Supplementary Material

for "Dynamic Spatial Panel Data Models with Interactive Fixed Effects: M-Estimation and Inference under Fixed or Relatively Small T"

The `Supplementary Material` contains two additional appendices, C and D, where Appendix C contains Lemmas C1-C7 for the proofs of Corollaries 3.1 and 3.2, and Appendix D contains additional technical Lemmas D1-D5 for the results in Appendix C.

Appendix C: Asymptotic Analysis with Relatively Large T

Recall $\|\cdot\|_{\mathtt{sp}}$, the spectrum norm of a matrix. Let $\dot{X}_j = B_{30}X_j$, for $j = 1, \ldots, k$, $\dot{X}_{k+1} = B_{30}Y_{-1}$, $\dot{X}_{k+2} = B_{30}W_1Y$, $\dot{X}_{k+3} = B_{30}W_2Y_{-1}$, and $\dot{\Gamma} = B_{30}\Gamma_0$. Denote $\tilde{F} = \hat{F}(\psi_0)$.

Lemma C.1. *Suppose Assumptions A-H hold and $T/n + n/T^2 \to 0$. Define the pseudo-inverses $\big(\dot{\Gamma}F_0'\big)^\dagger = F_0\big(F_0'F_0\big)^{-1}\big(\dot{\Gamma}'\dot{\Gamma}\big)^{-1}\dot{\Gamma}'$ and $\big(F_0\dot{\Gamma}'\big)^\dagger = \dot{\Gamma}\big(\dot{\Gamma}'\dot{\Gamma}\big)^{-1}\big(F_0'F_0\big)^{-1}F_0'$. The following expansion holds for the projection matrix $M_{\tilde{F}}$:*

$$M_{\tilde{F}} = M_{F_0} + M_{\tilde{F},v}^{(1)} + M_{\tilde{F},v}^{(2)} + M_{\tilde{F}}^{(rem)},$$

where the last three $T \times T$ matrices are $M_{\tilde{F},v}^{(1)} = -M_{F_0}\mathbb{V}'\big(F_0\dot{\Gamma}'\big)^\dagger - \big(\dot{\Gamma}F_0'\big)^\dagger\mathbb{V}M_{F_0}$,

$$\begin{aligned} M_{\tilde{F},v}^{(2)} = {} & M_{F_0}\mathbb{V}'\big(F_0\dot{\Gamma}'\big)^\dagger\mathbb{V}'\big(F_0\dot{\Gamma}'\big)^\dagger + \big(\dot{\Gamma}F_0'\big)^\dagger\mathbb{V}\big(\dot{\Gamma}F_0'\big)^\dagger\mathbb{V}M_{F_0} - M_{F_0}\mathbb{V}'M_{\dot{\Gamma}}\mathbb{V}\big(\dot{\Gamma}F_0'\big)^\dagger\big(F_0\dot{\Gamma}'\big)^\dagger \\ & - \big(\dot{\Gamma}F_0'\big)^\dagger\big(F_0\dot{\Gamma}'\big)^\dagger\mathbb{V}'M_{\dot{\Gamma}}\mathbb{V}M_{F_0} - M_{F_0}\mathbb{V}'\big(F_0\dot{\Gamma}'\big)^\dagger\big(\dot{\Gamma}F_0'\big)^\dagger\mathbb{V}M_{F_0} + \big(\dot{\Gamma}F_0'\big)^\dagger\mathbb{V}M_{F_0}\mathbb{V}'\big(F_0\dot{\Gamma}'\big)^\dagger, \end{aligned}$$

and $\|M_{\tilde{F}}^{(rem)}\|_{\mathtt{sp}} = O_p(T^{-3/2})$, which is the remainder term.

Lemma C.2. *Suppose Assumptions A-H hold and $T/n + n/T^2 \to 0$. The concentrated AQS vector (3.27) has the expansion $\tilde{S}_{nT}^*(\psi_0) = \tilde{S}_{nT} + o_p(\sqrt{nT})$. The leading term $\tilde{S}_{nT}$ has elements:*

$$\begin{aligned} \tilde{S}_{j,nT} = {} & \tfrac{1}{\sigma_{v0}^2}\,\mathrm{tr}(\dot{X}_jM_{F_0}\mathbb{V}'M_{\dot{\Gamma}}), \qquad \textit{for } j = 1, \ldots, k, \\ \tilde{S}_{k+1,nT} = {} & \tfrac{1}{\sigma_{v0}^2}\,\mathrm{tr}(\dot{X}_{k+1}M_{F_0}\mathbb{V}'M_{\dot{\Gamma}}) - \mathrm{tr}[(M_{F_0} \otimes M_{\dot{\Gamma}})\mathbf{D}_{-1}], \\ \tilde{S}_{k+2,nT} = {} & \tfrac{1}{\sigma_{v0}^2}\,\mathrm{tr}(\dot{X}_{k+2}M_{F_0}\mathbb{V}'M_{\dot{\Gamma}}) - \mathrm{tr}[(M_{F_0} \otimes M_{\dot{\Gamma}}W_1)\mathbf{D}], \\ \tilde{S}_{k+3,nT} = {} & \tfrac{1}{\sigma_{v0}^2}\,\mathrm{tr}(\dot{X}_{k+3}M_{F_0}\mathbb{V}'M_{\dot{\Gamma}}) - \mathrm{tr}[(M_{F_0} \otimes M_{\dot{\Gamma}}W_2)\mathbf{D}_{-1}], \\ \tilde{S}_{k+4,nT} = {} & \tfrac{1}{\sigma_{v0}^2}\,\mathrm{tr}(W_3B_{30}^{-1}\mathbb{V}M_{F_0}\mathbb{V}') - (T-r)\mathrm{tr}(W_3B_{30}^{-1}), \\ \tilde{S}_{k+5,nT} = {} & \tfrac{1}{2\sigma_{v0}^4}\,\mathrm{tr}(\mathbb{V}M_{F_0}\mathbb{V}') - \tfrac{n(T-r)}{2\sigma_{v0}^2}. \end{aligned}$$

In addition, we have $E(\tilde{S}_{j,nT}) = 0$ for all $j = 1, \ldots, k+5$.

Next lemma studies the properties of the derivative of the concentrated AQS function.

Lemma C.3. *Suppose Assumptions A-H hold and $T/n + n/T^2 \to 0$. We have,*

$$-\frac{1}{nT}\frac{\partial \tilde{S}^*_{nT}(\psi)}{\partial \psi'} \mid_{\psi_0} = \tilde{H}_{nT} + o_p(1).$$

The leading term $\tilde{H}_{nT}$ has elements $\tilde{H}_{nT,(j,q)}$, $j, q = 1, 2, \ldots, \dim(\psi)$:

(i) $\tilde{H}_{nT,(j,q)} = \frac{1}{nT\sigma^2_{v0}} \operatorname{tr}(\dot{X}'_j M_{\dot{\Gamma}} \dot{X}_q M_{F_0}) - \mathbf{1}\{j = q = k+2\} \cdot \frac{T-r}{nT} \operatorname{tr}(W_1 B^{-1}_{10} W_1 B^{-1}_{10})$, $j \le k+3$, $q \le k+3$; $\tilde{H}_{nT,(j,q)} = o_p(1)$, $q = k+4, k+5$;

(ii) $\tilde{H}_{nT,(j,q)} = \frac{1}{n} \operatorname{tr}(B^{-1\prime}_{30} W'_3 W_3 B^{-1}_{30} - W_3 B^{-1}_{30} W_3 B^{-1}_{30})$, $j = k+4$, $q = k+4$;

(iii) $\tilde{H}_{nT,(j,q)} = \frac{1}{2\sigma^4_{v0}} + o_p(1)$, $j = k+5$, $q = k+5$; $\tilde{H}_{nT,(j,q)} = o_p(1)$, $j = k+5$, $q \ne k+5$.

Next lemma studies the variance matrix of $\tilde{S}_{nT}$, for which we write

$$\tilde{S}_{nT} = \begin{cases} \tilde{\Pi}'_1 \mathbf{v} \\ \mathbf{v}'\tilde{\Psi}_1 \mathbf{y}_0 + \mathbf{v}'\tilde{\Phi}_1 \mathbf{v} + \tilde{\Pi}'_2 \mathbf{v} - \sigma^2_{v0} \operatorname{tr}(\tilde{\Phi}_1) \\ \mathbf{v}'\tilde{\Psi}_2 \mathbf{y}_0 + \mathbf{v}'\tilde{\Phi}_2 \mathbf{v} + \tilde{\Pi}'_3 \mathbf{v} - \sigma^2_{v0} \operatorname{tr}(\tilde{\Phi}_2) \\ \mathbf{v}'\tilde{\Psi}_3 \mathbf{y}_0 + \mathbf{v}'\tilde{\Phi}_3 \mathbf{v} + \tilde{\Pi}'_4 \mathbf{v} - \sigma^2_{v0} \operatorname{tr}(\tilde{\Phi}_3) \\ \mathbf{v}'\Phi_4 \mathbf{v} - \sigma^2_{v0} \operatorname{tr}(\Phi_4) \\ \mathbf{v}'\Phi_5 \mathbf{v} - \sigma^2_{v0} \operatorname{tr}(\Phi_5) \end{cases}$$

where $\tilde{\Psi}_1 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}} B_{30})\mathbf{Q}_{-1}$, $\tilde{\Psi}_2 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}} B_{30} W_1)\mathbf{Q}$,
$\tilde{\Psi}_3 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}} B_{30} W_2)\mathbf{Q}_{-1}$, $\tilde{\Pi}_1 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}})\mathbf{X}$,
$\tilde{\Pi}_2 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}} B_{30})[\boldsymbol{\eta}_{-1} + \mathbf{D}_{-1}\mathtt{vec}(\Gamma_0 F'_0)]$,
$\tilde{\Pi}_3 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}} B_{30} W_1)[\boldsymbol{\eta} + \mathbf{D}\mathtt{vec}(\Gamma_0 F'_0)]$,
$\tilde{\Pi}_4 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}} B_{30} W_2)[\boldsymbol{\eta}_{-1} + \mathbf{D}_{-1}\mathtt{vec}(\Gamma_0 F'_0)]$,
$\tilde{\Phi}_1 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}} B_{30})\mathbf{D}_{-1}\mathbf{B}^{-1}_{30}$, $\tilde{\Phi}_2 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}} B_{30} W_1)\mathbf{D}\mathbf{B}^{-1}_{30}$,
$\tilde{\Phi}_3 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes M_{\dot{\Gamma}} B_{30} W_2)\mathbf{D}_{-1}\mathbf{B}^{-1}_{30}$, $\Phi_4 = \frac{1}{\sigma^2_{v0}}(M_{F_0} \otimes W_3 B^{-1}_{30})$, $\Phi_5 = \frac{1}{2\sigma^4_{v0}}(M_{F_0} \otimes I_n)$.

Lemma C.4. *Suppose Assumptions A-H hold and $T/n + n/T^2 \to 0$. Denote the variance of $\tilde{S}_{nT}/\sqrt{nT}$ conditional on $\mathcal{D}$ as $\tilde{\Sigma}_{nT}$. We have $\tilde{\Sigma}_{nT} = \tilde{H}_{nT} + \tilde{\Xi}_{nT} + o_p(1)$. All three matrices are symmetric. The upper triangular part of $\tilde{\Xi}_{nT}$ has (j, q)-entries:*

$\tilde{\Xi}_{nT,(1:k,1:k)} = 0$; $\tilde{\Xi}_{nT,(1:k,q)} = \frac{\mu_3}{nT}\tilde{\Pi}'_{1,j}\mathtt{diag}(\tilde{\Phi}_a)$, $q = k+a$ *and* $a = 1, \ldots, 5$;

$\tilde{\Xi}_{nT,(k+1,k+1)} = \frac{2\mu_3}{nT}(\tilde{\Pi}_2 + \tilde{\Psi}_1 \mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_1) + \frac{\sigma^4_{v0}}{nT} \operatorname{tr}(\tilde{\Phi}_1 \tilde{\Phi}_1) + \frac{(\mu_4 - 3\sigma^4_{v0})}{nT} \|\mathtt{diag}(\tilde{\Phi}_1)\|^2$;

$\tilde{\Xi}_{nT,(k+1,k+2)} = \frac{\mu_3}{nT}(\tilde{\Pi}_2 + \tilde{\Psi}_1 \mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_2) + \frac{\mu_3}{nT}(\tilde{\Pi}_3 + \tilde{\Psi}_2 \mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_1)$
$\quad + \frac{\sigma^4_{v0}}{nT} \operatorname{tr}(\tilde{\Phi}_1 \tilde{\Phi}_2) + \frac{(\mu_4 - 3\sigma^4_{v0})}{nT}\mathtt{diag}(\tilde{\Phi}_1)'\mathtt{diag}(\tilde{\Phi}_2)$;

$\tilde{\Xi}_{nT,(k+1,k+3)} = \frac{\mu_3}{nT}(\tilde{\Pi}_3 + \tilde{\Psi}_2 \mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_3) + \frac{\mu_3}{nT}(\tilde{\Pi}_4 + \tilde{\Psi}_3 \mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_2) + \frac{\sigma^4_{v0}}{nT} \operatorname{tr}(\tilde{\Phi}_2 \tilde{\Phi}_3)$
$\quad + \frac{(\mu_4 - 3\sigma^4_{v0})}{nT}\mathtt{diag}(\tilde{\Phi}_2)'\mathtt{diag}(\tilde{\Phi}_3)$;

$\tilde{\Xi}_{nT,(k+1,k+4)} = \frac{\mu_3}{nT}(\tilde{\Pi}_2 + \tilde{\Psi}_1 \mathbf{y}_0)'\mathtt{diag}(\Phi_4) + \frac{\sigma^4_{v0}}{nT} \operatorname{tr}(\tilde{\Phi}_1 \Phi_4 + \tilde{\Phi}_1 \Phi'_4) + \frac{(\mu_4 - 3\sigma^4_{v0})}{nT}\mathtt{diag}(\tilde{\Phi}_1)'\mathtt{diag}(\Phi_4)$;

$$\tilde{\Xi}_{nT,(k+1,k+5)} = \frac{\mu_3}{nT}(\tilde{\Pi}_2+\tilde{\Psi}_1\mathbf{y}_0)'\mathtt{diag}(\Phi_5)+\frac{\sigma_{v0}^4}{nT}\operatorname{tr}(\tilde{\Phi}_1\Phi_5+\tilde{\Phi}_1\Phi_5')+\frac{(\mu_4-3\sigma_{v0}^4)}{nT}\mathtt{diag}(\tilde{\Phi}_1)'\mathtt{diag}(\Phi_5);$$

$$\tilde{\Xi}_{nT,(k+2,k+2)} = \frac{nT}{2\mu_3}(\tilde{\Pi}_3+\tilde{\Psi}_2\mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_2)+\frac{nT}{\sigma_{v0}^4}\operatorname{tr}(\tilde{\Phi}_2\tilde{\Phi}_2)+\frac{nT}{(\mu_4-3\sigma_{v0}^4)}\|\mathtt{diag}(\tilde{\Phi}_2)\|^2;$$

$$\tilde{\Xi}_{nT,(k+2,k+3)} = \frac{\mu_3}{nT}(\tilde{\Pi}_3+\tilde{\Psi}_2\mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_3)+\frac{\mu_3}{nT}(\tilde{\Pi}_4+\tilde{\Psi}_3\mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_2)+\frac{\sigma_{v0}^4}{nT}\operatorname{tr}(\tilde{\Phi}_2\tilde{\Phi}_3) + \frac{(\mu_4-3\sigma_{v0}^4)}{nT}\mathtt{diag}(\tilde{\Phi}_2)'\mathtt{diag}(\tilde{\Phi}_3);$$

$$\tilde{\Xi}_{nT,(k+2,k+4)} = \frac{\mu_3}{nT}(\tilde{\Pi}_4+\tilde{\Psi}_3\mathbf{y}_0)'\mathtt{diag}(\Phi_4)+\frac{\sigma_{v0}^4}{nT}\operatorname{tr}(\tilde{\Phi}_3\Phi_4+\tilde{\Phi}_3\Phi_4')+\frac{(\mu_4-3\sigma_{v0}^4)}{nT}\mathtt{diag}(\tilde{\Phi}_3)'\mathtt{diag}(\Phi_4);$$

$$\tilde{\Xi}_{nT,(k+2,k+5)} = \frac{\mu_3}{nT}(\tilde{\Pi}_4+\tilde{\Psi}_3\mathbf{y}_0)'\mathtt{diag}(\Phi_5)+\frac{\sigma_{v0}^4}{nT}\operatorname{tr}(\tilde{\Phi}_3\Phi_5+\tilde{\Phi}_3\Phi_5')+\frac{(\mu_4-3\sigma_{v0}^4)}{nT}\mathtt{diag}(\tilde{\Phi}_3)'\mathtt{diag}(\Phi_5);$$

$$\tilde{\Xi}_{nT,(k+3,k+3)} = \frac{2\mu_3}{nT}(\tilde{\Pi}_4+\tilde{\Psi}_3\mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_3)+\frac{\sigma_{v0}^4}{nT}\operatorname{tr}(\tilde{\Phi}_3\tilde{\Phi}_3)+\frac{(\mu_4-3\sigma_{v0}^4)}{nT}\|\mathtt{diag}(\tilde{\Phi}_3)\|^2;$$

$$\tilde{\Xi}_{nT,(k+3,k+4)} = \frac{\mu_3}{nT}(\tilde{\Pi}_4+\tilde{\Psi}_3\mathbf{y}_0)'\mathtt{diag}(\Phi_4)+\frac{\sigma_{v0}^4}{nT}\operatorname{tr}(\tilde{\Phi}_3\Phi_4+\tilde{\Phi}_3\Phi_4')+\frac{(\mu_4-3\sigma_{v0}^4)}{nT}\mathtt{diag}(\tilde{\Phi}_3)'\mathtt{diag}(\Phi_4);$$

$$\tilde{\Xi}_{nT,(k+3,k+5)} = \frac{\mu_3}{nT}(\tilde{\Pi}_4+\tilde{\Psi}_3\mathbf{y}_0)'\mathtt{diag}(\Phi_5)+\frac{\sigma_{v0}^4}{nT}\operatorname{tr}(\tilde{\Phi}_3\Phi_5+\tilde{\Phi}_3\Phi_5')+\frac{(\mu_4-3\sigma_{v0}^4)}{nT}\mathtt{diag}(\tilde{\Phi}_3)'\mathtt{diag}(\Phi_5);$$

$$\tilde{\Xi}_{nT,(k+4,k+4)} = \frac{1}{nT}\operatorname{tr}(\Phi_4\Phi_4')+\frac{(\mu_4-3\sigma_{v0}^4)}{nT}\|\mathtt{diag}(\Phi_4)\|^2;$$

$$\tilde{\Xi}_{nT,(k+4,k+5)} = \frac{\sigma_{v0}^4}{nT}\operatorname{tr}(\Phi_4\Phi_5+\Phi_4\Phi_5')+\frac{(\mu_4-3\sigma_{v0}^4)}{nT}\mathtt{diag}(\Phi_4)'\mathtt{diag}(\Phi_5);$$

$$\tilde{\Xi}_{nT,(k+5,k+5)} = \frac{(\mu_4-3\sigma_{v0}^4)}{nT}\|\mathtt{diag}(\Phi_5)\|^2.$$

In Section 3.4, we propose an estimate for $\Sigma_{nT}(\psi_0)$, the VC matrix of of the AQS funtion $S^*_{nT}(\psi_0)$. In the following, we use $\Sigma_{nT} = \Sigma_{nT}(\psi_0)$ for notation simplicity. Next Lemma finds the asymptotic leading term of Σ_{nT}.

Lemma C.5. *Suppose Assumptions A-H hold. Then, as $T/n + n/T^2 \to 0$, we have,*

(i) $\Sigma_{nT,(k+a,k+b))} = \frac{1}{nT\sigma_{v0}^2}\mathtt{vec}(\dot{F}_a')'(M_{F_0}\otimes\dot{\Gamma}'\dot{\Gamma})\mathtt{vec}(\dot{F}_b') + O_p(T^{-2})$, *for* $a, b = 1, \ldots, k_\phi$, $\|\Sigma_{nT,\phi\phi} - H_{nT,\phi\phi}\|_{\mathtt{sp}} = o_p(T^{-1})$, *and* $\|\hat{\Sigma}_{nT,\phi\phi} - \Sigma_{nT,\phi\phi}\|_{\mathtt{sp}} = o_p(T^{-1})$.

(ii) $\hat{\Sigma}_{nT,(k+5+a,j)} - \Sigma_{nT,(k+5+a,j)} = o_p(T^{-1})$ *uniformly,* $j \le k+5$, $a = 1, \ldots, k_\phi$;

$$\Sigma_{nT,(k+5+a,j)} = \frac{1}{\sigma_{v0}^2 nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0}\otimes\dot{\Gamma})\mathtt{vec}(\dot{F}_a') + o_p(T^{-1}),\ j = 1, \ldots, k;$$

$$\Sigma_{nT,(k+5+a,j)} = \frac{1}{\sigma_{v0}^2 nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0}\otimes\dot{\Gamma})\mathtt{vec}(\dot{F}_a') + \frac{\mu_3}{nT}\mathtt{diag}(\Phi_{j-k})'\Pi_{4+a} + o_p(T^{-1}),\quad j = k+1, \ldots, k+3;$$

$$\Sigma_{nT,(k+5+a,j)} = \frac{\mu_3}{nT}\mathtt{diag}(\Phi_{j-k})'\Pi_{4+a} + o_p(T^{-1}),\ j = k+4, k+5;$$

(iii) $\Sigma_{nT,\psi\psi} = H_{nT,\psi\psi} + \Xi_{nT}$, *where Ξ is the same as $\tilde{\Xi}$ with quantities $\tilde{\Pi}_a$'s, $\tilde{\Psi}_b$'s and $\tilde{\Phi}_c$'s replaced by Π_a's, Ψ_b's and Φ_c's.*

For the next lemma, we derive detailed expressions of the elements $H_{nT,(j,q)}$ of H_{nT}.

For $j = 1, \ldots, k$,

$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{X}_q M_{F_0}),\quad q \le k+3,$$

$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[X_j'(B_{30}'W_3 + W_3'B_{30})\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+4,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^4}\operatorname{tr}[\dot{X}_j'B_{30}\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+45,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[\dot{X}_j'B_{30}\mathbb{Z}(\theta_0)\dot{P}_{F,a}], \quad q = k+6,\ldots,k+5+k_\phi,$$

For $j = k+1$,

$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{X}_qM_{F_0}), \quad q \le k,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{X}_qM_{F_0}) - \frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes I_n)\dot{\mathbf{D}}_{-1,\psi_q}], \quad q = k+1, k+2, k+3$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[Y_{-1}'(B_{30}'W_3 + W_3'B_{30})\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+4,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^4}\operatorname{tr}[\dot{X}_j'B_{30}\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+5,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[\dot{X}_j'B_{30}\mathbb{Z}(\theta_0)\dot{P}_{F,s}] - \frac{1}{nT}\operatorname{tr}[(\dot{P}_{F,s}\otimes I_n)\mathbf{D}_{-1}], \quad q = k+6,\ldots,k+5+k_\phi.$$

For $j = k+2$,

$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{X}_qM_{F_0}), \quad q \le k,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{X}_qM_{F_0}) - \frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes W_1)\dot{\mathbf{D}}_{\psi_q}], \quad q = k+1, k+2, k+3$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[YW_1'(B_{30}'W_3 + W_3B_{30})\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+4,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^4}\operatorname{tr}[\dot{X}_j'B_{30}\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+5,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[Y'W_1'B_{30}'B_{30}\mathbb{Z}(\theta_0)\dot{P}_{F,s}] - \frac{1}{nT}\operatorname{tr}[(\dot{P}_{F,s}\otimes W_1)\mathbf{D}], \quad q = k+6,\ldots,k+5+k_\phi.$$

For $j = k+3$,

$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{X}_qM_{F_0}), \quad q \le k,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{X}_qM_{F_0}) - \frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes W_2)\dot{\mathbf{D}}_{-1,\rho}], \quad q = k+1,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{X}_qM_{F_0}) - \frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes W_2)\dot{\mathbf{D}}_{-1,\lambda_1}], \quad q = k+2,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{X}_qM_{F_0}) - \frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes W_2)\dot{\mathbf{D}}_{-1,\lambda_2}], \quad q = k+3,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[Y_{-1}'W_2'(B_{30}'W_3 + W_3'B_{30})\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+4,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^4}\operatorname{tr}[\dot{X}_j'B_{30}\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+5,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[\dot{X}_j'B_{30}\mathbb{Z}(\theta_0)\dot{P}_{F,s}] - \frac{1}{nT}\operatorname{tr}[(\dot{P}_{F,s}\otimes W_2)\mathbf{D}_{-1}], \quad q = k+6,\ldots,k+5+k_\phi.$$

For $j = k+4$.

$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[X_j'(B_{30}'W_3 + W_3'B_{30})\mathbb{Z}(\theta_0)M_{F_0}], \quad q \le k,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[Y_{-1}'(B_{30}'W_3 + W_3'B_{30})\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+1,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[YW_1'(B_{30}'W_3 + W_3B_{30})\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+2,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[Y_{-1}'W_2'(B_{30}'W_3 + W_3'B_{30})\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+3,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[\mathbb{Z}(\theta_0)'W_3'W_3\mathbb{Z}(\theta_0)M_{F_0}], \quad q = k+4,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^4}\operatorname{tr}[\mathbb{Z}(\theta_0)'B_{30}'W_3\mathbb{Z}(\theta_0)M_{F_0}] + \frac{(T-r)}{nT}\operatorname{tr}[W_3B_{30}^{-1}W_3B_{30}^{-1}], \quad q = k+5,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[\mathbb{Z}(\theta_0)'B_{30}'W_3\mathbb{Z}(\theta_0)\dot{P}_{F,s}], \quad q = k+6,\ldots,k+5+k_\phi.$$

For $j = k+5$,

$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[\dot{X}_j'B_{30}\mathbb{Z}(\theta_0)\dot{P}_{F,s}], \quad q \le k+3,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^4}\operatorname{tr}[\mathbb{Z}(\theta_0)'B_{30}'B_{30}\mathbb{Z}(\theta_0)M_{F_0}] - \frac{T-r}{2T\sigma_{v0}^2}, \quad q = k+4,$$
$$H_{nT,(j,q)} = o_p(1), \quad q = k+5,\ldots,k+5+k_\phi.$$

For $j = k + 5 + s$,

$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2} \operatorname{tr}[\dot{X}_j' B_{30} \mathbb{Z}(\theta_0) \dot{P}_{F,s}], \quad q \le k+3,$$
$$H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2} \operatorname{tr}[\mathbb{Z}(\theta_0)' B_{30}' W_3 \mathbb{Z}(\theta_0) \dot{P}_{F,s}], \quad q = k+4,$$
$$H_{nT,(j,q)} = o_p(1), \quad q \le k+5,$$
$$H_{nT,(j,q)} = \frac{1}{2nT\sigma_{v0}^2} \operatorname{tr}[\mathbb{Z}(\theta_0)' B_{30}' B_{30} \mathbb{Z}(\theta_0) \ddot{P}_{F,sq}].$$

Lemma C.6. *Suppose Assumptions A-H hold. Then, as $T/n + n/T^2 \to 0$, we have,*

(i) $H_{nT,(j,q)} = \frac{1}{nT\sigma_{v0}^2} \operatorname{tr}(\dot{X}_j' \dot{X}_q M_{F_0}) + o_p(1)$, $j, q \le k+3$ *except* $j = q = k+2$*;*

$H_{nT,(k+2,k+2)} = \frac{1}{nT\sigma_{v0}^2} \operatorname{tr}(\dot{X}_j' \dot{X}_j M_{F_0}) + \frac{T-r}{nT} \operatorname{tr}(W_1 B_{10}^{-1} W_1 B_{10}^{-1}) + o_p(1), ;$

$H_{nT,(k+4,k+4)} = \frac{1}{n} \operatorname{tr}(B_{30}^{-1\prime} W_3' W_3 B_{30}^{-1} + W_3 B_{30}^{-1} W_3 B_{30}^{-1}) + o_p(1)$*;*

$H_{nT,(k+5,k+5)} = \frac{1}{2\sigma_{v0}^2} + o_p(1)$; $H_{nT,(jq)} = o_p(1)$, *for other cases with* $j, q \le k+5$.

(ii) $H_{nT,(k+5+a,k+5+b)} = \frac{1}{nT\sigma_{v0}^2} \mathtt{vec}(\dot{F}_a')'(M_{F_0} \otimes \dot{\Gamma}'\dot{\Gamma}) \mathtt{vec}(\dot{F}_b') + o_p(T^{-1})$, $a, b = 1, \ldots, k_\phi$.

(iii) $H_{nT,(j,k+5+a)} = \frac{1}{nT\sigma_{v0}^2} \operatorname{tr}(\dot{X}_j' \dot{\Gamma} \dot{F}_a' M_{F_0}) + o_p(T^{-1})$, $j \le k+3$*;*

$H_{nT,(j,k+5+a)} = o_p(T^{-1})$, $j = k+4, k+5$, $a = 1, \ldots, k_\phi$.

(iv) $H_{nT,(k+5+a,q)} = \frac{1}{nT\sigma_{v0}^2} \operatorname{tr}(\dot{X}_q' \dot{\Gamma} \dot{F}_a' M_{F_0}) + o_p(T^{-1})$, $q \le k+3$*;*

$H_{nT,(k+5+a,q)} = o_p(T^{-1})$, $q = k+4, k+5$, $a = 1 \ldots, k_\phi$.

(v) $H_{*,nT} = \tilde{H}_{nT} + o_p(1)$ *and* $H_{nT,\phi\phi}^{-1}$ *exists with a leading term stated in the proof.*

The next Lemma calculates the matrix $[H_{nT}^{-1} \Sigma_{nT} H_{nT}^{-1\prime}]_{\psi\psi}$. Recall the partition matrix expression of H_{nT}^{-1} and write $\Sigma_{nT} = [\Sigma_{nT,\psi\psi}, \Sigma_{nT,\psi\phi}; \Sigma_{nT,\phi\psi}, \Sigma_{nT,\phi\phi}]$. We can verify that

$$[H_{nT}^{-1} \Sigma_{nT} H_{nT}^{\prime -1}]_{\psi\psi} = H_{*,nT}^{-1} \Sigma_{nT}^* H_{*,nT}^{-1},$$

where $\Sigma_{nT}^* = \Sigma_{nT,\psi\psi} - H_{nT,\psi\phi} H_{nT,\phi\phi}^{-1} \Sigma_{nT,\phi\psi} - \Sigma_{nT,\psi\phi} H_{nT,\phi\phi}^{-1} H_{nT,\phi\psi} + H_{nT,\psi\phi} H_{nT,\phi\phi}^{-1} H_{nT,\phi\psi}$.

Lemma C.7. *Suppose Assumptions A-H hold and $T/n + n/T^2 \to 0$. We have $\Sigma_{nT}^* = \tilde{\Sigma}_{nT} + o_p(1)$ and $H_{*,nT}^{-1} \Sigma_{nT}^* H_{*,nT}^{-1} - \tilde{H}_{nT}^{-1} \tilde{\Sigma}_{nT} \tilde{H}_{nT}^{-1} = o_p(1)$.*

Proofs of Lemmas C1-C7

Proof of Lemma C.1: This Lemma is a special case of Lemma S2 of Moon and Weidner (2015). Note that $\tilde{F}$ contains the first r eigenvectors of

$$(\Gamma F_0' + B_{30}^{-1} \mathbb{V})' B_{30}' B_{30} (\Gamma F_0' + B_{30}^{-1} \mathbb{V}) = F_0 \dot{\Gamma}' \dot{\Gamma} F_0' + F_0 \dot{\Gamma}' \mathbb{V} + \mathbb{V}' \dot{\Gamma} F_0' + \mathbb{V}' \mathbb{V}.$$

The first term is of rank r and its nonzero r eigenvalues are $O_p(\sqrt{nT})$. The remaining terms are of smaller order in terms of spectrum norm. We can apply the perturbation theory as in Kato (2013) to obtain this expansion. ■

Proof of Lemma C.2: First, consider the cases of $j \le k+3$. Using the identity $B_{30}\mathbb{Z}(\theta_0) = \dot{\Gamma}F_0' + \mathbb{V}$ and the fact that $M_{F_0}F_0 = 0$, we have

$$\mathrm{tr}[\dot{X}_j' B_{30}\mathbb{Z}(\theta_0)M_{\tilde{F}}] = \mathrm{tr}(\dot{X}_j'\mathbb{V}M_{F_0}) + \mathrm{tr}[\dot{X}_j'(\mathbb{V} + \dot{\Gamma}F_0')(M_{\tilde{F}} - M_{F_0})].$$

By Lemma D.1 (i-ii), we can readily show that $S^p_{j,nT}(\psi_0) = \tilde{S}_{j,nT} + o_p(\sqrt{nT})$, for $j \le k+3$. Second, consider the cases of $j = k+4, k+5$. We have,

$$\begin{aligned}\tilde{S}^*_{k+4,nT}(\psi_0) =& \tfrac{1}{\sigma^2_{v0}}\,\mathrm{tr}[(\dot{\Gamma}F_0' + \mathbb{V})'W_3B_{30}^{-1}(\dot{\Gamma}F_0' + \mathbb{V})M_{\tilde{F}}] - (T-r)\,\mathrm{tr}(W_3B_{30}^{-1}) \\ =& \tfrac{1}{\sigma^2_{v0}}\,\mathrm{tr}(\mathbb{V}'W_3B_{30}^{-1}\mathbb{V}M_{F_0}) - (T-r)\,\mathrm{tr}(W_3B_{30}^{-1}) \\ &+ \tfrac{1}{\sigma^2_{v0}}\,\mathrm{tr}[[(\dot{\Gamma}F_0' + \mathbb{V})'W_3B_{30}^{-1}[(\dot{\Gamma}F_0' + \mathbb{V})(M_{\tilde{F}} - M_{F_0})].\end{aligned}$$

Lemma D.1 (iii) has shown that the last term is $o_p(\sqrt{nT})$.
To show $E(\tilde{S}_{j,nT}) = 0$, one can follow an analysis similar to the proof of Theorem 3.1. ■

Proof of Lemma C.3: (i) There are three sub-cases: (i-a) $j \le k$ and $q \le k+3$; (i-b) $j \le k$ and $q \le k+4, \ldots, k+5$; and (i-c) $j = k+1, \ldots, k+3$.

First, consider the cases (i-a) and (i-b). We have $-\frac{\partial}{\partial \psi_q}\tilde{S}^*_{j,nT}(\psi)|_{\psi_0} =$

$$\begin{cases} \frac{1}{\sigma^2_{v0}}\,\mathrm{tr}(\dot{X}_j'\dot{X}_qM_{\tilde{F}}) - \frac{1}{\sigma^2_{v0}}\,\mathrm{tr}[\dot{X}_j'(\dot{\Gamma}F_0' + \mathbb{V})\frac{\partial}{\partial\psi_q}M_{\hat{F}(\psi)}|_{\psi_0}], \\ \frac{1}{\sigma^2_{v0}}\,\mathrm{tr}\{\dot{X}_j'[W_3'B_{30}^{-1} + (B_{30}^{-1})'W_3](\dot{\Gamma}F_0' + \mathbb{V})M_{\tilde{F}}\} - \frac{1}{\sigma^2_{v0}}\,\mathrm{tr}[\dot{X}_j'(\dot{\Gamma}F_0' + \mathbb{V})\frac{\partial}{\partial\lambda_3}M_{\hat{F}(\psi)}|_{\psi_0}], \\ \frac{1}{\sigma^4_{v0}}\,\mathrm{tr}[\dot{X}_j'(\dot{\Gamma}F_0' + \mathbb{V})M_{\tilde{F}}]. \end{cases}$$

By Lemma D.2, we can readily show $\frac{1}{nT\sigma^2_{v0}}\,\mathrm{tr}(\dot{X}_j'\dot{X}_qM_{\tilde{F}}) = \frac{1}{nT\sigma^2_{v0}}\,\mathrm{tr}(\dot{X}_j'\dot{X}_qM_{F_0}) + o_p(1)$ and

$$\frac{1}{nT\sigma^2_{v0}}\,\mathrm{tr}[\dot{X}_j'(\dot{\Gamma}F_0' + \mathbb{V})\frac{\partial}{\partial\psi_q}M_{\hat{F}(\psi)}|_{\psi_0}] = \frac{1}{nT\sigma^2_{v0}}\,\mathrm{tr}(\dot{X}_j'P_{\dot{\Gamma}}\dot{X}_qM_{F_0}) + o_p(1).$$

It follows that $-\frac{1}{nT}\frac{\partial}{\partial\psi_q}\tilde{S}^*_{j,nT}(\psi)|_{\psi_0} = \frac{1}{nT\sigma^2_{v0}}\,\mathrm{tr}(\dot{X}_j'P_{\dot{\Gamma}}\dot{X}_qM_{F_0}) + o_p(1)$. For the case $q = k+4$ and $k+5$, we can show it is $o_p(1)$.

Next, consider the case (i-c). For $j = k+1, \ldots, k+3$, derivatives of $S^p_{j,nT}(\psi)$ can be similarly studied. In addition, we need to consider terms like $\frac{\partial}{\partial\beta_q}\,\mathrm{tr}[\mathbf{M}_{\hat{F}(\psi)}\mathbf{W}_1\mathbf{D}(\rho, \lambda_1, \lambda_2)]|_{\psi_0}$. Lemma D.2 has found the asymptotic leading terms. Some algebra leads to the desired results.

(ii) For the case $j = k+4$ and $q \le k+3$, we have

$$\begin{aligned}-\frac{1}{nT}\frac{\partial\tilde{S}^*_{j,nT}(\psi)}{\partial\psi_q}\,|_{\psi_0} =& \frac{1}{nT\sigma^2_{v0}}\,\mathrm{tr}[\dot{X}_q'(W_3B_{30}^{-1} + (B_{30}')^{-1}W_3')(\dot{\Gamma}F_0' + \mathbb{V})M_{\tilde{F}}] \\ &- \frac{1}{nT\sigma^2_{v0}}\,\mathrm{tr}[(\dot{\Gamma}F_0' + \mathbb{V})'W_3B_{30}^{-1}(\dot{\Gamma}F_0' + \mathbb{V})\frac{\partial M_{\hat{F}(\psi)}}{\partial\psi_q}\,|_{\psi_0}].\end{aligned}$$

We can verity that both terms on the right hand side are $o_p(1)$. The same result holds for the case $q = k+5$.

For the case $j=q=k+4$, we have

$$
\begin{aligned}
-\frac{1}{nT}\frac{\partial \tilde{S}^*_{j,nT}(\psi)}{\partial \psi_q}\mid_{\psi_0}= & \frac{1}{nT\sigma^2_{v0}}\operatorname{tr}[(\dot{\Gamma}F_0'+\mathbb{V})'(B_{30}')^{-1}W_3'W_3B_{30}^{-1}(\dot{\Gamma}F_0'+\mathbb{V})M_{\tilde{F}}] \\
& -\frac{1}{nT\sigma^2_{v0}}\operatorname{tr}[(\dot{\Gamma}F_0'+\mathbb{V})'W_3B_{30}^{-1}(\dot{\Gamma}F_0'+\mathbb{V})\frac{\partial M_{\hat{F}(\psi)}}{\partial \psi_q}\mid_{\psi_0}]+\frac{T-r}{nT}\operatorname{tr}(W_3B_{30}^{-1}W_3B_{30}^{-1}) \\
= & \frac{1}{nT}\operatorname{tr}(\mathbb{V}'B_{30}^{-1\prime}W_3'W_3B_{30}^{-1}\mathbb{V}M_{F_0})+\frac{T-r}{nT\sigma^2_{v0}}\operatorname{tr}(W_3B_{30}^{-1}W_3B_{30}^{-1})+o_p(1) \\
= & \frac{1}{n}\operatorname{tr}(B_{30}^{-1\prime}W_3'W_3B_{30}^{-1}+W_3B_{30}^{-1}W_3B_{30}^{-1})+o_p(1).
\end{aligned}
$$

(iii) The derivation is similar to parts (i-ii). ■

Proof of Lemma C.4: First, consider the case $j,q\le k$. By Lemma D.3, we can show that

$$
\tilde{\Sigma}_{nT,(jq)}=\frac{\sigma^2_{v0}}{nT}\tilde{\Pi}_j'\tilde{\Pi}_q=\frac{1}{nT\sigma^2_{v0}}\operatorname{tr}(\dot{X}_j'M_{\dot{\Gamma}}\dot{X}_qM_{F_0})=\tilde{H}_{nT,(jq)}.
$$

Second, consider the case $j\le k, q=k+1$. We have

$$
\begin{aligned}
\operatorname{Cov}(\tilde{S}_{j,nT},\tilde{S}_{q,nT}\mid \mathcal{D})= & \sigma^2_{v0}\tilde{\Pi}_{1,j}'(\tilde{\Pi}_2+\tilde{\Psi}_1\mathbf{y}_0)+\mu_3\tilde{\Pi}_{1,j}'\texttt{diag}(\tilde{\Phi}_1) \\
= & \sigma^2_{v0}\tilde{\Pi}_{1,j}'[(\tilde{\Pi}_2+\tilde{\Psi}_1\mathbf{y}_0)+\tilde{\Phi}_1\mathbf{v}]+\mu_3\tilde{\Pi}_{1,j}'\texttt{diag}(\tilde{\Phi}_1)+o_p(nT) \\
= & \frac{1}{\sigma^2_{v0}}\operatorname{tr}(\dot{X}_j'M_{\dot{\Gamma}}\dot{X}_{k+1}M_{F_0})+\mu_3\tilde{\Pi}_{1,j}'\texttt{diag}(\tilde{\Phi}_1)+o_p(nT).
\end{aligned}
$$

It follows that $\tilde{\Sigma}_{nT,(jq)}-\tilde{H}_{nT,(jq)}=\mu_3\tilde{\Pi}_{1,j}'\texttt{diag}(\tilde{\Phi}_1)/(nT)+o_p(1)$. For the other cases where $j\le k, q=k+1,\ldots,k+5$, the analysis is identical to that of $q=k+1$ case.
Third, consider the case $j=q=k+1$. We have,

$$
\begin{aligned}
\operatorname{Var}(\tilde{S}_{k+1,nT}\mid\mathcal{D})= & \sigma^2_{v0}\|\tilde{\Pi}_2+\tilde{\Psi}_1\mathbf{y}_0\|^2+2\mu_3(\tilde{\Pi}_2+\tilde{\Psi}_1\mathbf{y}_0)'\texttt{diag}(\tilde{\Phi}_1) \\
& +\sigma^4_{v0}\operatorname{tr}(\tilde{\Phi}_1\tilde{\Phi}_1+\tilde{\Phi}_1\tilde{\Phi}_1')+(\mu_4-3\sigma^4_{v0})\|\texttt{diag}(\tilde{\Phi}_1)\|^2.
\end{aligned}
$$

And we have the identity that

$$
\begin{aligned}
\tilde{H}_{nT,(jj)}= & \frac{1}{\sigma^2_{v0}}\texttt{vec}(\dot{X}_{k+1})'(M_{F_0}\otimes M_{\dot{\Gamma}})\texttt{vec}(\dot{X}_{k+1}) \\
= & \sigma^2_{v0}\|(M_{F_0}\otimes M_{\dot{\Gamma}}B_{30})(\mathbf{Q}_{-1}\mathbf{y}_0+\boldsymbol{\eta}_{\mathbf{-1}}+\mathbf{D}_{-1}\mathbf{B}_{30}^{-1}\texttt{vec}(\dot{\Gamma}F_0'+\mathbb{V}))/\sigma^2_{v0}\|^2 \\
= & \sigma^2_{v0}\|\tilde{\Pi}_2+\tilde{\Psi}_1\mathbf{y}_0+\tilde{\Phi}_1\mathbf{v}\|^2 \\
= & \sigma^2_{v0}\|\tilde{\Pi}_2+\tilde{\Psi}_1\mathbf{y}_0\|^2+\sigma^2_{v0}\mathbf{v}'\tilde{\Phi}_1'\tilde{\Phi}_1\mathbf{v}'+2\sigma^2_{v0}\mathbf{v}'\tilde{\Phi}_1'(\tilde{\Pi}_2+\tilde{\Psi}_1\mathbf{y}_0) \\
= & \sigma^2_{v0}\|\tilde{\Pi}_2+\tilde{\Psi}_1\mathbf{y}_0\|^2+\sigma^4_{v0}\operatorname{tr}(\tilde{\Phi}_1\tilde{\Phi}_1')+o_p(nT).
\end{aligned}
$$

Combining the above two identities, we can find the leading term of $\tilde{\Xi}_{nT,(jj)}$. The rest of the results can be verified by similar analysis. ■

Proof of Lemma C.5: (i) By Lemma D.3 we have $\operatorname{Var}(\Pi_{4+a}'\mathbf{v})=\sigma^2_{v0}\|\Pi_{4+a}\|^2=O_p(n)$ and $\operatorname{Var}(\mathbf{v}'\Phi_{5+a}\mathbf{v})=(\mu_4-3\sigma^2_{v0})\|\texttt{diag}(\Phi_{5+a})\|^2=O_p(n/T^2)$. It follows that $\Pi_{4+a}'\mathbf{v}/\sqrt{nT}=$

$O_p(T^{-1/2})$ and $\mathbf{v}'\Phi_{5+a}\mathbf{v}/\sqrt{nT} = O_p(T^{-3/2})$. Hence, one can easily see that

$$\begin{aligned}\Sigma_{nT,(k+5+a,k+5+b)} =& \tfrac{\sigma_{v0}^2}{nT}\Pi_{4+a}'\Pi_{4+b} + O_p(T^{-2}) = \tfrac{1}{\sigma_{v0}^2 nT}\operatorname{tr}(\dot{\Gamma}'\dot{\Gamma}\dot{F}_a' M_{F_0}\dot{F}_b) + O_p(T^{-2}) \\ =& \tfrac{1}{\sigma_{v0}^2 nT}\mathtt{vec}(\dot{F}_a')'(M_{F_0}\otimes\dot{\Gamma}'\dot{\Gamma})\mathtt{vec}(\dot{F}_b') + O_p(T^{-2}).\end{aligned}$$

In addition, we show that the absolute column sums of $\Sigma_{nT,\phi\phi}$ and $H_{nT,\phi\phi}$ is $o_p(T^{-1})$. Therefore, we have the spectrum norm bound of their difference. For the estimator $\hat{\Sigma}_{nT,\phi\phi}$, we only need to find the asymptotic orders of $\frac{1}{nT}\sum_{i=1}^n(\hat{\mathbf{g}}_i\hat{\mathbf{g}}_i' - \mathbf{g}_i\mathbf{g}_i')$. The derivation is tedious but straightforward, and thus are omitted.

(ii) First consider the $j \le k$. Similar to part (i), we have that

$$\Sigma_{nT,(k+5+a,j)} = \tfrac{\sigma_{v0}^2}{nT}\Pi_{1,j}'\Pi_{4+a} + o_p(T^{-3/2})\tfrac{1}{\sigma_{v0}^2 nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0}\otimes\dot{\Gamma})\mathtt{vec}(\dot{F}_a') + o_p(T^{-3/2}).$$

Next, consider the case $j = k+1$. We have that

$$\begin{aligned}\Sigma_{nT,(k+5+a,j)} &= \tfrac{\sigma_{v0}^2}{nT}(\Pi_2 + \Psi_1\mathbf{y}_0)'\Pi_{4+a} + \tfrac{\mu_3}{nT}\mathtt{diag}(\Phi_1)'\Pi_{4+a} + O_p(T^{-3/2}) \\ &= \tfrac{\sigma_{v0}^2}{nT}(\Pi_2 + \Psi_1\mathbf{y}_0 + \Phi_1\mathbf{v})'\Pi_{4+a} + \tfrac{\mu_3}{nT}\mathtt{diag}(\Phi_1)'\Pi_{4+a} + o_p(T^{-1}) \\ &= \tfrac{1}{\sigma_{v0}^2 nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0}\otimes\dot{\Gamma})\mathtt{vec}(\dot{F}_a') + \tfrac{\mu_3}{nT}\mathtt{diag}(\Phi_1)'\Pi_{4+a} + o_p(T^{-1}).\end{aligned}$$

The cases $j = k+2,\ldots,k+5$ can be similarly studied.

(iii) The proof is identical to that of Lemma C.4. ■

Proof of Lemma C.6: (i) It suffices to study that terms like $\frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes I_n)\dot{\mathbf{D}}_{-1,\rho}]$. By Lemma D.4, we have

$$\begin{aligned}\tfrac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes I_n)\dot{\mathbf{D}}_{-1,\psi_j}] =& o_p(1),\ \text{for } j = k+1, k+2, k+3 \\ \tfrac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes I_n)\dot{\mathbf{D}}_{\psi_j}] =& o_p(1),\ \text{for } j = k+1, k+3 \\ \tfrac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes W_2)\dot{\mathbf{D}}_{-1,\psi_j}] =& o_p(1),\ \text{for } j = k+1, k+2, k+3,\end{aligned}$$

where $\psi_{k+1} = \rho$, $\psi_{k+2} = \lambda_1$, and $\psi_{k+3} = \lambda_2$.

(ii) Note that $H_{nT,(k+5+a,k+5+b)} = \frac{1}{2nT\sigma_{v0}^2}\operatorname{tr}[\mathbb{Z}(\theta_0)'B_{30}'B_{30}\mathbb{Z}(\theta_0)\ddot{P}_{F,ab}]$. We have,

$$\begin{aligned}\ddot{P}_{F,ab} = \tfrac{\partial^2 P_F}{\partial\phi_a\partial\phi_b}|_{\psi_0} =& F_0(F_0'F_0)^{-1}(\dot{F}_a' M_{F_0}\dot{F}_b + \dot{F}_b' M_{F_0}\dot{F}_a)(F_0'F_0)^{-1}F_0' \\ &- M_{F_0}[\dot{F}_a(F_0'F_0)^{-1}\dot{F}_b' + \dot{F}_b(F_0'F_0)^{-1}\dot{F}_a']M_{F_0} \\ &+ M_{F_0}\dot{F}_a(F_0'F_0)^{-1}F_0'\dot{F}_b(F_0'F_0)^{-1}F_0' + F_0(F_0'F_0)^{-1}\dot{F}_b'F_0(F_0'F_0)^{-1}\dot{F}_a' M_{F_0} \\ &+ M_{F_0}\dot{F}_b(F_0'F_0)^{-1}F_0'\dot{F}_a(F_0'F_0)^{-1}F_0' + F_0(F_0'F_0)^{-1}\dot{F}_a'F_0(F_0'F_0)^{-1}\dot{F}_b' M_{F_0}.\end{aligned}$$

Plugging in the expression of $\ddot{P}_{F,ab}$ into $H_{nT,(k+5+a,k+5+b)} = \frac{1}{2nT\sigma_{v0}^2}\operatorname{tr}[\mathbb{Z}(\theta_0)'B_{30}'B_{30}\mathbb{Z}(\theta_0)\ddot{P}_{F,ab}]$, we obtain the results after some tedious algebra.

(iii) First consider the $j \leq k$ case. We have,

$$\begin{aligned}
H_{nT,(j,k+5+a)} =& -\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{\Gamma}F_0'\dot{P}_{F,a}) - \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\mathbb{V}\dot{P}_{F,a}) \\
=&\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\dot{\Gamma}\dot{F}_a'M_{F_0}) + O_p(n^{-1/2}T^{-1}) \\
=&\frac{1}{nT\sigma_{v0}^2}\texttt{vec}(\dot{F}_a')'(M_{F_0}\otimes\dot{\Gamma}')\texttt{vec}(\dot{X}_j) + O_p(n^{-1/2}T^{-1}).
\end{aligned}$$

Next consider the cases $j = k+1$, we have,

$$H_{nT,(j,k+5+a)} = \frac{1}{T\sigma_{v0}^2}\operatorname{tr}[\dot{X}_j'B_{30}\mathbb{Z}(\theta_0)\dot{P}_{F,a}] + \frac{1}{nT}\operatorname{tr}[(\dot{P}_{F,a}\otimes I_n)\mathbf{D}_{-1}]$$

The analysis is similar to the $j \leq k$ case. However, one difference is that now we need to use the result $-\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_{k+1}'\mathbb{V}\dot{P}_{F,a}) + \frac{1}{nT}\operatorname{tr}[(\dot{P}_{F,a}\otimes I_n)\mathbf{D}_{-1}] = o_p(T^{-1})$. The key point is that $\dot{X}_{k+1}$ is $B_{30}Y_{-1}$ which is no longer exogenous. Its mean is $\frac{1}{nT}\operatorname{tr}[(\dot{P}_{F,a}\otimes I_n)\mathbf{D}_{-1}]$, which is not of small order. But the recentered term can be verified to be small. The same analysis procedure gives us the result for cases $j = k+2, \ldots, k+5$.

(iv) It suffices to find the exact expression of $-\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\mathbb{V}\dot{P}_{F,a})$:

$$\begin{aligned}
-\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'\mathbb{V}\dot{P}_{F,a}) =& \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[\dot{X}_j'\mathbb{V}(M_{F_0}\dot{F}_a(F_0'F_0)^{-1}F_0' + F_0(F_0'F_0)^{-1}\dot{F}_a'M_{F_0})] \\
=& \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[(F_0'F_0)^{-1}\dot{F}_a'M_{F_0}(\dot{X}_j'\mathbb{V} + \mathbb{V}'\dot{X}_j)F_0] \\
=& \frac{1}{nT\sigma_{v0}^2}\texttt{vec}[F_0'(\dot{X}_j'\mathbb{V} + \mathbb{V}'\dot{X}_j)]'[M_{F_0}\otimes(F_0'F_0)^{-1}]\texttt{vec}(\dot{F}_a').
\end{aligned}$$

This term is $O_p(T^{-1})$, for $j = k+1, \ldots, k+3$.

(v) Recall $H_{nT,*} = H_{nT,\psi\psi} - H_{nT,\psi\phi}H_{nT,\phi\phi}^{-1}H_{nT,\phi\psi}$, in the inverse of the partitioned H_{nT}:

$$H_{nT}^{-1} = \begin{bmatrix} H_{nT,*}^{-1} & -H_{nT,*}^{-1}H_{nT,\psi\phi}H_{nT,\psi\psi}^{-1} \\ -H_{nT,\phi\phi}^{-1}H_{nT,\phi\psi}H_{nT,*}^{-1} & H_{nT,\phi\phi}^{-1} + H_{nT,\phi\phi}^{-1}H_{nT,\phi\psi}H_{nT,*}^{-1}H_{nT,\psi\phi}H_{nT,\phi\phi}^{-1}\end{bmatrix}.$$

The key step to establishing the result is to study the second term $H_{nT,\psi\phi}(H_{nT,\phi\phi})^{-1}H_{nT,\phi\psi}$, which involves high-dimensional matrices.

First, consider $(H_{nT,\phi\phi})^{-1}$, we can show that $\mathcal{S} = [\texttt{vec}(\dot{F}_1'), \ldots, \texttt{vec}(\dot{F}_{r(T-r)'})] = S_* \otimes L$, where $S_* = [I_{T-r}, 0_{r\times(T-r)}']'$, $L = [e_{r,r}, \ldots, e_{1,r}]$, and $e_{j,r}$ is the jth unit vector of dimension r. Using the result $M_{F_0} = M_{F_0}\mathcal{S}_*(\mathcal{S}_*'M_{F_0}\mathcal{S}_*)^{-1}\mathcal{S}_*'M_{F_0}$ in Lemma D.5. We can obtain,

$$\|H_{nT,\phi\phi} - \frac{1}{nT\sigma_{v0}^2}(S_*'M_{F_0}S_*\otimes L\dot{\Gamma}'\dot{\Gamma}L)\|_{\texttt{sp}} = o_p(T^{-1}).$$

It follows that $\|(H_{nT,\phi\phi})^{-1} - nT\sigma_{v0}^2[(S_*'M_{F_0}S_*)^{-1}\otimes L(\dot{\Gamma}'\dot{\Gamma})^{-1}L]\|_{\texttt{sp}} = o_p(T)$.

Next, we consider $H_{nT,\psi\phi}$. For its jth row, $H_{nT,j\phi}$, we obtain

$$\begin{aligned}
&\|H_{nT,j\phi} - \frac{1}{nT\sigma_{v0}^2}\texttt{vec}(\dot{X}_j)'(M_{F_0}\mathcal{S}_*\otimes\dot{\Gamma}L)\| = o_p(T^{-1/2}), && \text{for } j = 1, \ldots, k+3; \\
&\|H_{nT,j\phi}\| = o_p(T^{-1/2}), && \text{for } j = k+4, k+5.
\end{aligned}$$

Finally, consider $H_{nT,\phi\psi}$. Its jth column has the following property,
$H_{nT,\phi j} = \frac{1}{nT\sigma_{v0}^2}\{(\mathcal{S}_*' M_{F_0} \otimes L\dot{\Gamma}')\texttt{vec}(\dot{X}_j) + [\mathcal{S}_*' M_{F_0} \otimes L(F_0'F_0)^{-1}]\texttt{vec}[F_0'(\dot{X}_j'\mathbb{V} + \mathbb{V}'\dot{X}_j)]\}$.
One can verify that

$$\begin{aligned}&\frac{1}{nT\sigma_{v0}^2}\texttt{vec}(\dot{X}_q)'[M_{F_0} \otimes \dot{\Gamma}(\dot{\Gamma}'\dot{\Gamma})^{-1}(F_0'F_0)^{-1}]\texttt{vec}[F_0'(\dot{X}_j'\mathbb{V} + \mathbb{V}'\dot{X}_j)]\\ &= \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[\dot{X}_q'\dot{\Gamma}(\dot{\Gamma}'\dot{\Gamma})^{-1}(F_0'F_0)^{-1}F_0'(\dot{X}_j'\mathbb{V} + \mathbb{V}'\dot{X}_j)] = o_p(1).\end{aligned}$$

Combine the above results, we have, the (j,q) element of $H_{nT,\psi\phi}(H_{nT,\phi\phi})^{-1}H_{nT,\phi\psi}$ is

$$\frac{1}{nT\sigma_{v0}^2}\texttt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})\texttt{vec}(\dot{X}_q) + o_p(1) \text{ for } j,q \leq k+3$$

and is $o_p(1)$ for $j,q = k+4, k+5$. Summing up, we have shown that $H_{nT,*} = \tilde{H}_{nT} + o_p(1)$. ■

Proof of Lemma C.7: We take five steps to establish the result and within each step we discuss several cases.

(i) First, we consider the term $H_{nT,\psi\phi}(H_{nT,\phi\phi})^{-1}\Sigma_{nT,\phi\psi}$. The leading term of its (j,q)th entire can be summarized as follows:

- (i-1) $\frac{1}{\sigma_{v0}^2 nT}\texttt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})\texttt{vec}(\dot{X}_q)$, for $j,q \leq k+3$;
- (i-2) $\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'P_{\dot{\Gamma}}\dot{X}_q M_{F_0}) + \frac{\mu_3}{nT}\texttt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})\texttt{diag}(\Phi_a) + o_p(1)$,
  for $j \leq k+3, q = k+a, a = 1,2,3$;
- (i-3) $\frac{\mu_3}{nT}\texttt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})\texttt{diag}(\Phi_a)$, for $j \leq k+3, q = k+a, a = 4,5$;
- (i-4) $o_p(1)$, for $j = k+4, k+5$.

To see the results, we take $q = k+1$ in case (i-2) as an example, the leading term of the (j,q)th entry of $H_{nT,\psi\phi}(H_{nT,\phi\phi})^{-1}\Sigma_{nT,\phi\psi}$ is in this case,

$$\begin{aligned}&\frac{1}{nT}\texttt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})[\Pi_2 + \Psi_1\mathbf{y}_0 + \mu_3\texttt{diag}(\Phi_1)]\\ &= \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'P_{\dot{\Gamma}}\dot{X}_q M_{F_0}) + \frac{\mu_3}{nT}\texttt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})\texttt{diag}(\Phi_1) + o_p(1).\end{aligned}$$

The equality is by the fact that $\frac{1}{nT}\texttt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})\Phi_1\mathbf{v} = 0$ and the decomposition of $\mathbf{Y}_{-1}$. Other cases can be shown similarly.

(ii) Second, we consider the term $\Sigma_{nT,\psi\phi}(H_{nT,\phi\phi})^{-1}H_{nT,\phi\psi}$. Compared to $H_{nT,\psi\phi}$, the $(*, k+1 : k+3)$ entries of $H_{nT,\phi\psi}$ has an additional term, as in Lemma C.6. However, we can show that these additional term only results in additional $o_p(1)$ terms in the final product $\Sigma_{nT,\psi\phi}(H_{nT,\phi\phi})^{-1}H_{nT,\phi\psi}$. The leading terms of the (j,q)th entry is summarized as:

- (ii-1) $\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'P_{\dot{\Gamma}}\dot{X}_q M_{F_0})$, for $j,q \leq k+3$;
- (ii-2) $\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'P_{\dot{\Gamma}}\dot{X}_q M_{F_0}) + \frac{\mu_3}{nT}\texttt{vec}(\dot{X}_q)'(M_{F_0} \otimes P_{\dot{\Gamma}})\texttt{diag}(\Phi_a)$,
  for, $j = k+a, a = 1,2,3$, $q \leq k+3$;
- (ii-3) $\frac{\mu_3}{nT}\texttt{vec}(\dot{X}_q)'(M_{F_0} \otimes P_{\dot{\Gamma}})\texttt{diag}(\Phi_a)$, for $j = k+a, a = 4,5$, $q \leq k+3$;

(ii-4) $o_p(1)$, for $q = k+4, k+5$.

(iii) Third, consider the last term $H_{nT,\psi\phi}(H_{nT,\phi\phi})^{-1}H_{nT,\phi\psi}$. We have already found the leading term of its (j,q)th entry in Lemma C.6, which can be summarized as follows:

(iii-1) $\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \dot{X}_q M_{F_0})$, for $j, q \le k+3$;

(iii-2) $o_p(1)$, for the other cases.

(iv) Then we derive the summation of three matrices studied in parts (i-iii).

(iv-1) For $j, q \le k$ the (j,q)th entry has leading term $-\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \dot{X}_q M_{F_0})$;

(iv-2) For $j \le k$, $q = k+a$, $a = 1,2,3$, the (j,q)th entry has the leading term

$$-\tfrac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \dot{X}_q M_{F_0}) - \tfrac{\mu_3}{nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})\mathtt{diag}(\Phi_a).$$

The leading term of (q,j)th entry is identical to the above one.

(iv-3) For $j \le k$, $q = k+a$, $a = 4,5$, the (j,q)th entry has the leading term

$$-\tfrac{\mu_3}{nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})\mathtt{diag}(\Phi_a).$$

The leading term of (q,j)th entry is identical to that of the (j,q)th entry.

(iv-4) For $j = k+a$ and $q = k+b$, where $a, b = 1,2,3$, the leading term is

$$-\tfrac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \dot{X}_q M_{F_0}) - \tfrac{\mu_3}{nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0} \otimes P_{\dot{\Gamma}})\mathtt{diag}(\Phi_b) - \tfrac{\mu_3}{nT}\mathtt{vec}(\dot{X}_q)'(M_{F_0} \otimes P_{\dot{\Gamma}})\mathtt{diag}(\Phi_a);$$

(iv-5) For $j = k+a$ and $q = k+b$, where $a, b = 4,5$, the terms are $o_p(1)$.

(v) Finally, we look at the difference between $\tilde{\Sigma}_{nT}$ and Σ_{nT}. It suffices to compare the leading terms as in Lemmas C.4 and C.5.

(v-1) For $j \le k$, the leading term of $\tilde{\Sigma}_{jq,nT}$ is

$\frac{\sigma_{v0}^2}{nT}\tilde{\Pi}_{1,j}'\tilde{\Pi}_{1,q}$, for $q \le k$;

$\frac{\sigma_{v0}^2}{nT}\tilde{\Pi}_{1,j}'\tilde{\Pi}_{k+a} + \frac{\mu_3}{nT}\tilde{\Pi}_{1,j}'\mathtt{diag}(\tilde{\Phi}_a)$, for $q = k+a$, $a = 1,2,3$;

$\frac{\mu_3}{nT}\tilde{\Pi}_{1,j}'\mathtt{diag}(\tilde{\Phi}_a)$, for $q = k+a$, $a = 4,5$.

By Lemma D.5, the above terms can be rewritten as

$\Sigma_{jq,nT} - \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \dot{X}_q M_{F_0}) + o_p(1)$, for $q \le k$;

$\Sigma_{jq,nT} - \frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \dot{X}_q M_{F_0}) - \frac{\mu_3}{nT}\Pi_{1,j}'(I_T \otimes P_{\dot{\Gamma}})\mathtt{diag}(\Phi_a) + o_p(1)$, for $q = k+a, a = 1,2,3$;

$\Sigma_{jq,nT} - \frac{\mu_3}{nT}\Pi_{1,j}'(I_T \otimes P_{\dot{\Gamma}})\mathtt{diag}(\Phi_a) + o_p(1)$, for $q = k+a$, $a = 4,5$.

(v-2) Consider cases with $j = k+a$, and $a = 1,2,3$. We show the case $q = k+1$.

$$\begin{aligned}\tilde{\Sigma}_{jj,nT} &= \tfrac{\sigma_{v0}^2}{nT}\|\tilde{\Pi}_2 + \tilde{\Psi}_1\mathbf{y}_0\|^2 + \tfrac{2\mu_3}{nT}(\tilde{\Pi}_2 + \tilde{\Psi}_1\mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_1) + \tfrac{\sigma_{v0}^4}{nT}\operatorname{tr}(\tilde{\Phi}_1\tilde{\Phi}_1 + \tilde{\Phi}_1\tilde{\Phi}_1') + \tfrac{(\mu_4 - 3\sigma_{v0}^4)}{nT}\|\mathtt{diag}(\tilde{\Phi}_1)\|^2 \\ &= \tfrac{1}{\sigma_{v0}^2 T}\operatorname{tr}[\dot{X}_j'\dot{X}_j M_{F_0}] + \tfrac{2\mu_3}{nT}(\Pi_2 + \Psi_1\mathbf{y}_0)'\mathtt{diag}(\tilde{\Phi}_1) + \tfrac{\sigma_{v0}^4}{nT}\operatorname{tr}(\Phi_1\Phi_1) + \tfrac{(\mu_4 - 3\sigma_{v0}^4)}{nT}\|\mathtt{diag}(\Phi_1)\|^2 \\ &\quad - \tfrac{1}{\sigma_{v0}^2 nT}\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \dot{X}_j M_{F_0}) - \tfrac{2\mu_3}{nT}\sigma_{v0}^2(\Pi_2 + \Psi_1\mathbf{y}_0)'(M_{F_0} \otimes P_{\dot{\Gamma}})\mathtt{diag}(\tilde{\Phi}_1) + o_p(1)\end{aligned}$$

$$= \Sigma_{jq,nT} - \frac{1}{\sigma_{v0}^2 nT}\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \dot{X}_j M_{F_0}) - \frac{2\mu_3}{nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0}\otimes P_{\dot{\Gamma}})\mathtt{diag}(\tilde{\Phi}_1) + o_p(1).$$

For other cases we have can follow a similar analysis to show the following results.

For $j = k + a, q = k + b,\ a, b \leq 3$,

$$\tilde{\Sigma}_{jq,nT} = \Sigma_{jq,nT} - \frac{1}{\sigma_{v0}^2 nT}\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \dot{X}_j M_{F_0}) - \frac{\mu_3}{nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0}\otimes P_{\dot{\Gamma}})\mathtt{diag}(\tilde{\Phi}_b) \\ - \frac{\mu_3}{nT}\mathtt{vec}(\dot{X}_q)'(M_{F_0}\otimes P_{\dot{\Gamma}})\mathtt{diag}(\tilde{\Phi})_a + o_p(1);$$

For $j = k + a, q = k + b,\ a \leq 3$, and $b = 4, 5$,

$$\tilde{\Sigma}_{jq,nT} = \Sigma_{jq,nT} - \frac{\mu_3}{nT}\mathtt{vec}(\dot{X}_j)'(M_{F_0}\otimes P_{\dot{\Gamma}})\mathtt{diag}(\Phi_b) + o_p(1).$$

(v-3) Consider the cases with $j = k + 4, k + 5$. We can show.

$$\tilde{\Sigma}_{jq,nT} = \Sigma_{jq,nT} + o_p(1), \text{ for } q = k + 4, k + 5.$$

Therefore,we have shown that $\Sigma^*_{nT} = \tilde{\Sigma}_{nT} + o_p(1)$. ■

Appendix D: Additional Technical Lemmas

Some technical lemmas are given in this appendix, which are used in Appendix C. The following results are useful in their proofs: $\mathbf{D} = \sum_{j=0}^{T-1}(J_T^j \otimes \mathcal{B}_0^j B_{10}^{-1})$ and $\mathbf{D}_{-1} = \sum_{j=1}^{T-1}(J_T^j \otimes \mathcal{B}_0^{j-1} B_{10}^{-1})$, where $J_T = [0_{1\times(T-1)}, 0; I_{T-1}, 0_{(T-1)\times 1}]$; $\|\mathbf{D}\|_F = O(\sqrt{nT})$ and $\|\mathbf{D}_{-1}\|_F = O(\sqrt{nT})$. Let $\mathtt{mat}(\cdot)$ be the reverse operator of $\mathtt{vec}$, that is $\mathtt{mat}(\mathtt{vec}(X_j)) = X_j$.

Lemma D.1. *Suppose Assumptions A-H hold and $T/n + n/T^2 \to 0$. Then we have*

(i) $\operatorname{tr}[\dot{X}_j'(\mathbb{V} + \dot{\Gamma}F_0')(M_{\tilde{F}} - M_{F_0})] = -\operatorname{tr}(\dot{X}_j' P_{\dot{\Gamma}} \mathbb{V} M_{F_0}) + o_p(\sqrt{nT})$,

(ii) $\operatorname{tr}[(M_{\tilde{F}} \otimes I_n)\mathbf{D}_{-1}] = \operatorname{tr}[(M_{F_0} \otimes M_{\dot{\Gamma}})\mathbf{D}_{-1}] + o_p(\sqrt{nT})$,
$\operatorname{tr}[(M_{\tilde{F}} \otimes W_1)\mathbf{D}] = \operatorname{tr}[(M_{F_0} \otimes M_{\dot{\Gamma}} W_1)\mathbf{D}] + o_p(\sqrt{nT})$
$\operatorname{tr}[(M_{\tilde{F}} \otimes W_2)\mathbf{D}_{-1}] = \operatorname{tr}[(M_{F_0} \otimes M_{\dot{\Gamma}} W_2)\mathbf{D}_{-1}] + o_p(\sqrt{nT})$

(iii) $\operatorname{tr}[(\mathbb{V} + \dot{\Gamma}F_0')'(\mathbb{V} + \dot{\Gamma}F_0')(M_{\tilde{F}} - M_{F_0})] = o_p(\sqrt{nT})$,
$\operatorname{tr}[(\mathbb{V} + \dot{\Gamma}F_0')' W_3 B_{30}^{-1}(\mathbb{V} + \dot{\Gamma}F_0')(M_{\tilde{F}} - M_{F_0})] = o_p(\sqrt{nT})$.

Lemma D.2. *Suppose Assumptions A-H hold and $T/n + n/T^2 \to 0$. Let $\dot{X}_{k+1} = B_{30}Y_{-1}$, $\dot{X}_{k+2} = B_{30}W_1 Y$, $\dot{X}_{k+3} = B_{30}W_2 Y_{-1}$, and $\dot{X}_{k+4} = W_3 B_{30}^{-1}(\dot{\Gamma}F_0' + \mathbb{V})$. We have,*

(i) $\frac{\partial}{\partial\psi_q} M_{\hat{F}(\psi)}\big|_{\psi_0} = M_{F_0}\dot{X}_p'(F_0\dot{\Gamma}')^{\dagger} + (\dot{\Gamma}F_0')^{\dagger}\dot{X}_q M_{F_0}$*, for $p = 1, \ldots, k + 4$;*

(ii) $\frac{\partial}{\partial\psi_p}\operatorname{tr}[\mathbf{M}_{\hat{F}(\psi)}\mathbf{D}_{-1}(\rho, \lambda_1, \lambda_2)]\big|_{\psi_0}$ *and* $\frac{\partial}{\partial\psi_p}\operatorname{tr}[\mathbf{M}_{\hat{F}(\psi)}\mathbf{W}_2\mathbf{D}_{-1}(\rho, \lambda_1, \lambda_2)]\big|_{\psi_0}$ *are both $o_p(nT)$, for $p = 1, \ldots, k + 4$;*

(iii) $\frac{\partial}{\partial\psi_p}\operatorname{tr}[\mathbf{M}_{\hat{F}(\psi)}\mathbf{W}_1\mathbf{D}(\rho, \lambda_1, \lambda_2)]\big|_{\psi_0} = (T - r)\operatorname{tr}(W_1 B_{10}^{-1} W_1 B_{10}^{-1}) + o_p(nT)$*, for $p = k + 2$, and* $\frac{\partial}{\partial\psi_p}\operatorname{tr}[\mathbf{M}_{\hat{F}(\psi)}\mathbf{W}_1\mathbf{D}(\rho, \lambda_1, \lambda_2)]\big|_{\psi_0} = o_p(nT)$*, for $p \neq k + 2$.*

(iv) $\frac{\partial}{\partial\psi_p}\operatorname{tr}[W_3 B_3^{-1}(\lambda_3)]\big|_{\psi_0} = \operatorname{tr}(W_3 B_{30}^{-1} W_3 B_{30}^{-1})$*, for $p = k + 4$ and* $\frac{\partial}{\partial\psi_p}\operatorname{tr}[W_3 B_3^{-1}(\lambda_3)]\big|_{\psi_0} = 0$*, for $p \neq k + 4$.*

Lemma D.3. *For a random vector* $\mathbf{v}$ *with i.i.d. zero mean entries, whose third and fourth moments are denoted as* μ_3 *and* μ_4*, we can obtain that*

$$\begin{aligned}&cov(\Pi_a'\mathbf{v}+\mathbf{v}'\Phi_a\mathbf{v}-\sigma_{v0}^2\operatorname{tr}(\Phi_a),\Pi_b'\mathbf{v}+\mathbf{v}'\Phi_b\mathbf{v}-\sigma_{v0}^2\operatorname{tr}(\Phi_b))\\=&\sigma_{v0}^2\Pi_a'\Pi_b+\mu_3[\Pi_a'\mathtt{diag}(\Phi_b)+\Pi_b'\mathtt{diag}(\Phi_a)]+\sigma_{v0}^4\operatorname{tr}(\Phi_a\Phi_b+\Phi_a'\Phi_b)\\&+(\mu_4-3\sigma_{v0}^4)\mathtt{diag}(\Phi_a)'\mathtt{diag}(\Phi_b)\end{aligned}$$

Lemma D.4. *Suppose Assumptions A-H hold and* $T/n+n/T^2\to 0$*. We have,*

(i) $\frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes I_n)\dot{\mathbf{D}}_{-1,\psi_j}]=o_p(1)$ *and* $\frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes W_2)\dot{\mathbf{D}}_{-1,\psi_j}]=o_p(1)$*, for* $j=k+1,k+2,k+3$*;*

(ii) $\frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes I_n)\dot{\mathbf{D}}_{\psi_j}]=o_p(1)$*, for* $j=k+1,k+3$*.*

(iii) $\frac{1}{nT}\operatorname{tr}[(M_{F_0}\otimes W_1)\dot{\mathbf{D}}_{\lambda_1}]=\frac{T-r}{nT}\operatorname{tr}[W_1B_{10}^{-1}W_1B_{10}^{-1}]+o_p(1)$*.*

Lemma D.5. *Suppose Assumptions A-H hold and* $T/n+n/T^2\to 0$*.*

(i) $\tilde{\Pi}_a=(I_T\otimes M_{\dot{\Gamma}})\Pi_a=\Pi_a-(I_T\otimes P_{\dot{\Gamma}})\Pi_a$*, for* $a=1,\dots,4$*,*
$\tilde{\Psi}_b=(I_T\otimes M_{\dot{\Gamma}})\Psi_b=\Psi_b-(I_T\otimes P_{\dot{\Gamma}})\Psi_b$*, for* $b=1,2,3$*, and*
$\tilde{\Phi}_c=(I_T\otimes M_{\dot{\Gamma}})\Phi_c=\Phi_c-(I_T\otimes P_{\dot{\Gamma}})\Phi_c$*, for* $c=1,2,3$*;*

(ii) $\|\mathtt{diag}[(I_T\otimes P_{\dot{\Gamma}})\Phi_a]\|_\infty=O_p(n^{-1/2})$*, and*
$\|\mathtt{diag}[(I_T\otimes P_{\dot{\Gamma}})\Phi_a]\|=O_p(\sqrt{T/n})$*, for* $a=1,2,3$*;*

(iii) $\tilde{\Pi}_a'\tilde{\Pi}_b=\Pi_a'\Pi_b-\Pi_a'(I_T\otimes P_{\dot{\Gamma}})\Pi_b$*,*
$\tilde{\Pi}_a'\mathtt{diag}(\tilde{\Phi}_b)=\Pi_a'\mathtt{diag}(\Phi_b)-\Pi_a'(I_T\otimes P_{\dot{\Gamma}})\mathtt{diag}(\Phi_b)+o_p(nT)$*, and*
$\mathtt{diag}(\tilde{\Phi}_a)'\mathtt{diag}(\tilde{\Phi}_b)=diag(\Phi_a)'\mathtt{diag}(\Phi_b)+o_p(nT)$*;* $\operatorname{tr}(\tilde{\Phi}_a\tilde{\Phi}_b)=\operatorname{tr}(\Phi_a\Phi_b)+o_p(nT)$*;*

(iv) $\frac{\sigma_{v0}^2}{nT}\tilde{\Pi}_{1,j}'\tilde{\Pi}_{1,q}=\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'M_{\dot{\Gamma}}\dot{X}_qM_{F_0})$*, for* $j,q\le k$*,*
$\frac{\sigma_{v0}^2}{nT}\tilde{\Pi}_{1,j}'(\tilde{\Pi}_{a+1}+\tilde{\Psi}_a\mathbf{y}_0)=\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_j'M_{\dot{\Gamma}}\dot{X}_{k+a}M_{F_0})+o_p(1)$*, for* $j\le k$*,* $a=1,2,3$*,*
$\frac{\sigma_{v0}^2}{nT}(\tilde{\Pi}_{b+1}+\tilde{\Psi}_b\mathbf{y}_0)'(\tilde{\Pi}_{a+1}+\tilde{\Psi}_a\mathbf{y}_0)+\frac{\sigma_{v0}^4}{nT}\operatorname{tr}(\tilde{\Phi}_a'\tilde{\Phi}_b)=\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}(\dot{X}_{k+a}'M_{\dot{\Gamma}}\dot{X}_{k+b}M_{F_0})+o_p(1)$*, for* $a,b=1,2,3$*;*

(v) Let Π_{4+s} *be defined as in Section 6. We have* $[\Pi_{4+1},\dots,\Pi_{4+\phi_s}]=\frac{1}{\sigma_{v0}^2}(M_{F_0}\mathcal{S}_*\otimes\dot{\Gamma}L)$*. In addition, we have* $M_{F_0}=M_{F_0}\mathcal{S}_*(\mathcal{S}_*'M_{F_0}\mathcal{S}_*)^{-1}\mathcal{S}_*'M_{F_0}$*.*

Proofs of Lemmas D1-D5

Proof of Lemma D.1: (i) First, we study $\operatorname{tr}[\dot{X}_j'\dot{\Gamma}F_0'(M_{\tilde{F}}-M_{F_0})]$, and $\operatorname{tr}[\dot{X}_j'\mathbb{V}(M_{\tilde{F}}-M_{F_0})]$.

(i-1) Consider the first term. The decomposition of $M_{\tilde{F}}-M_{F_0}$ in Lemma C.1 leads to

$$\operatorname{tr}[\dot{X}_j'\dot{\Gamma}F_0'(M_{\tilde{F}}-M_{F_0})]=\operatorname{tr}(\dot{X}_j'\dot{\Gamma}F_0'M_{\tilde{F},v}^{(1)})+\operatorname{tr}(\dot{X}_j'\dot{\Gamma}F_0'M_{\tilde{F},v}^{(2)})+\operatorname{tr}(\dot{X}_j'\dot{\Gamma}F_0'M_{\tilde{F},v}^{(rem)}),$$

where the first term $\operatorname{tr}(\dot{X}_j'\dot{\Gamma}F_0'M_{\tilde{F},v}^{(1)})=-\operatorname{tr}(\dot{X}_j'P_{\dot{\Gamma}}\mathbb{V}M_{F_0})$, and the third term $\operatorname{tr}(\dot{X}_j'\dot{\Gamma}F_0'M_{\tilde{F},v}^{(rem)})=O_p(n/\sqrt{T})=o_p(\sqrt{nT})$.

For the second term, plugging the expression of $M^{(2)}_{\tilde{F},v}$ given in Lemma C.1 into $\mathrm{tr}(\dot{X}_j'\dot{\Gamma}F_0'M^{(2)}_{\tilde{F},v})$, we obtain $\mathrm{tr}(\dot{X}_j'\dot{\Gamma}F_0'M^{(2)}_{\tilde{F},v}) \equiv A = A_1 + A_2 + A_3$, where

$$A_1 = \mathrm{tr}[\dot{X}_j M_{F^0}\mathbb{V}'(F_0\dot{\Gamma}')^\dagger \mathbb{V}'(F_0\dot{\Gamma}')^\dagger F_0\dot{\Gamma}'];$$
$$A_2 = \mathrm{tr}[\dot{X}_j M_{F^0}\mathbb{V}' M_{\dot{\Gamma}}\mathbb{V}(\dot{\Gamma}F_0')^\dagger (F_0\dot{\Gamma}')^\dagger F_0\dot{\Gamma}'];$$
$$A_3 = \mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger \mathbb{V} M_{F_0}\mathbb{V}'(F_0\dot{\Gamma}')^\dagger F_0\dot{\Gamma}'].$$

For A_1, we have by Cauchy-Schwarz inequality,

$$\begin{aligned}|A_1| &= |\,\mathrm{tr}[\dot{\Gamma}'\dot{X}_j M_{F^0}\mathbb{V}'\dot{\Gamma}(\dot{\Gamma}'\dot{\Gamma})^{-1}(F_0'F_0)^{-1}F_0'\mathbb{V}'\dot{\Gamma}(\dot{\Gamma}'\dot{\Gamma})^{-1}]| \\ &\le \sqrt{T}\tfrac{\|\dot{\Gamma}'\dot{X}_j M_{F_0}\|}{n\sqrt{T}}\tfrac{\|\mathbb{V}'\dot{\Gamma}\|}{\sqrt{nT}}\|(\tfrac{\dot{\Gamma}'\dot{\Gamma}}{n})^{-1}\|_F^2\|(\tfrac{F_0'F_0}{T})^{-1}\|_F\tfrac{\|F_0'\mathbb{V}'\dot{\Gamma}\|}{\sqrt{nT}} = O_p(\sqrt{T}).\end{aligned}$$

For A_2, we have,

$$\begin{aligned}A_2 &= \mathrm{tr}[\dot{X}_j M_{F^0}\mathbb{V}' M_{\dot{\Gamma}}\mathbb{V}\ (\dot{\Gamma}F_0')^\dagger (F_0\dot{\Gamma}')^\dagger F_0\dot{\Gamma}'] \\ &= \mathrm{tr}[\dot{X}_j M_{F^0}\mathbb{V}' M_{\dot{\Gamma}}\mathbb{V}\ (\dot{\Gamma}F_0')^\dagger] \\ &= \mathrm{tr}[\dot{X}_j M_{F^0}\mathbb{V}'\mathbb{V}(\dot{\Gamma}F_0')^\dagger] - \mathrm{tr}[\dot{X}_j M_{F^0}\mathbb{V}' P_{\dot{\Gamma}}\mathbb{V}(\dot{\Gamma}F_0')^\dagger] \\ &= \mathrm{tr}[\dot{X}_j M_{F^0}(\mathbb{V}'\mathbb{V} - n\sigma_{v0}^2 I_T)(\dot{\Gamma}F_0')^\dagger] + O_p(\sqrt{T}) = O_p(T/n + \sqrt{T}).\end{aligned}$$

The fourth equality used the fact that $M_{F_0}(\dot{\Gamma}F_0')^\dagger = 0$. Applying Cauchy-Schwarz inequality on $\mathrm{tr}(\dot{X}_j M_{F^0}\mathbb{V}' P_{\dot{\Gamma}}\mathbb{V}(\dot{\Gamma}F_0')^\dagger)$ shows that it is $O_p(\sqrt{T})$. The last equality used the assumption that $\|\mathbb{V}'\mathbb{V}/n - \sigma_{v0}^2 I_T\|_{\mathtt{sp}} = O_p(T/n)$ and the inequality $\|AB\| \le \|A\|_{\mathtt{sp}}\|B\|_F$. For A_3, we have,

$$\begin{aligned}A_3 &= \mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger \mathbb{V} M_{F_0}\mathbb{V}'(F_0\dot{\Gamma}')^\dagger F_0\dot{\Gamma}'] \\ &= \mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger \mathbb{V}\mathbb{V}' P_{\dot{\Gamma}}] - \mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger \mathbb{V} P_{F_0}\mathbb{V}' P_{\dot{\Gamma}}] \\ &= \mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger \mathbb{V}\mathbb{V}' P_{\dot{\Gamma}}] + O_p(1).\end{aligned}$$

It follows that $A_3 = \mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger \mathbb{V}\mathbb{V}' P_{\dot{\Gamma}}] + o_p(\sqrt{nT})$,

(i-2) Consider the second term $\mathrm{tr}[\dot{X}_j'\mathbb{V}(M_{\tilde{F}} - M_{F_0})]$. Again we use result in Lemma C.1 to split it into three terms. For the first term $\mathrm{tr}(\dot{X}_j'\mathbb{V}M^{(1)}_{\tilde{F},v})$, we have

$$\begin{aligned}\mathrm{tr}\,\dot{X}_j M^{(1)}_{\tilde{F},v}\mathbb{V}' &= -\,\mathrm{tr}[\dot{X}_j M_{F_0}\mathbb{V}'(F_0\dot{\Gamma}')^\dagger \mathbb{V}'] - \mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger \mathbb{V} M_{F_0}\mathbb{V}'] \\ &= -\,\mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger \mathbb{V}\mathbb{V}'] + o_p(\sqrt{nT}).\end{aligned}$$

For the last two terms we have $\mathrm{tr}[\dot{X}_j'\mathbb{V}(M^{(2)}_{\tilde{F},v} + M^{(rem)}_{\tilde{F},v})] = o_p(\sqrt{nT})$.

Combining the leading terms of (i-1) and (i-2), we have

$$\mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger \mathbb{V}\mathbb{V}' M_{\dot{\Gamma}}] = \mathrm{tr}[\dot{X}_j(\dot{\Gamma}F_0')^\dagger(\mathbb{V}\mathbb{V}' - T\sigma_{v0}^2 I_n)M_{\dot{\Gamma}}] = O_p(T/n).$$

Summing up the terms, we have $\mathrm{tr}[\dot{X}_j'(\dot{\Gamma}F_0' + \mathbb{V})(M_{\tilde{F}} - M_{F_0})] = \mathrm{tr}(\dot{X}_j' P_{\dot{\Gamma}}\mathbb{V}M_{F_0}) + o_p(\sqrt{nT})$.

(ii) Recall that $\mathbf{D}_{-1} = \sum_{j=1}^{T-1}(J_T^j \otimes \mathcal{B}_0^{j-1}B_{10}^{-1})$. We can write

$$\begin{aligned}\operatorname{tr}(\mathbf{M}_{\tilde{F}}\mathbf{D}_{-1}) &= \textstyle\sum_{j=1}^{T-1}\operatorname{tr}(M_{\tilde{F}}J_T^j \otimes \mathcal{B}_0^{j-1}B_{10}^{-1}) \\ &= \operatorname{tr}(\mathbf{M}_{F_0}\mathbf{D}_{-1}) + \textstyle\sum_{j=1}^{T-1}\operatorname{tr}[(M_{\tilde{F}} - M_{F_0})J_T^j \otimes \mathcal{B}_0^{j-1}B_{10}^{-1}] \\ &= \operatorname{tr}(\mathbf{M}_{F_0}\mathbf{D}_{-1}) + A_1 + A_2 + A_3,\end{aligned}$$

where $\|M_{\tilde{F}} - M_{F_0}\|_{\mathtt{sp}} = o_p(1)$ and

$$\begin{aligned}A_1 &= \textstyle\sum_{j=1}^{T-1}\operatorname{tr}(M_{\tilde{F},v}^{(1)}J_T^j \otimes \mathcal{B}_0^{j-1}B_{10}^{-1}) = \sum_{j=1}^{T-1}\operatorname{tr}(M_{\tilde{F},v}^{(1)}J_T^j)\operatorname{tr}(\mathcal{B}_0^{j-1}B_{10}^{-1}), \\ A_2 &= \textstyle\sum_{j=1}^{T-1}\operatorname{tr}(M_{\tilde{F},v}^{(2)}J_T^j \otimes \mathcal{B}_0^{j-1}B_{10}^{-1}) = \sum_{j=1}^{T-1}\operatorname{tr}(M_{\tilde{F},v}^{(2)}J_T^j)\operatorname{tr}(\mathcal{B}_0^{j-1}B_{10}^{-1}), \\ A_3 &= \textstyle\sum_{j=1}^{T-1}\operatorname{tr}(M_{\tilde{F},v}^{(rem)}J_T^j \otimes \mathcal{B}_0^{j-1}B_{10}^{-1}) = \sum_{j=1}^{T-1}\operatorname{tr}(M_{\tilde{F},v}^{(rem)}J_T^j)\operatorname{tr}(\mathcal{B}_0^{j-1}B_{10}^{-1}).\end{aligned}$$

We have $|\operatorname{tr}(\mathcal{B}_0^{j-1}B_{10}^{-1})| \leq \|\mathcal{B}_0^{j-2}B_{10}^{-1}\|\|\mathcal{B}_0\| < C\rho^j\|B_{10}\| \cdot \|\mathcal{B}_0\| = \rho^j O_p(n)$. Now,

$$\operatorname{tr}(M_{\tilde{F},v}^{(1)}J_T^j) = -\operatorname{tr}[M_{F_0}\mathbb{V}'(F_0\dot{\Gamma}')^\dagger J_T^j] - \operatorname{tr}[(\dot{\Gamma}F_0')^\dagger\mathbb{V}M_{F_0}J_T^j] = O_p(1/\sqrt{n}),$$

uniformly over j. Summing up, we have $A_1 = o_p(\sqrt{nT})$. By similar arguments, we can show $A_2 + A_3 = o_p(\sqrt{nT})$. We can also show that $\operatorname{tr}[(M_{F_0} \otimes P_{\dot{\Gamma}})\mathbf{D}_{-1}] = o_p(\sqrt{nT})$. Therefore, we have $\operatorname{tr}(\mathbf{M}_{\tilde{F}}\mathbf{D}_{-1}) = \operatorname{tr}[(M_{F_0} \otimes M_{\dot{\Gamma}})\mathbf{D}_{-1}] + o_p(\sqrt{nT})$. The analysis of the other two terms are symmetric and are therefore omitted.

(iii) We prove the first result. Again we use the result in Lemma C.1 to obtain

$$\operatorname{tr}[(\mathbb{V} + \dot{\Gamma}F_0')'(\mathbb{V} + \dot{\Gamma}F_0')(M_{\tilde{F}} - M_{F_0})] = A_1 + A_2 + A_3,$$

where $A_1 = \operatorname{tr}[(\mathbb{V} + \dot{\Gamma}F_0')'(\mathbb{V} + \dot{\Gamma}F_0')M_{F,v}^{(1)}]$, $A_2 = \operatorname{tr}[(\mathbb{V} + \dot{\Gamma}F_0')'(\mathbb{V} + \dot{\Gamma}F_0')M_{F,v}^{(2)}]$, and $A_3 = \operatorname{tr}[(\mathbb{V} + \dot{\Gamma}F_0')'(\mathbb{V} + \dot{\Gamma}F_0')M_{F,v}^{(rem)}]$. We show that

$$\begin{aligned}A_1 &= -2\operatorname{tr}[(\mathbb{V} + \dot{\Gamma}F_0')'\mathbb{V}M_{F_0}\mathbb{V}'(F_0\dot{\Gamma})^\dagger] \\ &= -2\operatorname{tr}[\mathbb{V}'\mathbb{V}M_{F_0}\mathbb{V}'\dot{\Gamma}(\dot{\Gamma}'\dot{\Gamma})^{-1}(F_0'F_0)^{-1}F_0'] - 2\operatorname{tr}(P_{\dot{\Gamma}}\mathbb{V}M_{F_0}\mathbb{V}') \\ &= O_p(\sqrt{n} + T/n) = o_p(\sqrt{nT}); \\ A_2 &= -\operatorname{tr}[\mathbb{V}'\mathbb{V}M_{F_0}\mathbb{V}'\dot{\Gamma}(\dot{\Gamma}'\dot{\Gamma})^{-1}(F_0'F_0)^{-1}(\dot{\Gamma}'\dot{\Gamma})^{-1}\dot{\Gamma}'\mathbb{V}M_{F_0}] \\ &\quad + 2\operatorname{tr}[(\mathbb{V} + \dot{\Gamma}F_0')'\mathbb{V}M_{F_0}\mathbb{V}(F_0\dot{\Gamma}')^\dagger\mathbb{V}'(F_0\dot{\Gamma}')^\dagger] \\ &\quad - 2\operatorname{tr}[(\mathbb{V} + \dot{\Gamma}F_0')'\mathbb{V}M_{F_0}\mathbb{V}'M_{\dot{\Gamma}}\mathbb{V}F_0(F_0'F_0)^{-1}(\dot{\Gamma}'\dot{\Gamma})^{-1}(F_0'F_0)^{-1}F_0'] \\ &\quad + \operatorname{tr}(P_{\dot{\Gamma}}\mathbb{V}M_{F_0}\mathbb{V}') + 2\operatorname{tr}[\mathbb{V}(\dot{\Gamma}F_0')^\dagger\mathbb{V}M_{F_0}\mathbb{V}'P_{\dot{\Gamma}}] + \operatorname{tr}[\mathbb{V}'\mathbb{V}(F_0\dot{\Gamma}')^\dagger\mathbb{V}M_{F_0}\mathbb{V}'(\dot{\Gamma}F_0')^\dagger] \\ &= o_p(\sqrt{nT});\end{aligned}$$

and $A_3 = o_p(\sqrt{nT})$. The first result thus follows. The second result is proved similarly. ■

Proof of Lemma D.2: (i) Note that the columns of $\hat{F}(\psi)$ are the first r eigenvectors of

$$\mathcal{S}(\theta) = [B_3(\lambda_3)\mathtt{mat}(Z(\theta))]'[B_3(\lambda_3)\mathtt{mat}(Z(\theta))].$$

We obtain,

$$\begin{aligned} B_3(\lambda_3)\texttt{mat}(Z(\theta)) &= \dot\Gamma F_0' + \mathbb{V} - \textstyle\sum_{j=1}^k(\beta_j-\beta_{j0})\dot X_j - (\rho-\rho_0)\dot X_{k+1} \\ &\quad - \textstyle\sum_{\ell=1}^3(\lambda_j-\lambda_{j0})\dot X_{k+1+\ell} - (\lambda_3-\lambda_{30})[\texttt{mat}(Z(\theta)) - \texttt{mat}(Z(\theta_0))] \\ &= \dot\Gamma F_0' + \mathbb{V} + \Delta(\theta), \end{aligned}$$

with $\|\texttt{mat}(Z(\theta)) - \texttt{mat}(Z(\theta_0))\|_{\texttt{sp}} \le \|\theta-\theta_0\|O_p(\sqrt{nT})$, and $\|\Delta(\theta)\|_{\texttt{sp}} = \|\theta-\theta_0\|O_p(\sqrt{nT})$. For small enough $\theta-\theta_0$, we have,

$$\begin{aligned} \mathcal{S}(\theta) =& F_0\dot\Gamma'\dot\Gamma F_0' + F_0\dot\Gamma'\mathbb{V} + \mathbb{V}'\dot\Gamma F_0' + \mathbb{V}'\mathbb{V} \\ &+ F_0\dot\Gamma'\Delta(\theta) + \Delta'(\theta)\dot\Gamma F_0' + \Delta'(\theta)\mathbb{V} + \mathbb{V}'\Delta(\theta) + \Delta'(\theta)\Delta(\theta). \end{aligned}$$

The leading term of $\mathcal{S}(\theta)$ is $F_0\dot\Gamma'\dot\Gamma F_0'$, which is low rank and $O_p(NT)$ non-zero eigenvalues values. The remainder terms are of smaller order in terms of spectrum norm. One can use the perturbation theory to show the desired results.

(ii) Note $\mathbf{M}_{\hat F(\psi)}\mathbf{D}_{-1}(\rho,\lambda_1,\lambda_2) = \sum_{j=1}^{T-1} M_{\hat F(\psi)}J_T^j \otimes \mathcal{B}^{j-1}(\rho\lambda_1,\lambda_2)B_1^{-1}(\lambda_1)$, then we have

$$\begin{aligned} \tfrac{\partial}{\partial\psi_p}\operatorname{tr}[\mathbf{M}_{\hat F(\psi)}\mathbf{D}_{-1}(\rho,\lambda_1,\lambda_2)]\big|_{\psi_0} &= \textstyle\sum_{j=1}^{T-1}\tfrac{\partial}{\partial\psi_p}\big\{\operatorname{tr}[M_{\hat F(\psi)}J_T^j]\operatorname{tr}[\mathcal{B}^{j-1}(\rho,\lambda_1,\lambda_2)B_1^{-1}(\lambda_1)]\big\}\big|_{\psi_0} \\ &= \textstyle\sum_{j=1}^{T-1}\tfrac{\partial}{\partial\psi_p}\operatorname{tr}(M_{\hat F(\psi)}J_T^j)\big|_{\psi_0}\operatorname{tr}(\mathcal{B}_0^{j-1}B_{10}^{-1}) \\ &\quad + \textstyle\sum_{j=1}^{T-1}\operatorname{tr}(M_{\hat F(\psi_0)}J_T^j)\tfrac{\partial}{\partial\psi_p}\operatorname{tr}[\mathcal{B}^{j-1}(\rho,\lambda_1,\lambda_2)B_1^{-1}(\lambda_1)]\big|_{\psi_0} \\ &\equiv A + B. \end{aligned}$$

For term A, we derive the case for $p = 1,,k+3$, we have,

$$\tfrac{\partial}{\partial\psi_p}\operatorname{tr}(M_{\hat F(\psi)}J_T^j)\big|_{\psi_0} = -\operatorname{tr}[M_{F_0}\dot X_k'(F_0\dot\Gamma')^\dagger J_T^j] - \operatorname{tr}[(\dot\Gamma F_0')^\dagger \dot X_k M_{F_0}J_T^j] = O_p(1)$$

uniformly over j, and $|\operatorname{tr}(\mathcal{B}_0^{j-1}B_{10}^{-1})| < \bar\rho^j O_p(n)$. It follows that $A = O_p(n)$. For term B, first note that $B = 0$ for $p = 1,\dots,k$ and $p = k+4$. For $p = k+1, k+2$, and$k+3$, we can show that $\frac{\partial}{\partial\psi_p}\operatorname{tr}[\mathcal{B}^{j-1}(\rho,\lambda_1,\lambda_2)B_1^{-1}(\lambda_1)]\big|_{\psi_0} = \bar\rho^j O_p(n)$. For $\operatorname{tr}(M_{\tilde F}J_T^j)$, we have,

$$|\operatorname{tr}(M_{\tilde F}J_T^j)| = |\operatorname{tr}(P_{\tilde F}J_T^j)| \le \|\tilde F' J_T^j \tilde F\| \cdot \|(\tilde F'\tilde F)^{-1}\| = O_p(1), \text{ for any } j > 0.$$

The first equality is by the fact that $\operatorname{tr}(J_T^j) = 0$. Hence, we can conclude that $B = O_p(n)$.

(iii) Recall the representation of $\mathbf{D}(\rho,\lambda_1,\lambda_2)$ given at the beginning of Appendix D. We have,

$$\begin{aligned} &\operatorname{tr}[\mathbf{M}_{\hat F(\psi)}\mathbf{W}_1\mathbf{D}(\rho,\lambda_1,\lambda_2)] = \textstyle\sum_{j=0}^{T-1}\operatorname{tr}(M_{\hat F(\psi)}J_T^j)\operatorname{tr}[W_1\mathcal{B}^{j-1}(\rho,\lambda_1,\lambda_2)B_1^{-1}(\lambda_1)] \\ &= (T-r)\operatorname{tr}[W_1B_1^{-1}(\lambda_1)] + \textstyle\sum_{j=1}^{T-1}\operatorname{tr}(M_{\hat F(\psi)}J_T^j)\operatorname{tr}[W_1\mathcal{B}^{j-1}(\rho,\lambda_1,\lambda_2)B_1^{-1}(\lambda_1)]. \end{aligned}$$

The analysis of the second term is similar to part (ii). For the first term is a function of λ_1 only and its derivative equals $(T-r)\operatorname{tr}[W_1B_1^{-1}(\lambda_1)W_1B_1^{-1}(\lambda_1)]$. ■

Proof of Lemma D.3: This lemma can be proved by direct derivation. ■

Proof of Lemma D.4: The proof is similar to the proof of Lemma D.2. ■

Proof of Lemma D.5: (i) The results can be verified by direct calculation.
(ii) We take $\texttt{diag}[(I_T \otimes P_{\dot{\Gamma}})\Phi_1]$ as an example. We have,

$$
\begin{aligned}
&(M_{F_0} \otimes P_{\dot{\Gamma}} B_{30})\mathbf{D}_{-1}\mathbf{B}_{30}^{-1}/\sigma_{v0}^2 \\
=&\frac{1}{\sigma_{v0}^2}(M_{F_0} \otimes P_{\dot{\Gamma}}) \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ B_{30} & 0 & \cdots & 0 & 0 \\ B_{30}\mathcal{B}_0 & B_{30} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{30}\mathcal{B}_0^{T-2} & B_{30}\mathcal{B}_0^{T-3} & \cdots & B_{30} & 0 \end{pmatrix} (I_T \otimes B_{10}^{-1}B_{30}^{-1})I_{nT}.
\end{aligned}
$$

Its jth diagonal elements can be thought of product of jth row of $M_{F_0} \otimes P_{\dot{\Gamma}}$, the middle part matrix and jth column of I_{nT}. Rows of $M_{F_0} \otimes P_{\dot{\Gamma}}$ are uniformly $O_p(n^{-1/2})$ in Euclidean norm and each column of I_{nT} is a unit vector. The middle part matrix is of spectrum norm $O_p(1)$. It follows that the diagonal elements are uniformly $O_p(n^{-1/2})$.
(iii) The results can be verified easily. The derivation is similar to that of part (ii).
(iv) The first two results can be verified easily. We prove the third result. Note that $(M_{F_0} \otimes M_{\dot{\Gamma}})\texttt{vec}(\dot{X}_{k+a})/\sigma_{v0}^2 = \tilde{\Pi}_{1+a} + \tilde{\Psi}_a\mathbf{y}_0 + \tilde{\Phi}_a\mathbf{v}$. We can show

$$
\begin{aligned}
&\frac{1}{nT\sigma_{v0}^2}\operatorname{tr}[\dot{X}_{k+a}' M_{\dot{\Gamma}} \dot{X}_{k+b} M_{F_0}] \\
=&\frac{\sigma_{v0}^2}{nT}(\tilde{\Pi}_{b+1} + \tilde{\Psi}_b\mathbf{y}_0)'(\tilde{\Pi}_{a+1} + \tilde{\Psi}_a\mathbf{y}_0) + \frac{\sigma_{v0}^2}{nT}\mathbf{v}'\tilde{\Phi}_a'\tilde{\Phi}_b\mathbf{v} + \frac{\sigma_{v0}^2}{nT}(\tilde{\Pi}_{b+1} + \tilde{\Psi}_b\mathbf{y}_0)'\tilde{\Phi}_a\mathbf{v} \\
&+ \frac{\sigma_{v0}^2}{nT}(\tilde{\Pi}_{a+1} + \tilde{\Psi}_a\mathbf{y}_0)'\tilde{\Phi}_b\mathbf{v} \\
=&\frac{\sigma_{v0}^2}{nT}(\tilde{\Pi}_{b+1} + \tilde{\Psi}_b\mathbf{y}_0)'(\tilde{\Pi}_{a+1} + \tilde{\Psi}_a\mathbf{y}_0) + \frac{\sigma_{v0}^2}{nT}\mathbf{v}'\tilde{\Phi}_a'\tilde{\Phi}_b\mathbf{v} + o_p(1).
\end{aligned}
$$

One can verify that $\frac{\sigma_{v0}^2}{nT}\mathbf{v}'\tilde{\Phi}_a'\tilde{\Phi}_b\mathbf{v} = \frac{\sigma_{v0}^4}{nT}\operatorname{tr}[\tilde{\Phi}_a'\tilde{\Phi}_b]$. Hence, the desired result follows.
(v) The first result follows from $\Pi_{4+s} = \frac{1}{\sigma_{v0}^2}(M_{F_0} \otimes \dot{\Gamma})\texttt{vec}(\dot{F}_s')$.

Consider the second result. Recall that $\mathcal{S}_* = [I_{T-r}, 0_{(T-r)\times r}]'$. Write M_{F_0} as a partitioned matrix $M_{F_0} = [M_1, M_{12}; M_{21}, M_2]$, where M_1 is $(T-r)\times(T-r)$ and M_2 is $r \times r$. We have that $\mathcal{S}_*' M_{F_0}\mathcal{S}_* = M_1$ and $M_{F_0}\mathcal{S}_* = [M_1', M_{21}']'$. It follows that $M_{F_0}\mathcal{S}_*(\mathcal{S}_*' M_{F_0}\mathcal{S}_*)^{-1}\mathcal{S}_*' M_{F_0} = [M_1, M_{12}; M_{21}, M_{21}M_1^{-1}M_{12}]$. Then, it suffices to show $M_{21}M_1^{-1}M_{12} = M_2$.

As $F_0 = [F^{*\prime}, I_r]'$, one can easily see that $M_1 = I_{T-r} - F^*(I_r + F^{*\prime}F^*)^{-1}F^{*\prime}$, $M_{21} = M_{12}' = -(I_r + F^{*\prime}F^*)^{-1}F^{*\prime}$, $M_2 = I_r - (I_r + F^{*\prime}F^*)^{-1}$, and $M_1^{-1} = I_{T-r} + F^*F^{*\prime}$. It follows that

$$
\begin{aligned}
M_{21}M_1^{-1}M_{12} &= (I_r + F^{*\prime}F^*)^{-1}F^{*\prime}(I_{T-r} + F^*F^{*\prime})F^*(I_r + F^{*\prime}F^*)^{-1} \\
&= (I_r + F^{*\prime}F^*)^{-1}F^{*\prime}F^* = I_r - (I_r + F^{*\prime}F^*)^{-1} = M_2.
\end{aligned}
$$

■