)$ both converge pointwise to a Gaussian distribution by the central limit theorem (CLT) in Lemma A.3. This can be extended to any finite collection of γ to yield the convergence of the finite-dimensional distributions.

Thus, it is left to establish the tightness of $J_{s,nT}(\gamma)$ for $s = 1, 2$. We show $J_{1,nT}(\gamma)$ by verifying the conditions for Theorem 15.5 of Billingsley (1968). In the following, we claim that there are finite constants c_3 and c_4 such that for all $\gamma_1 \in \Gamma$, $\eta > 0$ and $\varphi \geqslant (nT)^{-1}$, if $\sqrt{nT} \geqslant c_4/\eta$,

$$P\left(\sup_{\gamma_1 \leqslant \gamma \leqslant \gamma_1 + \varphi} |J_{s,nT}(\gamma) - J_{s,nT}(\gamma_1)| > \eta \right) \leqslant c_3 \varphi^2 \eta^{-4},$$

Now suppose the above results are ture for $s = 1, 2$. Then, fix $\epsilon > 0$ and $\eta > 0$, and let $\varphi = \epsilon \eta^4 / c_3$. The above results imply there is a large enough nT such that for any $\gamma_1 \in \Gamma$,

$$P\left(\sup_{\gamma_1 \leqslant \gamma \leqslant \gamma_1 + \varphi} |J_{s,nT}(\gamma) - J_{s,nT}(\gamma_1)| > \eta \right) \leqslant c_3 \varphi^2 \eta^{-4} = \varphi \epsilon,$$

establishing the conditions for Theorem 15.5 of Billingsley (1968).

Since $\varphi \geqslant (nT)^{-1}$, we can let m be an integer satisfying $nT\varphi/2 \leqslant m \leqslant nT\varphi$. Set $\varphi_m = \varphi/m$. For $k = 1, \ldots, m+1$, set $\gamma_k = \gamma_1 + (k-1)\varphi_m$, $f_{it,k} = f_{it}(\gamma_k, \gamma_{k+1})$, and $f_{it,jk} = f_{it}(\gamma_k, \gamma_j)$. We let $F_{nT,k} = \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} f_{it,k}$, and thus for $\gamma_k \leqslant \gamma \leqslant \gamma_{k+1}$,

$$|J_{1,nT}(\gamma) - J_{1,nT}(\gamma_1)| \leqslant \sqrt{nT} F_{nT,k} \leqslant \sqrt{nT} |F_{nT,k} - \mathrm{E}F_{nT,k}| + \sqrt{nT} \mathrm{E}F_{nT,k}.$$

It follows that

$$\begin{aligned} &\sup_{\gamma_1 \leqslant \gamma \leqslant \gamma_1 + \varphi} |J_{1,nT}(\gamma) - J_{1,nT}(\gamma_1)| \\ &\leqslant \max_{1 \leqslant k \leqslant m} \sup_{\gamma_k \leqslant \gamma \leqslant \gamma_{k+1}} |J_{1,nT}(\gamma_k) - J_{1,nT}(\gamma_1) + J_{1,nT}(\gamma) - J_{1,nT}(\gamma_k)| \\ &\leqslant \max_{2 \leqslant k \leqslant m+1} |J_{1,nT}(\gamma_k) - J_{1,nT}(\gamma_1)| + \max_{1 \leqslant k \leqslant m} \sqrt{nT} |F_{nT,k} - \mathrm{E}F_{nT,k}| + \max_{1 \leqslant k \leqslant m} \sqrt{nT} \mathrm{E}F_{nT,k}. \qquad \text{(C.5)} \end{aligned}$$

In the following analysis, we consider bounding each term of the above equation to show the final result. For any $1 \leqslant j < k \leqslant m+1$, by the Burkholder's inequality (see (Hall and Heyde, 1980, p.23)) for some constant $\bar{c}_1 < \infty$,

$$\begin{aligned}\mathrm{E}|J_{1,nT}(\gamma_k) - J_{1,nT}(\gamma_j)|^4 &\leqslant \bar{c}_1 \mathrm{E}|\tfrac{1}{nT}\textstyle\sum_{i=1}^n \sum_{t=1}^T f_{it,jk}^2|^2 \\ &= \bar{c}_1 \mathrm{E}|\tfrac{1}{nT}\textstyle\sum_{i=1}^n \sum_{t=1}^T (f_{it,jk}^2 - \mathrm{E}f_{it,jk}^2) + \tfrac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T \mathrm{E}f_{it,jk}^2|^2.\end{aligned}$$

By Minkowski's inequality, (iv) of Lemma C.1 and (C.4), the above expression is bounded by

$$\begin{aligned}&\bar{c}_1 \big[(\mathrm{E}|\tfrac{1}{nT}\textstyle\sum_{i=1}^n \sum_{t=1}^T (f_{it,jk}^2 - \mathrm{E}f_{it,jk}^2)|^2)^{1/2} + c_1(k-j)\varphi_m\big]^2 \\ \leqslant\ &\bar{c}_1[(\tfrac{c_2(k-j)\varphi_m}{nT})^{1/2} + c_1(k-j)\varphi_m]^2 \leqslant\ \bar{c}_1(c_1+\sqrt{c_2})^2(k-j)^2\varphi_m^2,\end{aligned}$$

where we use the fact that $(nT)^{-1} \leqslant \varphi_m$ and $(k-j)^{1/2} \leqslant (k-j)$. Given the above result, Theorem 12.2 of Billingsley (1968, p. 94) implies that there is a finite constant $\bar{c}_2$ such that

$$P\Big(\max_{2\leqslant k\leqslant m+1}|J_{1,nT}(\gamma_k) - J_{1,nT}(\gamma_1)| > \eta/3\Big) \leqslant 81\bar{c}_2(m\varphi_m)^2\eta^{-4} = 81\bar{c}_2\varphi^2\eta^{-4}, \tag{C.6}$$

which bounds the first term of (C.5).

Next, we consider the second term of (C.5). By Lemma C.1, Lemma C.2 and $(nT)^{-1} \leqslant \varphi_m$,

$$\begin{aligned}\mathrm{E}|\sqrt{nT}(F_{nT,k} - \mathrm{E}F_{nT,k})|^4 &= \mathrm{E}\big|\tfrac{1}{\sqrt{nT}}\textstyle\sum_{i=1}^n \sum_{t=1}^T (f_{it,k} - \mathrm{E}f_{it,k})\big|^4 \\ &\leqslant \tfrac{1}{(nT)^2}\textstyle\sum_{i=1}^n \sum_{t=1}^T \mathrm{E}f_{it,k}^4 + 3[\tfrac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T \mathrm{E}f_{it,k}^2]^2 \\ &\leqslant \tfrac{1}{nT}c_1\varphi_m + 3c_1^2\varphi_m^2 \leqslant\ (c_1+3c_1^2)\varphi_m^2.\end{aligned}$$

By Markov's inequality, the above inequality implies

$$\begin{aligned}P\Big(\max_{1\leqslant k\leqslant m}\sqrt{nT}|F_{nT,k} - \mathrm{E}F_{nT,k}| > \eta/3\Big) &\leqslant\ \textstyle\sum_{k=1}^m P\Big(\sqrt{nT}|F_{nT,k} - \mathrm{E}F_{nT,k}| > \eta/3\Big) \\ &\leqslant 81m(c_1+3c_1^2)\varphi_m^2\eta^{-4} \leqslant\ 81(c_1+3c_1^2)\varphi^2\eta^{-4},\end{aligned}$$

where the final equality uses $m\varphi_m = \varphi$ and $\varphi_m \leqslant \varphi$.

Finally, we consider the last term of (C.5). By (iv) of Lemma C.1 and $\varphi_m \leqslant \frac{2}{nT}$,

$$\sqrt{nT}\mathrm{E}F_{nT,k} = \sqrt{nT}\mathrm{E}f_{it,k} \leqslant \sqrt{nT}c_1\varphi_m \leqslant 2c_1(nT)^{-1/2}.$$

Aggregating the above results for the three terms of (C.5), we have if $2c_1(nT)^{-1/2} \leqslant \eta/3$,

$$P\Bigg(\sup_{\gamma_1\leqslant\gamma\leqslant\gamma_1+\varphi}|J_{1,nT}(\gamma) - J_{1,nT}(\gamma_1)| > \eta\Bigg) \leqslant\ 81(\bar{c}_2+c_1+3c_1^2)\varphi^2\eta^{-4}. \tag{C.7}$$

By setting $c_3 = 81(\bar{c}_2 + c_1 + 3c_1^2)$ and $c_4 = 6c_1$, we achieve the desired result.

The proof on the tightness of $J_{2,nT}(\gamma)$ follows the same reasoning as that of result (a) in Lemma A.8 of Li and Lin (2024). As the details are analogous, they are omitted here for brevity. Finally, the derivation of their asymptotic variances follows Lemma B.5 of Yang (2015) for each γ. This concludes the proof of Lemma B.1. ■

Proof of Lemma B.2: We show the result for $\mathcal{F}_{nT}(v)$, as the other two can be shown similarly. For notation simplicity, let $m_{it} = \delta_0' h_{it}$ and $m_{it}(v) = \delta_0' h_{it} d_{it}(\gamma_0, \gamma_{nT})$. Hence,

$$\begin{aligned}\mathcal{F}_{nT}(v) &= \tfrac{a_{nT}}{nT}\textstyle\sum_{i=1}^n\sum_{t=1}^T m_{it}^2(v) - \tfrac{a_{nT}}{nT^2}\sum_{i=1}^n\sum_{k=1}^T\sum_{t=1}^T m_{it}(v)m_{ik}(v)\\ &\quad - \tfrac{a_{nT}}{n^2T}\textstyle\sum_{i=1}^n\sum_{j=1}^n\sum_{t=1}^T m_{it}(v)m_{jt}(v) + \tfrac{a_{nT}}{n^2T^2}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^T\sum_{t=1}^T m_{it}(v)m_{jk}(v)\\ &\equiv \textstyle\sum_{s=1}^4 \mathcal{F}_{s,nT}(v).\end{aligned} \tag{C.8}$$

Consider the case where v is positive first. Observe that since $\gamma_1 = \gamma_{nT} \to \gamma_0$,

$$a_{nT}P(\gamma_0 < q_{it} \leqslant \gamma_1) = v\frac{P(q_{it} \leqslant \gamma_1) - P(q_{it} \leqslant \gamma_0)}{\gamma_1 - \gamma_0} \to f|v| \tag{C.9}$$

as simple size increases. Symmetrically, we can show that $a_{nT}P(\gamma_1 < q_{it} \leqslant \gamma_0) \to f|v|$, when v is negative. In the following argument, we only consider the case where v is positive, as the negative case can be studied symmetrically. Thus,

$$\begin{aligned}\mathrm{E}\mathcal{F}_{1,nT}(v) &= \tfrac{a_{nT}}{nT}\textstyle\sum_{i=1}^n\sum_{t=1}^T \mathrm{E}[m_{it}^2\mathbb{1}\{\gamma_0 < q_{it} \leqslant \gamma_1\}]\\ &= \tfrac{1}{nT}\textstyle\sum_{i=1}^n\sum_{t=1}^T \mathrm{E}(m_{it}^2|\gamma_0 < q_{it} \leqslant \gamma_1)a_{nT}P(\gamma_0 < q_{it} \leqslant \gamma_1)\\ &\to \tfrac{1}{nT}\textstyle\sum_{i=1}^n\sum_{t=1}^T \mathrm{E}(m_{it}^2|q_{it} = \gamma_0)f|v| = \delta_0' M\delta_0 f|v|.\end{aligned}$$

Besides, by (C.1),

$$\begin{aligned}\mathrm{E}|\mathcal{F}_{1,nT}(v) - \mathrm{E}\mathcal{F}_{1,nT}(v)|^2 &\leqslant \tfrac{a_{nT}^2}{nT}\|\delta_0\|^4\mathrm{E}\big|\tfrac{1}{\sqrt{nT}}\textstyle\sum_{i=1}^n\sum_{t=1}^T[h_{it}^2(\gamma_0,\gamma_1) - \mathrm{E}h_{it}^2(\gamma_0,\gamma_1)]\big|^2\\ &\leqslant \tfrac{a_{nT}}{nT}\|\delta_0\|^4 c_2|v| = o(1).\end{aligned}$$

Hence, the Markov's inequality implies that $\mathcal{F}_{1,nT}(v) - \delta_0' M\delta_0 f|v| \xrightarrow{p} 0$.

We next consider the second term of (C.8). By (C.9), for $i \neq j$ or $t \neq k$, $a_{nT}P(\gamma_0 < q_{it} \leqslant$

$\gamma_1, \gamma_0 < q_{jk} \leqslant \gamma_1) = a_{nT} P(\gamma_0 < q_{it} \leqslant \gamma_1) P(\gamma_0 < q_{jk} \leqslant \gamma_1) \to 0$. Hence,

$$\mathrm{E}\mathcal{F}_{2,nT}(v) = \tfrac{a_{nT}}{nT^2} \textstyle\sum_{i=1}^n \sum_{k=1}^T \sum_{t=1}^T \mathrm{E}[m_{it} m_{ik} \mathbb{1}\{\gamma_0 < q_{it} \leqslant \gamma_1\} \mathbb{1}\{\gamma_0 < q_{ik} \leqslant \gamma_1\}]$$
$$= \tfrac{1}{nT^2} \textstyle\sum_{i=1}^n \sum_{k=1}^T \sum_{t=1}^T \mathrm{E}(m_{it} m_{ik} | \gamma_0 < q_{it} \leqslant \gamma_1, \gamma_0 < q_{ik} \leqslant \gamma_1) a_{nT} P(\gamma_0 < q_{it} \leqslant \gamma_1, \gamma_0 < q_{ik} \leqslant \gamma_1)$$
$$= \tfrac{1}{nT^2} \textstyle\sum_{i=1}^n \sum_{t=1}^T \mathrm{E}(m_{it}^2 | \gamma_0 < q_{it} \leqslant \gamma_1) a_{nT} P(\gamma_0 < q_{it} \leqslant \gamma_1) + o_p(1) \to \tfrac{1}{T} \delta_0' M \delta_0 f |v|.$$

Similarly, we have

$$\mathrm{E}|\mathcal{F}_{2,nT}(v) - \mathrm{E}\mathcal{F}_{2,nT}(v)|^2 \leqslant \mathrm{E}\mathcal{F}_{2,nT}^2(v) = \tfrac{a_{nT}^2}{n^2T^4} \textstyle\sum_{i=1}^n \sum_{t=1}^T \mathrm{E} m_{it}^4(v) + o_p(1) \to 0.$$

Hence, the Markov's inequality implies $\mathcal{F}_{2,nT}(v) - \frac{1}{T}\delta_0' M \delta_0 f|v| \xrightarrow{p} 0$.

Similarly, we have

$$\tfrac{a_{nT}}{n^2T} \textstyle\sum_{i=1}^n \sum_{j=1}^n \sum_{t=1}^T m_{it}(v) m_{jt}(v) = \tfrac{1}{n} \delta_0' M \delta_0 f |v| + o_p(1) \xrightarrow{p} 0$$

and

$$\tfrac{a_{nT}}{n^2T^2} \textstyle\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^T \sum_{t=1}^T m_{it}(v) m_{jk}(v) = \tfrac{1}{nT} \delta_0' M \delta_0 f |v| + o_p(1) \xrightarrow{p} 0.$$

Since $\mathcal{F}_{nT}(v)$ is monotonically increasing on $[0, \bar{v}]$ and decreasing on $[-\bar{v}, 0]$, and the limit function is continuous, the convergence is uniform over Υ. ■

Proof of Lemma B.3: The uniform convergence follows if

(a) The finite dimensional distributions of $\mathcal{R}_{nT}(v)$ converge to those of $B(v)$;

(b) $\mathcal{R}_{nT}(v)$ is tight.

We show (a) first. With Assumptions A to D, the conditions for the CLT in Lemma A.3 are well established. Hence, for $v \in \Upsilon$, we have $\mathcal{R}_{nT}(v) \xrightarrow{D} N(0, \sigma_{\mathcal{R}}^2(v))$, where $\sigma_{\mathcal{R}}^2(v)$ is the variance of $\mathcal{R}_{nT}(v)$. Then, it is left to show $\sigma_{\mathcal{R}}^2(v) = |v| \Xi f$. Let $\mathbf{H}^*(v) = \mathbf{D}(\gamma_{nT}, \gamma_0)\mathbf{H}$, $\mathbf{G}^*(v) = \mathbf{D}(\gamma_{nT}, \gamma_0)\mathbf{G}$ and $\mathbf{q}(v) = \mathtt{diagv}[\mathbf{Q}_{nT}\mathbf{D}(\gamma_{nT}, \gamma_0)\mathbf{G}]$. By Lemma B.5 of Yang (2015), we have

$$\sigma_{\mathcal{R}}^2(v) = \sigma_0^2 \mathrm{E}\mathcal{F}_{nT}(v) + 2 l_0 \sigma_0^3 \kappa_3 \tfrac{a_{nT}}{nT} \mathrm{E}[\delta_0' \mathbf{H}^{*\prime}(v) \mathbf{Q}_{nT} \mathbf{q}(v)] + l_0^2 \sigma_0^4 \kappa_4 \tfrac{a_{nT}}{nT} \mathrm{E}[\mathbf{q}'(v)\mathbf{q}(v)]$$
$$+ l_0^2 \sigma_0^4 \tfrac{a_{nT}}{nT} \mathrm{E}[\mathtt{tr}(\mathbf{Q}_{nT} \mathbf{G}^*(v)(\mathbf{G}^{*\prime}(v) + \mathbf{Q}_{nT}\mathbf{G}^*(v))] \equiv \textstyle\sum_{s=1}^4 \mathcal{C}_s.$$

By Lemma B.2, we have $(\mathcal{C}_1 + \mathcal{C}_4) - \sigma_0^2 \Xi_1 f|v| \xrightarrow{p} 0$. Similar to the proof of Lemma B.2, we can also show $(\mathcal{C}_2 + \mathcal{C}_3) - \sigma_0^2 \Xi_2 f |v| \xrightarrow{p} 0$. Hence, we conclude that $\mathcal{R}_{nT}(v) \xrightarrow{D} N(0, \Xi f|v|)$. This argument can be extended to include any finite collection $[v_1, \ldots, v_k]$ to yield the convergence of the finite dimensional distributions of $\mathcal{R}_{nT}(v)$ to those of $B(v)$.

We now show (b). By Lemma B.1, for all $\gamma_j \in \Gamma$, $\eta > 0$ and $\varphi \geqslant (nT)^{-1}$, there exist finite constant c_3 and c_4 such that if $\eta \geqslant c_4/\sqrt{nT}$,

$$P\bigg(\sup_{\gamma_j \leqslant \gamma \leqslant \gamma_j+\varphi} \|\delta_0'(\mathcal{J}_{1,nT}(\gamma,\gamma_j)) + l_0(\mathcal{J}_{2,nT}(\gamma,\gamma_j))\| > \eta\bigg) \leqslant \tfrac{1}{\eta^4} c_3 \varphi^2. \tag{C.10}$$

Fix $\epsilon > 0, \eta_1 > 0$. Set $\varphi_1 = \epsilon\eta_1^4/c_3$, $\varphi = \varphi_1/a_{nT}$,$\eta = \eta_1/\sqrt{a_{nT}}$ and $N_1 = (\max(\varphi^{-1/2}, c_4/\eta_1))^{1/\tau}$. Hence, for $nT \geqslant N_1$, we have $\varphi = \frac{\epsilon\eta^4}{nTc_3}(nT)^{2\tau} \geqslant \frac{\epsilon\eta^4}{nT\varphi c_3} = (nT)^{-1}$ and $\eta \geqslant c_4/\sqrt{nT}$. Set $\gamma_1 = \gamma_0 + v_1/a_{nT}$. By (C.10), for $nT \geqslant N_1$,

$$\begin{aligned}
&P\bigg(\sup_{v_1 \leqslant v \leqslant v_1+\varphi_1} |\mathcal{R}_{nT}(v) - \mathcal{R}_{nT}(v_1)| > \eta_1\bigg)\\
&= P\bigg(\sup_{\gamma_1 \leqslant \gamma \leqslant \gamma_1+\varphi} \|\delta_0'\mathcal{J}_{1,nT}(\gamma,\gamma_j) + l_0\mathcal{J}_{2,nT}(\gamma,\gamma_1)\| > \eta\bigg)\\
&\leqslant \tfrac{1}{\eta_1^4} c_3 a_{nT}^2 (\varphi_1/a_{nT})^2 = \varphi_1\epsilon.
\end{aligned}$$

As discussed in the proof of Lemma B.1, this shows that (b) holds. ■

Proof of Lemma B.4: Firstly, we show (a) when $r = 1$, and the proofs of the other results in (a)-(d) are similar and thus omitted. Note that $D_{1,nT}(\gamma)$ is just a linear transformation of $D_{11,nT}(\gamma) = \frac{1}{nT}\delta_0'\mathbf{H}'\mathbf{D}(\gamma_0,\gamma)\mathbf{H}\delta_0$. It suffices to show

$$P\bigg(\sup_{\gamma\in\mathcal{N}_{nT}} \frac{D_{11,nT}(\gamma)}{|\gamma-\gamma_0|} < (1-\eta)k\bigg) \leqslant \epsilon.$$

Without loss of generality (WLOG), we assume $\gamma > \gamma_0$, as a symmetric argument can be established for the case of $\gamma < \gamma_0$. Hence,

$$d\mathrm{E}D_{11,nT}(\gamma)/d\gamma = \delta_0' M(\gamma) f(\gamma)\delta_0.$$

Since $\delta_0' M(\gamma) f(\gamma)\delta_0 > 0$ (Assumption B(v)) and $\delta_0' M(\gamma) f(\gamma)\delta_0$ is continuous at γ_0 (Assumption B(iv)), then there is a B sufficiently small such that

$$k = \min_{|\gamma-\gamma_0|\leqslant B} \delta_0' M(\gamma) f(\gamma)\delta_0 > 0,$$

Because $\mathrm{E}D_{11,nT}(\gamma_0) = 0$, a first-order Taylor series expansion about γ_0 yields

$$\inf_{|\gamma-\gamma_0|\leqslant B} \mathrm{E}D_{11,nT}(\gamma) \geqslant k|\gamma-\gamma_0|. \tag{C.11}$$

Then, (C.1) implies

$$\begin{aligned}\mathrm{E}|D_{11,nT}(\gamma)-\mathrm{E}D_{11,nT}(\gamma)|^2 &\leqslant \|\delta_0\|^4\mathrm{E}|\tfrac{1}{nT}\textstyle\sum_{i=1}^n\sum_{t=1}^T[h_{it}^2(\gamma_1,\gamma_2)-\mathrm{E}h_{it}^2(\gamma_1,\gamma_2)]|^2\\ &\leqslant \|\delta_0\|^4(nT)^{-1}c_2|\gamma-\gamma_0|. \qquad (C.12)\end{aligned}$$

For any η and ϵ, set

$$b=\frac{1-\eta/2}{1-\eta}>1, \quad \text{and} \qquad (C.13)$$

$$\bar{v}=\frac{8\|\delta_0\|^4c_2}{\epsilon\eta^2k^2(1-1/b)}. \qquad (C.14)$$

We may assume that (n,T) is large enough so that $\frac{\bar{v}}{a_{nT}}\leqslant B$, else the inequality (a) is trival. For $l=1,2,\ldots,\bar{N}+1$, set $\gamma_j=\gamma_0+\bar{v}b^{j-1}/a_{nT}$, where $\bar{N}$ is the integer such that $\gamma_{\bar{N}}-\gamma_0=\bar{v}b^{\bar{N}-1}/a_{nT}\leqslant B$ and $\gamma_{\bar{N}+1}-\gamma_0=\bar{v}b^{\bar{N}}/a_{nT}>B$. (Note that $\bar{N}\geqslant 1$ since $\frac{\bar{v}}{a_{nT}}\leqslant B$.)

Markov's inequality, (C.11) and (C.12) yield

$$\begin{aligned}P\Big(\sup_{1\leqslant j\leqslant\bar{N}}\Big|\frac{D_{11,nT}(\gamma_j)-\mathrm{E}D_{11,nT}(\gamma_j)}{\mathrm{E}D_{11,nT}(\gamma_j)}\Big|>\frac{\eta}{2}\Big) &\leqslant \frac{4}{\eta^2}\sum_{j=1}^{\bar{N}}\frac{\mathrm{E}|D_{11,nT}(\gamma_j)-\mathrm{E}D_{11,nT}(\gamma_j)|^2}{|\mathrm{E}D_{11,nT}(\gamma_j)|^2}\\ &\leqslant \frac{4}{\eta^2}\sum_{j=1}^{\bar{N}}\frac{\|\delta_0\|^4(nT)^{-1}c_2}{k^2|\gamma_j-\gamma_0|}\\ &\leqslant (nT)^{-2\tau}\frac{4\|\delta_0\|^4c_2}{\eta^2k^2\bar{v}}\sum_{j=1}^{\infty}\frac{1}{b^{j-1}}\\ &\leqslant \frac{4\|\delta_0\|^4c_2}{\eta^2k^2\bar{v}(1-1/b)}=\frac{\epsilon}{2}, \qquad (C.15)\end{aligned}$$

where the final equation is based on (C.14). Thus, with probability exceeding $1-2/\epsilon$, $\Big|\frac{D_{11,nT}(\gamma_j)}{\mathrm{E}D_{11,nT}(\gamma_j)}-1\Big|\leqslant\frac{\eta}{2}$ for all $1\leqslant j\leqslant\bar{N}$. So for any $\gamma\in[\gamma_0+\bar{v}/a_{nT},\gamma_0+B]$, there is some $1\leqslant j\leqslant\bar{N}$ such that $\gamma_j<\gamma<\gamma_{j+1}$ and

$$\frac{D_{11,nT}(\gamma)}{|\gamma-\gamma_0|}\geqslant\frac{D_{11,nT}(\gamma_j)}{\mathrm{E}D_{11,nT}(\gamma_j)}\frac{\mathrm{E}D_{11,nT}(\gamma_j)}{|\gamma_{j+1}-\gamma_0|}\geqslant(1-\frac{\eta}{2})\frac{k|\gamma_j-\gamma_0|}{|\gamma_{j+1}-\gamma_0|}=(1-\frac{\eta}{2})\frac{k}{b}$$

with probability exceeding $1-\epsilon/2$, according to (C.15). Based on the definition of b, (C.13), the above inequality can be simplified as $\frac{D_{11,nT}(\gamma)}{|\gamma-\gamma_0|}\geqslant(1-\eta)k$. Since this event has a probability exceeding $1-\epsilon/2$, we have established

$$P\Big(\inf_{\gamma\in\mathcal{N}_{nT}}\frac{D_{11,nT}(\gamma)}{|\gamma-\gamma_0|}<(1-\eta)k\Big)\leqslant\frac{\epsilon}{2}.$$

A symmetric argument applies to the case $-B\leqslant\gamma-\gamma_0\leqslant-\frac{\bar{v}}{a_{nT}}$.

Secondly, we show the results in (e). WLOG, we assume $\gamma > \gamma_0$. Let $\gamma_j = \gamma_0 + \bar{v}b^{j-1}/a_{nT}$ for $l = 1, 2, \ldots, \bar{N}+1$, where b and $\bar{N}$ are defined as before. By definition, it is seen that there are at most $\log_b(a_{nT}B/\bar{v})+2$ points in the interval $\gamma-\gamma_0 \in [\frac{\bar{v}}{a_{nT}}, B]$, i.e., $\bar{N} \leqslant \log_b(a_{nT}B/\bar{v})+2$. Then, for $r = 1, 2, 3$,

$$P\Big(\sup_{\gamma\in\mathcal{N}_{nT}} \frac{\|P_{r,nT}(\gamma)\|}{|\gamma-\gamma_0|} > \eta\Big) = P\Big(\max_{1\leqslant j\leqslant\bar{N}} \frac{\|P_{r,nT}(\gamma_j)\|}{|\gamma_j-\gamma_0|} > \eta\Big) \leqslant \sum_{j=1}^{\bar{N}} P\Big(\frac{\|P_{r,nT}(\gamma_j)\|}{|\gamma_j-\gamma_0|} > \eta\Big).$$

Following the proof of Lemma C.2, for any j, we have $\mathrm{E}\|P_{r,nT}(\gamma_j)\|^2 \leqslant \frac{c_2}{nT}|\gamma_j-\gamma_0|$. Thus, Chebyshev inequality implies that

$$\sum_{j=1}^{\bar{N}} P\Big(\frac{\|P_{r,nT}(\gamma_j)\|}{|\gamma_j-\gamma_0|} > \eta\Big) \leqslant \sum_{j=1}^{\bar{N}} \frac{\mathrm{E}\|P_{r,nT}(\gamma_j)\|^2}{|\gamma_j-\gamma_0|^2\eta^2} \leqslant \sum_{j=1}^{\infty} \frac{c_2 a_{nT}}{nT\bar{v}b^{j-1}\eta^2} \leqslant \frac{c_2(nT)^{-2\tau}}{\bar{v}(1-1/b)\eta^2} \to 0.$$

A symmetric argument establishes a similar result for $\gamma < \gamma_0$.

Finally, we consider the two results in (f). As their proofs follow the same manner, we use general notation $\mathcal{J}_{s,nT}(\gamma)$ to denote either of them. Fix $\eta > 0$. For $j = 1, 2, \ldots,$ set $\gamma_j - \gamma_0 = \bar{v}2^{j-1}/a_{nT}$, where $\bar{v} < \infty$ will be determined later. By the similar analysis as used in the proof of Lemma B.1, for all $\gamma_j \in \Gamma$, $\eta > 0$ and $\varphi \geqslant (nT)^{-1}$, there exist $c_3, c_4 < \infty$ such that if $\eta \geqslant c_4/\sqrt{nT}$,

$$\mathrm{E}\|\mathcal{J}_{s,nT}(\gamma_j) - \mathcal{J}_{s,nT}(\gamma_0)\|^2 \leqslant c_1|\gamma_j-\gamma_0|, \quad \text{and} \tag{C.16}$$

$$P\Big(\sup_{\gamma_j\leqslant\gamma\leqslant\gamma_j+\varphi} \|\mathcal{J}_{s,nT}(\gamma) - \mathcal{J}_{s,nT}(\gamma_j)\| > \eta\Big) \leqslant c_3\varphi^2\eta^{-4}. \tag{C.17}$$

Next, we do the following decomposition

$$\begin{aligned}
&\sup_{\gamma\in\mathcal{N}_{nT}} \frac{\|\mathcal{J}_{s,nT}(\gamma) - \mathcal{J}_{s,nT}(\gamma_0)\|}{\sqrt{a_{nT}}|\gamma-\gamma_0|} \\
&= \sup_j \sup_{\gamma_j\leqslant\gamma\leqslant\gamma_{j+1}} \frac{\|\mathcal{J}_{s,nT}(\gamma) - \mathcal{J}_{s,nT}(\gamma_0)\|}{\sqrt{a_{nT}}|\gamma_j-\gamma_0|}\frac{|\gamma_j-\gamma_0|}{|\gamma-\gamma_0|} \\
&\leqslant \sup_j \sup_{\gamma_j\leqslant\gamma\leqslant\gamma_{j+1}} \frac{\|\mathcal{J}_{s,nT}(\gamma) - \mathcal{J}_{s,nT}(\gamma_j)\|}{\sqrt{a_{nT}}|\gamma_j-\gamma_0|} + \sup_j \frac{\|\mathcal{J}_{s,nT}(\gamma_j) - \mathcal{J}_{s,nT}(\gamma_0)\|}{\sqrt{a_{nT}}|\gamma_j-\gamma_0|}.
\end{aligned} \tag{C.18}$$

For the first term of (C.18), we set $\varphi_j = \gamma_{j+1} - \gamma_j$ and $\eta_j = \sqrt{a_{nT}}|\gamma_j-\gamma_0|\eta/2$, and then

$$P\Big(\sup_j \sup_{\gamma_j\leqslant\gamma\leqslant\gamma_{j+1}} \frac{\|\mathcal{J}_{s,nT}(\gamma) - \mathcal{J}_{s,nT}(\gamma_j)\|}{\sqrt{a_{nT}}|\gamma_j-\gamma_0|} > \eta/2\Big) \leqslant \sum_{j=1}^{\infty} P\Big(\sup_{\gamma_j\leqslant\gamma\leqslant\gamma_j+\varphi_j} \|\mathcal{J}_{s,nT}(\gamma)-\mathcal{J}_{s,nT}(\gamma_j)\| > \eta_j\Big).$$

Note that if $\bar{v} \geqslant 1$, then $\varphi_j \geqslant 1/a_{nT} \geqslant 1/n$. In addition , if $\bar{v} \geqslant 12c_1/\eta$, then $\eta_j = \bar{v}2^{j-2}\eta/\sqrt{a_{nT}} \geqslant c_4/\sqrt{a_{nT}} \geqslant c_4/\sqrt{nT}$. Thus, if $\bar{v} \geqslant \max(1, 12c_1/\eta)$, using (C.17), the right hand side of above inequality is bounded by

$$\sum_{j=1}^{\infty} \frac{c_3 \varphi_j^2}{\eta_j^4} = \sum_{j=1}^{\infty} \frac{16 c_3 |\gamma_{j+1} - \gamma_j|^2}{a_{nT}^2 |\gamma_j - \gamma_0|^4 \eta^4} = \frac{64 c_3}{3 \bar{v}^2 \eta^4}.$$

For the second term of (C.18), Markov's inequality and (C.16) imply

$$\begin{aligned} P\left(\sup_j \frac{\|\mathcal{J}_{s,nT}(\gamma_j) - \mathcal{J}_{s,nT}(\gamma_0)\|}{\sqrt{a_{nT}}|\gamma_j - \gamma_0|} > \eta/2\right) &\leqslant \sum_{j=1}^{\infty} P\left(\frac{\|\mathcal{J}_{s,nT}(\gamma_j) - \mathcal{J}_{s,nT}(\gamma_0)\|}{\sqrt{a_{nT}}|\gamma_j - \gamma_0|} > \eta/2\right) \\ &\leqslant \sum_{j=1}^{\infty} \frac{4\mathrm{E}\|\mathcal{J}_{s,nT}(\gamma_j) - \mathcal{J}_{s,nT}(\gamma_0)\|^2}{a_{nT}|\gamma_j - \gamma_0|^2 \eta^2} \\ &\leqslant \sum_{j=1}^{\infty} \frac{4 c_1 |\gamma_j - \gamma_0|}{a_{nT}|\gamma_j - \gamma_0|^2 \eta^2} = \frac{8 c_1}{\bar{v} \eta^2}. \end{aligned}$$

Together, if $\bar{v} \geqslant \max(1, 12c_1/\eta)$ we have

$$P\left(\sup_{\gamma \in \mathcal{N}_{nT}} \frac{\|\mathcal{J}_{s,nT}(\gamma) - \mathcal{J}_{s,nT}(\gamma_0)\|}{\sqrt{a_{nT}}|\gamma - \gamma_0|} > \eta\right) \leqslant \frac{64 c_3}{3 \bar{v}^2 \eta^4} + \frac{8 c_1}{\bar{v} \eta^2},$$

which can be made arbitrarily small by picking suitably large $\bar{v}$. Thus, results in (f) hold. ■

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