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LM Tests of Spatial Dependence Based on Bootstrap Critical Values

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Abstract

To test the existence of spatial dependence in an econometric model, a convenient test is the Lagrange Multiplier (LM) test. However, evidence shows that, in finite samples, the LM test referring to asymptotic critical values may suffer from the problems of size distortion and low power, which become worse with a denser spatial weight matrix. In this paper, residual-based bootstrap methods are introduced for asymptotically refined approximations to the finite sample critical values of the LM statistics. Conditions for their validity are clearly laid out and formal justifications are given in general, and in details under several popular spatial LM tests using Edgeworth expansions. Monte Carlo results show that when the conditions are not fully met, bootstrap may lead to unstable critical values that change significantly with the alternative, whereas when all conditions are met, bootstrap critical values are very stable, approximate much better the finite sample critical values than those based on asymptotics, and lead to significantly improved size and power. The methods are further demonstrated using more general spatial LM tests, in connection with local misspecification and unknown heteroskedasticity.

Key Words: Asymptotic refinements; Bootstrap; Edgeworth expansion; LM tests; Spatial dependence; Size; Power; Local misspecification; Heteroskedasticity; Wild bootstrap.

JEL Classification: C12, C15, C18, C21

1 Introduction

To test the existence of spatial dependence in an econometric model, a convenient test is the Lagrange Multiplier (LM) test as it requires model estimation only under the null hypothesis (Anselin, 1988b). However, evidence shows that, in finite samples, the true sizes of the LM test referring to the asymptotic critical values can be quite different from their nominal sizes, and more so with a denser spatial weight matrix and with one-sided tests. As a result, the LM tests in such circumstances may have low power in detecting a 'negative' or 'positive' spatial dependence. Also, LM tests may not be robust against the misspecification in error distribution. Standardization (Koenker, 1981; Robinson, 2008; Yang, 2010; Yang and Shen, 2011; Baltagi and Yang, 2013) robustifies the LM tests. It also helps alleviate the problem of size distortion for two-sided tests, but not for one-sided tests. Furthermore, standardization does not solve the problem of low power in detecting a negative or positive spatial

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dependence. The reason is that a denser spatial weight matrix makes the shape of the finite-sample distribution of the LM statistic deviate more from the shape of its limiting distribution (rendering the asymptotic critical values less accurate). In the special case where the LM test is univariate and asymptotically standard normal under the null, a denser spatial weight makes its finite sample distribution more skewed to the left or right depending on the design of the regressors. Standardization only changes the location and scale, but not the shape of the distribution of the LM test. This is why it cannot solve the problems of size distortion and low power for one-sided tests. However, we demonstrate in this paper that standardization coupled with bootstrap provide a satisfactory solution to these problems.

It is well documented in the econometrics literature that bootstrap method is able to provide asymptotic refinements on the critical values of a test statistic if this statistic is asymptotically pivotal under the null hypothesis. See, among others, Beran (1988), Hall (1992), Horowitz (1994, 1997), Hall and Horowitz (1996), Davidson and MacKinnon (1999, 2006), van Giersbergen and Kiviet (2002), MacKinnon (2002), Cameron and Trivedi (2005, Ch. 11), and Godfrey (2009), for theoretical analyses and numerical evidence for many different type of econometric models. However, as pointed out by Davidson (2007) and reiterated in Godfrey (2009, p. 82), it is not always the case that the asymptotic analysis seems to provide a good explanation of what is observed in finite samples. For the residual-based bootstrap method which is followed in this paper, Godfrey (2009, Ch. 3), based on the work of van Giersbergen and Kiviet (2002) and MacKinnon (2002), give a detailed discussion on the type of residuals (restricted under the null hypothesis or unrestricted) to be resampled and the type of estimates (restricted or unrestricted) of the nuisance parameters to be used as parameters in the bootstrap world. However, the debate on the choices of parameter estimates and residuals does not seem to have been settled. These issues carry over to spatial models. In contrast to the vast literature on the bootstrap tests in general econometrics, such a literature in spatial econometrics is rather thin, in both applications (e.g., Burridge and Fingleton, 2010; Lin et al., 2007, 2009, 2011; and Burridge, 2012), and in theory (e.g., Yang, 2011; and Jin and Lee, 2012a,b). This research completes Yang (2011) by providing second-order asymptotic analyses in LM test setting, which provides a good explanation of what is observed in finite sample and settles the debate over the choice of bootstrap parameters.

Residual-based bootstrap methods are introduced for asymptotically refined approximations to the finite sample critical values of the LM statistics. Conditions for their validity are clearly laid out and formal justifications are given in general, and in details under several popular spatial LM tests, namely, LM tests for spatial error dependence (SED), LM tests for spatial lag dependence (SLD), and LM tests for spatial error components (SEC). The key methodologies used in the proofs are asymptotic expansions (Beran, 1988) for general LM tests, and Edgeworth expansions (Hall, 1992, Ch. 3) for the three specific LM tests. The validity of the proposed methods is further demonstrated using more complicated spatial LM tests: joint LM test for SLD and SED, LM test of SED allowing SLD and vise versa, spatial LM tests under local misspecification, and spatial LM tests with unknown heteroskedasticity. Our results show that with the unrestricted estimates/residuals, bootstrap is able to provide critical values that are stable (with respect to the true value of the tested parameters) and achieve *full* asymptotically refined approximations to the finite sample critical values of the test statistic, leading to correct size and reliable power. In contrast, use of restricted estimates/residuals, the bootstrap critical values can be either smaller or larger (in absolute value) than the 'true' values when the null is false, leading to the power of the test that is either higher or lower than the 'true' underlining power. However, we show that while use of restricted estimates/residuals does not lead to full asymptotic refinements, it does provide partial asymptotic refinements. This explains why in certain situations bootstrap based

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on restricted estimates/residuals still leads to improved results over the large sample approximations.

The proposed bootstrap methods are applicable to a wide class of LM testing situations, not only the LM tests for spatial dependence. We demonstrate that for these methods to work well, it is important that (*i*) the bootstrap DGP resembles the null model, (*ii*) the LM statistic is asymptotically pivotal under the null or its robustified/standardized version must be used, (*iii*) the estimates of the nuisance parameters, to be used as parameters in the bootstrap world, are consistent whether or not the null hypothesis is true, (*iv*) the empirical distribution function (EDF) of the residuals to be resampled consistently estimates the error distribution whether or not the null hypothesis is true, and (*v*) calculation of the bootstrapped values of the LM statistic is done under the null hypothesis.

Among these points, (*i*) and (*ii*) are well understood and agreed among the researchers, (*v*) follows the nature of LM or score tests, and (*iii*) and (*iv*) lead in general to the use of unrestricted parameter estimates and unrestricted residuals. Points (*iii*) and (*iv*), related to the major subjects of debate, make sense because in reality one does not know whether or not the null hypothesis is true. In order for the bootstrap world to be able to mimic the real world at the null, it must be set up such that the 'parameters' in the bootstrap world mimic (converge to) the nuisance parameters in the real world, and the errors in the bootstrap world mimic the true errors in the real world whether the null is true or false. These can only be guaranteed in general if the unrestricted estimates and residuals are used. Clearly, (*ii*) is typically true when the error distribution is correctly specified, but may not be so when it is misspecified. In this case, bootstrap may not be able to provide the desired level of improvement on the critical values, and a robust version of the LM statistic needs to be in place.

We conclude that *the general validity of the proposed bootstrap methods lies upon the use of unrestricted estimates of the nuisance parameters, unrestricted residuals, and an LM-type statistic robust against distributional misspecification.* There are special cases where it doesn't matter whether to use the unrestricted or restricted estimates and residuals (see Section 3 for details), which is perhaps the reason why some authors advocate the use of the restricted estimates and residuals as they are often simpler computationally than their unrestricted counterparts. However, the additional computational cost of the proposed procedure occurs only at the initial estimation stage, not in the bootstrap process.

Section 2 presents the results under a general LM test framework. Section 3 considers the three special cases (LM tests for SLD, SED and SEC) where each case is supplemented with a set of Monte Carlo results. Section 4 provides further demonstrations of the proposed methods using more complicated spatial LM tests. Section 5 concludes the paper. Appendix A contains some derivations and fundamental lemmas, Appendix B provides some additional details of the proofs for Section 3, and Appendix C describes the general setting of the Monte Carlo experiments.

2 Bootstrap Critical Values for LM Tests

Consider an LM test statistic $\text{LM}_n(\lambda) \equiv \text{LM}_n(Y_n, X_n, W_n; \lambda)$ for testing the spatial dependence represented by the parameter (vector) λ, in a model with dependent variable Y_n conditional on a set of independent variables X_n, a spatial weight matrix W_n[2], and parameters θ and λ, where the parameter vector θ may contain the regression coefficients, error standard deviations, etc., depending on the model considered. Typically, $\text{LM}_n(\lambda)$ is not a pivotal quantity as its finite sample distribution depends on the parameters θ and λ, but is asymptotically pivotal if the error distribution is correctly specified, in

[2]It is a known matrix that specifies the relationship (distance) among the spatial units. In case when λ is a vector, each component of it may associate with a different spatial weight matrix.

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the sense that its limiting distribution is free of parameters, such as standard normal or chi-square, depending on whether λ is a scalar or a vector. However, if the error distribution is misspecified, $\text{LM}_n(\lambda)$ may not even be an asymptotically pivotal quantity as its limiting distribution may depend on the unknown error distribution, as well as the model parameters (see Section 3.3 for such a case), and this will have important implications on the performance of the bootstrap procedures (Beran, 1988).

The most interesting inference in a spatial model is perhaps to test $H_0 : \lambda = 0$, i.e., non-existence of spatial dependence, versus $H_a : \lambda \neq 0$ $(< 0, > 0)$, i.e., existence of spatial dependence (negative spatial dependence or positive spatial dependence). To test this hypothesis using the test statistic $\text{LM}_n(0)$, one often refers to the asymptotic critical values of $\text{LM}_n(0)$ at H_0, or $\text{LM}_n(0)|_{H_0}$. However, as argued in the introduction, these asymptotic critical values may give poor approximations in cases of heavy spatial dependence.[3] It is thus desirable to find better approximations to the finite sample critical values of $\text{LM}_n(0)|_{H_0}$. As $\text{LM}_n(\lambda)$ is not a pivotal quantity, it is not possible to find the exact finite sample critical values. However, if $\text{LM}_n(\lambda)$ is asymptotically pivotal, the bootstrap approach can be used to obtain critical values that are more accurate than the asymptotic critical values, according to Beran (1988), Hall (1992), Horowitz (1994) and Hall and Horowitz (1996). See also Cameron and Trivedi (2005, Ch. 11) and Godfrey (2009, Ch. 2 & 3) for detailed descriptions on bootstrap tests.

Our discussions above and below are for the LM tests of spatial regressions models. However, they can be applied to the LM tests of other types of models as well. It is the unique feature of LM tests (requiring the estimation of the null model only) and the unique feature of the spatial models (finite sample behavior of the LM tests of spatial dependence can be heavily affected by the spatial weight matrix W_n) that make it more appealing to study bootstrap methods in approximating the finite sample critical values of spatial LM statistics.

2.1 The methods

To facilitate our discussions, suppose that the model can be written as,

$$q(Y_n, X_n, W_n; \theta, \lambda) = e_n, \tag{1}$$

where e_n is an n-vector of model errors, with iid elements $\{e_{n,i}\}$, of zero mean, unit variance, and cumulative distribution function (CDF) $\mathcal{F}$. The error standard deviation σ is absorbed into θ.[4] Suppose that the model can be inverted to give

$$Y_n = h(X_n, W_n; \theta, \lambda; e_n). \tag{2}$$

Consider a general hypothesis: $H_0 : \lambda = \lambda_0$ versus $H_a : \lambda \neq \lambda_0$ $(< \lambda_0, > \lambda_0)$. The test statistic to be used is the $\text{LM}_n(\lambda_0)$, derived under a 'specified' error distribution, typically $N(0, 1)$, although the true $\mathcal{F}$ may not be the CDF of $N(0, 1)$. We are interested in the finite sample null CDF $\mathcal{G}_n(\cdot, \theta, \mathcal{F})$

[3] The denser the matrix W_n is, the more skewed is the finite sample null distribution of the LM test, e.g., the first three cumulants of $\text{LM}_{\text{SED}}|_{H_0}$ considered in Section 3.1 are shown to be $O(\sqrt{h_n/n})$, $1 + O(h_n/n)$, and $O(\sqrt{h_n/n})$, respectively, where h_n can be understood as a dense measure of W_n as it corresponds to the number of non-zero elements in each row of W_n. This suggests that for a fixed n the first three cumulants of LM_{SED} can be quite different from their asymptotic values 0, 1 and 0, and more so with a larger h_n (denser W_n). See Section 3.1 and the proof of Proposition 3.1 for details. The same results hold for the other spatial LM tests considered in Subsections 3.2 and 3.3, and in Section 4.

[4] Model (1) encompasses many popular spatial models, linear or nonlinear, such as SAR, SARAR, SEC, spatial probit, spatial Tobit, etc.; see Kelejian and Prucha (2001). It can be extended to include more than one spatial weight matrix and to have non-spherical disturbances of the form $u_n \sim (0, \sigma^2\Omega_n(\rho))$, where $\Omega_n(\rho)$ is an $n \times n$ positive definite matrix, known up to a finite number of parameters ρ. Writing $u_n = \sigma\Omega_n^{1/2}(\rho)e_n$ and merging σ and ρ into θ give the form of (1).

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of $\text{LM}_n(\lambda_0)$, in particular the finite sample critical values of $\text{LM}_n(\lambda_0)|_{H_0}$, $c_n(\alpha;\theta,\mathcal{F})$, $0<\alpha<1$, and investigate how bootstrap can provide a valid method for approximating these critical values.

In what follows, $\tilde{\theta}_n$ denotes the restricted estimate of θ under H_0, and $(\hat{\theta}_n,\hat{\lambda}_n)$ the unrestricted estimates of (θ,λ). The observable counterpart of e_n is referred to as *residuals*. If the residuals are obtained from the null model, i.e., $\tilde{e}_n = q(Y_n,X_n,W_n;\tilde{\theta}_n,\lambda_0)$, they are called the *restricted residuals*; if they are obtained from the full model, i.e., $\hat{e}_n = q(Y_n,X_n,W_n;\hat{\theta}_n,\hat{\lambda}_n)$, they are called the *unrestricted residuals*. The corresponding empirical distribution function (EDF) of the restricted residuals is denoted as $\tilde{\mathcal{F}}_n$, and that of the unrestricted residuals as $\hat{\mathcal{F}}_n$.

Note that the null model is determined by the pair $\{\theta,\mathcal{F}\}$, and that under the LM framework only the estimation of the null model is required. In order to approximate the finite sample null distribution (or critical values) of $\text{LM}_n(\lambda_0)$, the bootstrap world must be set up so that it is able to mimic the real world at the null. Thus, the bootstrap DGP should take the following form:

$$Y_n^* = h(X_n,W_n;\ddot{\theta}_n,\lambda_0;e_n^*),\quad e_n^* \overset{iid}{\sim} \ddot{\mathcal{F}}_n, \tag{3}$$

where $\ddot{\theta}_n$ is the bootstrap parameter vector (an estimate of the nuisance parameter vector based on the original data) which mimics (consistently estimates) θ, and $\ddot{\mathcal{F}}_n$ is the bootstrap error distribution (the EDF of some type of residuals) mimicking (consistently estimating) $\mathcal{F}$. The steps for finding the bootstrap critical values for $\text{LM}_n(\lambda_0)|_{H_0}$ is summarized as follows:

(a) Draw a bootstrap sample e_n^* from $\ddot{\mathcal{F}}_n$;

(b) Compute $Y_n^* = h(X_n,W_n;\ddot{\theta}_n,\lambda_0;\ e_n^*)$ to obtain the bootstrap data $\{Y_n^*,X_n,W_n\}$;

(c) Estimate the **null model** based on $\{Y_n^*,X_n,W_n\}$, and then compute a bootstrapped value $\text{LM}_n^b(\lambda_0)$ of $\text{LM}_n(\lambda_0)|_{H_0}$;

(d) Repeat (a)-(c) B times to obtain the EDF of $\{\text{LM}_n^b(\lambda_0)\}_{b=1}^B$, and its α-quantile gives a bootstrap estimate of $c_n(\alpha;\theta,\mathcal{F})$, the true finite sample α-quantile of $\text{LM}_n(\lambda_0)|_{H_0}$.[5]

In reality, one does not know whether or not H_0 is true, thus it incurs an important issue: the choice of the pair $\{\ddot{\theta}_n,\ddot{\mathcal{F}}_n\}$. We argue in this paper that for the bootstrap DGP $Y_n^* = h(X_n,W_n;\ddot{\theta}_n,\lambda_0;e_n^*)$ to be able to mimic the real world null DGP $Y_n = h(X_n,W_n;\theta,\lambda_0;e_n)$ in general, $\{\ddot{\theta}_n,\ddot{\mathcal{F}}_n\}$ must be consistent for $\{\theta,\mathcal{F}\}$ whether or not H_0 is true. In this spirit, the only choice for $\{\ddot{\theta}_n,\ddot{\mathcal{F}}_n\}$ that can be correct in general is $\{\hat{\theta}_n,\hat{\mathcal{F}}_n\}$. As this resampling scheme is based on the unrestricted estimates of the nuisance parameters and the unrestricted residuals, it is termed as the *unrestricted resampling scheme*, or the **r**esampling **s**cheme with **u**nrestricted estimates and **u**nrestricted residuals ($\texttt{RS}_{uu}$).

There are many special cases where $\tilde{\theta}_n$ and/or $\tilde{\mathcal{F}}_n$ are consistent whether or not H_0 is true. This leads to other choices for the pair $\{\ddot{\theta}_n,\ddot{\mathcal{F}}_n\}$: $\{\tilde{\theta}_n,\tilde{\mathcal{F}}_n\}$, $\{\hat{\theta}_n,\tilde{\mathcal{F}}_n\}$, or $\{\tilde{\theta}_n,\hat{\mathcal{F}}_n\}$, giving, respectively, the so-called the *restricted resampling scheme* ($\texttt{RS}_{rr}$), and the *hybrid resampling schemes* 1 ($\texttt{RS}_{ur}$) and the *hybrid resampling schemes* 2 ($\texttt{RS}_{ru}$), to adopt the similar terms as in Godfrey (2009).

Alternative to the bootstrap method based on $\texttt{RS}_{uu}$, one may consider the bootstrap analog of H_0, $H_0^*: \lambda = \hat{\lambda}_n$. To test H_0^*, one generates the response values through the estimated full model, and performs bootstrap estimation conditional on $\hat{\lambda}_n$. Thus, the bootstrap critical values of $\text{LM}_n(\lambda_0)|_{H_0}$ are simply the empirical quantiles of the bootstrap distribution of $\text{LM}_n(\hat{\lambda}_n)$ conditional on $\hat{\lambda}_n$. This resampling scheme is denoted as $\texttt{RS}_{uf}$, and the corresponding bootstrap procedure is as follows:

[5] By choosing an arbitrarily large B, the EDF of $\{\text{LM}_n^b(\lambda_0)\}_{b=1}^B$ gives an arbitrarily accurate approximation to the true bootstrap CDF of $\text{LM}_n(\lambda_0)|_{H_0}$ and its quantiles (Efron, 1978; Beran, 1988). Hence, in the subsequent discussions on the validity of the proposed bootstrap method this type of approximation errors are ignored.

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(a) Draw a bootstrap sample $\hat{e}_n^*$ from the EDF $\hat{\mathcal{F}}_n$ of $\hat{e}_n$,

(b) Compute $Y_n^* = h(X_n, W_n; \hat{\theta}_n, \hat{\lambda}_n; \hat{e}_n^*)$ to obtain the bootstrap data $\{Y_n^*, X_n, W_n\}$,

(c) Conditional on $\hat{\lambda}_n$, estimate the model based on $\{Y_n^*, X_n, W_n\}$, and then compute $\mathrm{LM}_n(\hat{\lambda}_n)$ and denote its value as $\mathrm{LM}_n^b(\hat{\lambda}_n)$,

(d) Repeat (a)-(c) B times to obtain the EDF of $\{\mathrm{LM}_n^b(\hat{\lambda}_n)\}_{b=1}^B$, and the quantiles of it give the bootstrap critical values of $\mathrm{LM}_n(\lambda_0)|_{H_0}$.

Among the five resampling schemes ($\mathtt{RS}_{uu}$, $\mathtt{RS}_{rr}$, $\mathtt{RS}_{ur}$, $\mathtt{RS}_{ru}$, $\mathtt{RS}_{uf}$) described above, $\mathtt{RS}_{rr}$ is the simplest as the estimation of λ is not required in both the model estimation based on the original data and the model estimation based on the bootstrap data. This method is attractive, but it is valid only under special scenarios. Other schemes all require the estimation of λ based on the original data, but not based on the bootstrapped data, to be in line with the LM principle. The proposed bootstrap methods preserve the feature of LM tests in the process of bootstrapping the values of the test statistic, thus greatly alleviate the computational burden as compared with bootstrapping, e.g., a Wald type test, or a likelihood ratio type test where the full model is estimated in every bootstrap sample. This point is particularly relevant to the tests of spatial dependence as spatial parameters often enter the model in a nonlinear fashion, and hence the estimation of them must be through a numerical optimization, which is avoided by the LM tests.

2.2 Validity of the bootstrap methods

When do the bootstrap methods described above offer asymptotically refined (higher-order) approximation to the finite sample critical values of the LM statistic? To address this issue, we need the following general assumptions regarding the LM test statistic $\mathrm{LM}_n(\lambda_0)$ and its finite sample null distribution $\mathcal{G}_n(\cdot, \theta, \mathcal{F})$ at the true $(\theta, \mathcal{F})$. Let $\mathcal{N}_{\theta,\mathcal{F}}$ denote a neighborhood of $(\theta, \mathcal{F})$. When the 'specified' CDF for $e_{n,i}$ (i.e., the CDF under which $\mathrm{LM}_n(\lambda_0)$ is developed) is the same as $\mathcal{F}$, we say $\mathcal{F}$ is correctly specified, otherwise misspecified.

Assumption G1. *$\mathcal{F}$ is correctly specified such that (i) $\mathrm{LM}_n(\lambda_0)$ developed under $\mathcal{F}$ is asymptotically pivotal when H_0 is true; (ii) $(\tilde{\theta}_n, \tilde{\mathcal{F}}_n)$ is $\sqrt{n}$-consistent for $(\theta, \mathcal{F})$ under H_0; and (iii) $(\hat{\theta}_n, \hat{\mathcal{F}}_n)$ is $\sqrt{n}$-consistent for $(\theta, \mathcal{F})$ whether or not H_0 is true.*

Assumption G2. *$\mathcal{F}$ is misspecified but Assumptions G1(ii)-(iii) remain. Furthermore, either $\mathrm{LM}_n(\lambda_0)$ is robust (i.e., it remains to be asymptotically pivotal at H_0) or its robust version, denoted as $\mathrm{SLM}_n(\lambda_0)$, exists and is used.*

Assumption G3. *For $(\vartheta, F) \in \mathcal{N}_{\theta,\mathcal{F}}$, the null CDF $\mathcal{G}_n(\cdot, \vartheta, F)$ converges weakly to a limit null CDF $\mathcal{G}(\cdot, \vartheta, F)$ as n increases, and admits the following asymptotic expansion uniformly in t and locally uniformly for $(\vartheta, F) \in \mathcal{N}_{\theta,\mathcal{F}}$:*

$$\mathcal{G}_n(t, \vartheta, F) = \mathcal{G}(t, \vartheta, F) + n^{-\frac{1}{2}} g(t, \vartheta, F) + O(n^{-1}), \tag{4}$$

where $\mathcal{G}(\cdot, \vartheta, F)$ is differentiable and strictly monotone over its support, and $g(t, \vartheta, F)$ is a functional of (t, ϑ, F) differentiable in (ϑ, F).

Assumption G1 is standard for likelihood-based inferences. Assumption G2 (consistency part) is also standard for quasi-likelihood-based inferences (see, e.g., White, 1982; White, 1994). Assumption G3 is adapted from Beran (1988). The difference is that the θ in our set-up contains only the nuisance

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parameters. Clearly, the limit null CDF $\mathcal{G}(t,\theta,\mathcal{F})$ depends on $(\theta,\mathcal{F})$ in general, unless $\mathcal{F}$ is correctly specified. In this case, an asymptotically robust version, $\text{SLM}_n(\lambda_0)$, has to be used for the bootstrap methods to be effective. In an important special case where λ_0 is a scalar and the test statistic is asymptotic $N(0,1)$, the asymptotic expansion (4) at $(\theta,\mathcal{F})$ reduces to:

$$\mathcal{G}_n(t,\theta,\mathcal{F}) = \Phi(t) + n^{-\frac{1}{2}}\phi(t)p(t,\theta,\mathcal{F}) + O(n^{-1}), \tag{5}$$

where Φ and ϕ are, respectively, the CDF and pdf of $N(0,1)$, *provided* that the jth cumulant $\boldsymbol{\kappa}_{j,n} \equiv \boldsymbol{\kappa}_{j,n}(\theta,\mathcal{F})$ of $\text{LM}_n(\lambda_0)_{H_0}$ can be expanded as a power series in n^{-1}:

$$\boldsymbol{\kappa}_{j,n} = n^{-\frac{j-2}{2}}(k_{j,1} + n^{-1}k_{j,2} + n^{-2}k_{j,3} + \cdots), \tag{6}$$

from which one has $p(t,\theta,\mathcal{F}) = -k_{1,2} + \frac{1}{6}k_{3,1}(1-t^2)$. See Hall (1992, Sec. 2.3) and Section 3 below for details. The validity of the bootstrap methods given above is summarized below.

Proposition 2.1. *Under Assumptions G1 and G3, the bootstrap methods under* $\texttt{RS}_{uu}$ *and* $\texttt{RS}_{uf}$ *are generally valid in that they are both able to provide full asymptotic refinements on the critical values of the LM tests, with an error of approximation of order* $O(n^{-1})$.

Proposition 2.2. *Under Assumptions G2 and G3, if further* $\frac{\partial}{\partial\mathcal{F}}g(t,\theta,\mathcal{F}) = O(n^{-\frac{1}{2}})$,[6] *then* $\tilde{\mathcal{F}}_n$ *can be used in place of* $\hat{\mathcal{F}}_n$*, and thus the bootstrap method with* $\texttt{RS}_{ur}$ *is also valid.*

Proposition 2.3. *Under Assumption G1 or G2, and Assumption G3, if either* $\tilde{\theta}_n$ *is also consistent when* H_0 *is false or* LM *or* SLM *test is invariant of* θ*, then* $\tilde{\theta}_n$ *can be used in place of* $\hat{\theta}_n$ *and thus the bootstrap method with* $\texttt{RS}_{ru}$ *is also valid.*

Proposition 2.4. *Under Assumptions G2 and G3, if the conditions for both Propositions 2.2 and 2.3 hold, then all the five bootstrap methods are valid.*

Remark 2.1: The four propositions give general principles on the proper ways to set up the bootstrap DGP in bootstrapping the critical values of LM tests, and settle the debate on the choices of residuals and parameter estimates (e.g., van Giersbergen and Kiviet (2002), MacKinnon (2002), and Godfrey (2009)) within the LM test framework. For related works on other type of tests, see, e,g,, Horowitz (1994) and Hall and Horowitz (1996).[7]

Proof. We present proofs in the main text to facilitate the understanding of the results. We prove these propositions collectively. Based on the general model specified in (1) and (2), the general hypothesis stated therein, and the LM statistic $\text{LM}_n(\lambda_0)$, we have by (2) and under H_0, i.e., under the real world null DGP: $Y_n = h(X_n, W_n, \theta, \lambda_0; e_n)$,

$$\begin{aligned} \text{LM}_n(\lambda_0)|_{H_0} &\equiv \text{LM}_n(Y_n, X_n, W_n; \lambda_0) \\ &= \text{LM}_n[h(X_n, W_n, \theta, \lambda_0; e_n), X_n, W_n; \lambda_0] \\ &\equiv \text{LM}_n(X_n, W_n, \theta, \lambda_0; e_n). \end{aligned}$$

The bootstrap DGP that mimics the real world null DGP is $Y_n^* = h(X_n, W_n; \ddot{\theta}_n, \lambda_0; e_n^*)$, where $e_n^* \overset{iid}{\sim} \ddot{\mathcal{F}}_n$. Based on the bootstrap data (Y_n^*, X_n, W_n), estimating the null model and computing the bootstrap

[6] This implies that the terms involving $\mathcal{F}$ in $\mathcal{G}_n(t,\theta,\mathcal{F})$ are smaller in magnitude than their neighboring terms. See Sections 3 and 4 for such cases.

[7] Godfrey (2009, p. 82) remarked that there are many published results on the asymptotic refinements associated with bootstrap tests. This literature is technical and sometimes involves relatively complex asymptotic analysis. However, it is not always the case that such asymptotic analysis seems to provide a good explanation of what is observed in finite samples. See also Davidson (2007) for some similar remarks.

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analogue of $\mathrm{LM}_n(\lambda_0)$, we have

$$\begin{aligned}\mathrm{LM}_n^*(\lambda_0) &\equiv \mathrm{LM}_n(Y_n^*, X_n, W_n; \lambda_0)\\ &= \mathrm{LM}_n[h(X_n, W_n, \ddot{\theta}_n, \lambda_0; e_n^*), X_n, W_n; \lambda_0]\\ &\equiv \mathrm{LM}_n(X_n, W_n, \ddot{\theta}_n, \lambda_0; e_n^*).\end{aligned}$$

Thus, $\mathrm{LM}_n^*(\lambda_0)$ is identical in structure to $\mathrm{LM}_n(\lambda_0)|_{H_0}$, suggesting that the *bootstrap CDF* of $\mathrm{LM}_n^*(\lambda_0)$ has the form $\mathcal{G}_n(\cdot, \ddot{\theta}_n, \ddot{\mathcal{F}}_n)$, identical in form to the finite sample CDF $\mathcal{G}_n(\cdot, \theta, \mathcal{F})$ of $\mathrm{LM}_n(\lambda_0)|_{H_0}$.[8] If $(\ddot{\theta}_n, \ddot{\mathcal{F}}_n)$ is consistent for $(\theta, \mathcal{F})$ and $\mathcal{G}_n(\cdot, \theta, \mathcal{F})$ converges weakly to the limit CDF $\mathcal{G}(\cdot, \theta, \mathcal{F})$ (Assumptions G1-G3), it can be easily argued based on the triangular-array convergence that $\mathcal{G}_n(\cdot, \ddot{\theta}_n, \ddot{\mathcal{F}}_n)$ converges weakly to $\mathcal{G}(\cdot, \theta, \mathcal{F})$. This shows that the test based on the bootstrap critical values has correct sizes asymptotically. When do the bootstrap methods offer asymptotically refined approximations?

Clearly, under Assumption G3, the asymptotic expansion (4) holds for $(\theta, \mathcal{F})$, which gives,

$$\mathcal{G}_n(t, \theta, \mathcal{F}) = \mathcal{G}(t, \theta, \mathcal{F}) + n^{-\frac{1}{2}} g(t, \theta, \mathcal{F}) + O(n^{-1}). \tag{7}$$

Assume (W.L.O.G.) $\mathrm{plim}_{n\to\infty}(\ddot{\theta}_n, \ddot{\mathcal{F}}_n) \in \mathcal{N}_{\theta,\mathcal{F}}$. As (4) holds locally uniformly for any $(\vartheta, F) \in \mathcal{N}_{\theta,\mathcal{F}}$, the bootstrap CDF admits the following asymptotic expansion:

$$\mathcal{G}_n(t, \ddot{\theta}_n, \ddot{\mathcal{F}}_n) = \mathcal{G}(t, \ddot{\theta}_n, \ddot{\mathcal{F}}_n) + n^{-\frac{1}{2}} g(t, \ddot{\theta}_n, \ddot{\mathcal{F}}_n) + O_p(n^{-1}). \tag{8}$$

Comparing (8) with (7), the scenarios under which the bootstrap is able to provide asymptotic refinements on the critical values are clear.

First, for Proposition 2.1, as $\mathcal{F}$ is correctly specified, $\mathcal{G}(t, \theta, \mathcal{F}) = \mathcal{G}(t)$, i.e., the limit null CDF is independent of the unknown parameter (θ). As (7) holds locally uniformly in $(\theta, \mathcal{F})$, it follows that $\mathcal{G}(t, \hat{\theta}_n, \hat{\mathcal{F}}_n) = \mathcal{G}(t)$. Taking difference between (7) and (8), we have whether or not H_0 is true,

$$\mathcal{G}_n(t, \theta, \mathcal{F}) - \mathcal{G}_n(t, \hat{\theta}_n, \hat{\mathcal{F}}_n) = n^{-\frac{1}{2}}[g(t, \theta, \mathcal{F}) - g(t, \hat{\theta}_n, \hat{\mathcal{F}}_n)] + O_p(n^{-1}) = O_p(n^{-1}),$$

where the latter equality is due to the differentiability of $g(\cdot, \theta, \mathcal{F})$ and the $\sqrt{n}$-consistency of $(\hat{\theta}_n, \hat{\mathcal{F}}_n)$. It follows that $c_n(\alpha, \theta, \mathcal{F}) - c_n(\alpha, \hat{\theta}_n, \hat{\mathcal{F}}_n) = O_p(n^{-1})$. However, $c_n(\alpha, \theta, \mathcal{F}) - c(\alpha) = O_p(n^{-\frac{1}{2}})$, where $c(\alpha)$ is the asymptotic critical value of $\mathrm{LM}_n(\lambda_0)|_{H_0}$ or the α-quantile of $\mathcal{G}(t)$, showing that the bootstrap critical value gives a higher-order approximation to the finite sample critical value of $\mathrm{LM}_n(\lambda_0)|_{H_0}$ than does the $c(\alpha)$. Thus, the $\mathtt{RS}_{uu}$ scheme is valid. Similar arguments lead to the validity of the $\mathtt{RS}_{fu}$ scheme. Finally, when the pair $(\hat{\theta}_n, \tilde{\mathcal{F}}_n)$, or $(\tilde{\theta}_n, \hat{\mathcal{F}}_n)$, or $(\tilde{\theta}_n, \tilde{\mathcal{F}}_n)$ is used for $(\ddot{\theta}_n, \ddot{\mathcal{F}}_n)$, i.e., $\mathrm{LM}_n^*(\lambda_0)$ is constructed as if $(\ddot{\theta}_n, \ddot{\mathcal{F}}_n) = (\theta, \mathcal{F})$, neither $\mathcal{G}(t, \ddot{\theta}_n, \ddot{\mathcal{F}}_n) = \mathcal{G}(t)$ nor $g(t, \ddot{\theta}_n, \ddot{\mathcal{F}}_n) - g(t, \theta, \mathcal{F}) = O_p(n^{-\frac{1}{2}})$ holds in general, because neither $\tilde{\theta}_n$ nor $\tilde{\mathcal{F}}_n$ is generally consistent when H_0 is false. This shows that the remaining resampling schemes cannot be valid in general.

To prove Proposition 2.2, we have in view of (7),

$$\mathcal{G}_n(t, \hat{\theta}_n, \tilde{\mathcal{F}}_n) = \mathcal{G}(t, \hat{\theta}_n, \tilde{\mathcal{F}}_n) + n^{-\frac{1}{2}} g(t, \hat{\theta}_n, \tilde{\mathcal{F}}_n) + O_p(n^{-1}).$$

The fact that $\mathrm{LM}_n(\lambda_0)|_{H_0}$ (or its robust version) is asymptotically pivotal even if $\mathcal{F}$ is misspecified implies that $\mathcal{G}(t, \theta, \mathcal{F}) = \mathcal{G}(t)$ and that $\mathcal{G}(t, \hat{\theta}_n, \tilde{\mathcal{F}}_n) = \mathcal{G}(t)$. Since $\frac{\partial}{\partial \mathcal{F}} g(t, \theta, \mathcal{F}) = O(n^{-\frac{1}{2}})$ and $\hat{\theta}_n$ is consistent, it follows that $g(t, \hat{\theta}_n, \tilde{\mathcal{F}}_n) - g(t, \theta, \mathcal{F}) = O_p(n^{-\frac{1}{2}})$. The result of Proposition 2.2 thus follows. The proofs of Propositions 3 and 4 are evident.

[8] Clearly, $\mathcal{G}_n(\cdot, \theta, \mathcal{F})$ does not have a closed-form expression in general and hence cannot be evaluated. However, as remarked in Footnote 5 the EDF of $\mathrm{LM}_n^*(\lambda_0)$ offers an arbitrarily accurate approximation to $\mathcal{G}_n(\cdot, \ddot{\theta}_n, \ddot{\mathcal{F}}_n)$ for sufficiently large B. The question that remains is thus how close can $\mathcal{G}_n(\cdot, \ddot{\theta}_n, \ddot{\mathcal{F}}_n)$ be to $\mathcal{G}_n(\cdot, \theta, \mathcal{F})$.

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Remark 2.2: When $\mathcal{F}$ is misspecified and $\text{LM}_n(\lambda_0)$ is not robust such that its limit null CDF $\mathcal{G}(t, \theta, \mathcal{F})$ depends on $(\theta, \mathcal{F})$, one can easily see from (7) and (8) that $\mathcal{G}(t, \hat{\theta}_n, \hat{\mathcal{F}}_n)$ typically differs from $\mathcal{G}(t, \theta, \mathcal{F})$ by an order of $O_p(n^{-\frac{1}{2}})$, and that $\mathcal{G}(t, \tilde{\theta}_n, \tilde{\mathcal{F}}_n)$ may differ from $\mathcal{G}(t, \theta, \mathcal{F})$ by an order of $O_p(1)$ when H_0 is false. This shows that bootstrapping a non-robust LM statistic in case of misspecification does not improve the critical values, or may even give wrong results.

Remark 2.3: The idea of using a robust/standardized statistic in case of misspecification in error distribution is in line with the 'prepivoting' idea of Beran (1988). Standardization can be viewed as analytical prepivoting. As indicated by Beran (1988), prepivoting can be iterated in the bootstrap way to attain further refinements on the critical values.

Remark 2.4: The arguments used in the proofs of the propositions rest on the formal asymptotic expansion (4) the existence of which can be shown in many cases (see Sec. 3).

3 Bootstrap LM Tests for Spatial Dependence

In this section, we consider several popular spatial LM tests to demonstrate the general methodology described in the last section. These include the LM tests for spatial error dependence (SED), the LM tests for spatial lag dependence (SLD), and the LM tests for spatial error components (SEC), presented respectively in Subsections 3.1-3.3. In each subsection, we present the LM tests (existing or new), formal arguments for the validity of the five bootstrap methods to supplement the general theoretical arguments presented in Section 2, and Monte Carlo results to support these arguments.

In what follows, the set of notation used above will be followed closely. Specifically, Y_n denotes an $n \times 1$ vector of response values, X_n an $n \times k$ matrix containing the values of nonstochastic regressors with its first column being a column of ones, W_n is an $n \times n$ spatial weight matrix, and $\mathcal{F}$ the CDF of the standardized errors $\{e_{n,i}\}$, with following conditions maintained.

Assumption S1. *The innovations $\{e_{n,i}\}$ are iid random draws from $\mathcal{F}$ with mean zero, variance 1, and finite cumulants $\kappa_j \equiv \kappa_j(\mathcal{F}), j = 3, 4, 5, 6$.*

Assumption S2. *The elements of X_n are uniformly bounded for all n, and $\lim_{n\to\infty} \frac{1}{n} X_n' X_n$ exists and is nonsingular. (These conditions are to be replaced by their stochastic versions if X_n is stochastic. The results are then interpreted conditionally, given the exogenous X_n.)*

Assumption S3. *The elements $\{w_{n,ij}\}$ of W_n are at most of order h_n^{-1} uniformly for all i, j, with the rate sequence $\{h_n\}$ satisfying $h_n/n \to 0$ as $n \to \infty$. $\{W_n\}$ are uniformly bounded in both row and column sums with $w_{n,ii} = 0$ and $\sum_j w_{n,ij} = 1$ for all i.*

We adopt the notation: E^*, Var^*, $\xrightarrow{D^*}$, $\xrightarrow{p^*}$, $o_{p^*}(\cdot)$, etc., to indicate that the expectation, variance, convergence in distribution, convergence in probability, smaller order of magnitude in probability, etc., are with respect to the bootstrap error distribution $\ddot{\mathcal{F}}_n$, to distinguish from the usual notation corresponding to $\mathcal{F}$. We assume throughout $\ddot{\mathcal{F}}_n$ has a zero mean and a unit variance (which are achievable through centering and scaling), and jth cumulant $\ddot{\kappa}_{jn} \equiv \ddot{\kappa}_j(\ddot{\mathcal{F}}_n), j = 3, 4, 5, 6$. Further, let $\text{tr}(A)$ denote the trace of a square matrix A, and $\text{diagv}(A)$ the column vector formed by the diagonal elements of A. Denote n/h_n by n_r. Recall: '˜' means `restricted`, and 'ˆ' means `unrestricted`.

3.1 Linear Regression with Spatial Error Dependence

We consider the LM test of Burridge (1980) (or Moran's I) and the standardized LM test of Baltagi and Yang (2013). As shown in Baltagi and Yang (2013), these tests are robust against nonnormality.

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Also, they are invariant of the nuisance parameters. According to the general principles laid in Section 2, the three bootstrap methods using the unrestricted residuals are valid for any $\mathcal{F}$. Indeed, this result is shown to be true, and further the two bootstrap methods using the restricted residuals are also valid if the error distribution is symmetric, but if not they are still able to achieve partial asymptotic refinements. Monte Carlo results confirm these findings and show that the gains from using the bootstrap critical values can be great. While the literature does contain some works on bootstrap tests for this model (Lin et al., 2011; Jin and Lee, 2012b) it seems to be lacking on both theoretical justifications and detailed comparisons on various bootstrap methods.

3.1.1 The model and the LM tests.

The linear regression model with spatial error dependence (SED) takes the form:

$$Y_n = X_n\beta + u_n, \qquad u_n = \rho W_n u_n + \varepsilon_n, \qquad \varepsilon_n = \sigma e_n, \tag{9}$$

where ρ is the spatial parameter, β is a $k \times 1$ vector of regression coefficients, and σ is the error standard deviation. Clearly, this model falls into the general framework of Model (1) with $\theta = \{\beta', \sigma^2\}'$, $e_n = q(Y_n, X_n, W_n; \theta, \rho) = B_n(\rho)(Y_n - X_n\beta)/\sigma$, and its inverse $Y_n = X_n\beta + \sigma B_n^{-1}(\rho)e_n$, where $B_n(\rho) = I_n - \rho W_n$ and I_n is an $n \times n$ identity matrix. Given ρ, the restricted QMLEs of β and σ^2 under Gaussian likelihood are $\tilde\beta_n(\rho) = [X_n'B_n(\rho)'B_n(\rho)X_n]^{-1}X_n'B_n(\rho)'B_n(\rho)Y_n$ and $\tilde\sigma_n^2(\rho) = \frac{1}{n}Y_n'B_n(\rho)'M_n(\rho)B_n(\rho)Y_n$, where $M_n(\rho) = I_n - B_n(\rho)X_n[X_n'B_n(\rho)'B_n(\rho)X_n]^{-1}X_n'B_n(\rho)'$. Maximizing the concentrated quasi Gaussian likelihood of ρ numerically leads to the unrestricted QMLE $\hat\rho_n$ of ρ, which upon substitutions gives the unrestricted QMLEs $\hat\beta_n \equiv \tilde\beta_n(\hat\rho_n)$ and $\hat\sigma_n^2 \equiv \tilde\sigma_n^2(\hat\rho_n)$ of β and σ^2.

We are interested in testing the lack of SED in the model, i.e., $H_0 : \rho = 0$ vs $H_a : \rho \neq 0$ (< 0, > 0), based on the LM principle. The LM test of Burridge (1980) takes the form:

$$\text{LM}_{\texttt{SED}} = \frac{n}{\sqrt{K_n}} \frac{\tilde\varepsilon_n' W_n \tilde\varepsilon_n}{\tilde\varepsilon_n'\tilde\varepsilon_n}, \tag{10}$$

where $\tilde\varepsilon_n$ is the vector of restricted (or OLS) residuals under H_0 and $K_n = \text{tr}(W_n'W_n + W_nW_n)$. To improve the finite sample performance and to enhance the robustness of $\text{LM}_{\texttt{SED}}$, Baltagi and Yang (2013) introduced a standardized version:

$$\text{SLM}_{\texttt{SED}} = \frac{n}{\sqrt{K_n^\dagger + \tilde\kappa_{4n} a_n' a_n}} \frac{\tilde\varepsilon_n' W_n^\circ \tilde\varepsilon_n}{\tilde\varepsilon_n'\tilde\varepsilon_n}, \tag{11}$$

where $W_n^\circ = W_n - \frac{1}{n-k}\text{tr}(W_nM_n)I_n$, $M_n = M_n(0)$, $K_n^\dagger = \text{tr}[\mathcal{A}_n(\mathcal{A}_n + \mathcal{A}_n')]$, $a_n = \text{diagv}(\mathcal{A}_n)$, $\mathcal{A}_n = M_nW_n^\circ M_n$, and $\tilde\kappa_{4n}$ is the 4th cumulant of $\tilde e_n = \tilde\sigma_n^{-1}\tilde\varepsilon_n$ (or $\tilde{\mathcal{F}}_n$). Baltagi and Yang (2013) show that both $\text{LM}_{\texttt{SED}}$ and $\text{SLM}_{\texttt{SED}}$ have limiting null distribution $N(0,1)$ and are robust to $\mathcal{F}$. To implement the bootstrap method under the $\texttt{RS}_{uf}$ scheme, we derived an LM statistic for a nonzero ρ, $\text{LM}_{\texttt{SED}}(\rho)$, and a standardized version of it, $\text{SLM}_{\texttt{SED}}(\rho)$, which are given in (B-2) and (B-3) of Appendix B.

3.1.2 Validity of the bootstrap methods

To see the validity of the various bootstrap methods presented in Section 2, we concentrate on $\text{LM}_{\texttt{SED}}$. Under the real world null DGP: $Y_n = X_n\beta + \sigma e_n$, $\tilde\varepsilon_n = \sigma M_n e_n$, and

$$\text{LM}_{\texttt{SED}}|_{H_0} = \frac{n}{\sqrt{K_n}} \frac{e_n' M_n W_n M_n e_n}{e_n' M_n e_n}. \tag{12}$$

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which shows $\text{LM}_{\text{SED}}|_{H_0}$ is invariant of the nuisance parameters, and thus a pivot if $\mathcal{F}$ is known, normal or nonnormal. In this situation, one can simply use Monte Carlo method to find the finite sample critical values of $\text{LM}_{\text{SEC}}|_{H_0}$ to any level of accuracy. To be exact, one draws e_n from the known distribution $\mathcal{F}$ repeatedly to give a sequence of values for $\text{LM}_{\text{SED}}|_{H_0}$, and then find the quantiles of this sequence that serve as approximations to the finite sample quantiles of $\text{LM}_{\text{SEC}}|_{H_0}$. When $\mathcal{F}$ is unknown and possibly misspecified, however, $\text{LM}_{\text{SED}}|_{H_0}$ is not an exact pivot, hence the Monte Carlo method just described does not work and the bootstrap methods need to be called for to provide asymptotically refined approximations to the finite sample critical values of $\text{LM}_{\text{SED}}|_{H_0}$.

In the bootstrap world, the bootstrap DGP that mimics the real world null DGP is $Y_n^* = X_n\ddot{\beta}_n + \ddot{\sigma}_n e_n^*$, where the elements of e_n^* are random draws from $\ddot{\mathcal{F}}_n$, the EDF of standardized residuals. Based on the bootstrap data (Y_n^*, X_n), computing the OLS estimates of $(\ddot{\beta}_n, \ddot{\sigma}_n)$, the OLS residuals and the LM test (10), we have the bootstrap analogue of $\text{LM}_{\texttt{SED}}|_{H_0}$:

$$\text{LM}^*_{\texttt{SED}} = \frac{n}{\sqrt{K_n}} \frac{e_n^{*\prime} M_n W_n M_n e_n^*}{e_n^{*\prime} M_n e_n^*}, \tag{13}$$

which shows that $\text{LM}^*_{\texttt{SED}}$ is invariant of $\ddot{\beta}_n$ and $\ddot{\sigma}_n^2$. Thus, whether $\ddot{\beta}_n$ and $\ddot{\sigma}_n^2$ correspond to the restricted or unrestricted estimates of β and σ makes no difference on the performance of the bootstrap procedures.

Comparing (13) with (12), it is intuitively quite clear that if e_n^* are drawn from an EDF $\ddot{\mathcal{F}}_n$ that consistently estimates $\mathcal{F}$ whether or not H_0 is true, then the EDF of LM^*_{SED} offers a consistent estimate of the finite sample distribution of $\text{LM}_{\text{SED}}|_{H_0}$. This is just like the Monte Carlo approach under a known $\mathcal{F}$. However, with $\ddot{\mathcal{F}}_n$ the finite sample distribution of $\text{LM}_{\texttt{SED}}|_{H_0}$ is estimated nonparametrically. With this in mind, the attractiveness of the bootstrap approach becomes clearer.

Proposition 3.1. *Suppose Model (9) satisfies Assumptions S1-S3. If (i) $\hat{\rho}_n$ is $\sqrt{n/h_n}$-consistent, and (ii) $|\text{LM}_{\texttt{SED}}|_{H_0}| \le U$ a.e., and $E(U^4)$ exists, then the bootstrap methods under $\texttt{RS}_{uu}, \texttt{RS}_{uf}$ and $\texttt{RS}_{ru}$ are valid for $\text{LM}_{\texttt{SED}}$, $\forall \mathcal{F}$. If, in addition, $\kappa_3 = 0$, the bootstrap methods under $\texttt{RS}_{ur}$ and $\texttt{RS}_{rr}$ are valid as well. The same conclusions apply to $\text{SLM}_{\texttt{SED}}$.*

Proof: We highlight the key arguments here for a quick appreciation of the results. Details are given in Appendix B (Lemma A8 and Proof of Proposition 3.1 (Cont'd)). We have $\text{LM}_{\texttt{SED}}|_{H_0} \xrightarrow{D} N(0,1), \forall \mathcal{F}$, and the following Edgeworth expansion for the finite sample null CDF of $\text{LM}_{\texttt{SED}}$:

$$\mathcal{G}_n(t, \mathcal{F}) = \Phi(t) + n_r^{-\frac{1}{2}} c_0^{-\frac{3}{2}} \phi(t)\ p(t, \mathcal{F}) + O(n_r^{-1}), \tag{14}$$

where $p(t,\mathcal{F}) = -c_0 c_1 + (\frac{1}{6}\kappa_3^2 T_4 + T_5)(1-t^2)$, $n_r = \frac{n}{h_n}$, $c_0 = \lim_{n\to\infty} n_r^{-1} K_n$, $c_1 = \lim_{n\to\infty} \text{tr}(M_n W_n M_n)$, and T_4 and T_5, given in Appendix B, are $O(1)$ and are free of θ. Similarly, we show that $\text{LM}^*_{\texttt{SED}} \xrightarrow{D^*} N(0,1), \forall \ddot{\mathcal{F}}_n$, and that the bootstrap CDF of $\text{LM}^*_{\texttt{SED}}$ admits the following asymptotic expansion:

$$\mathcal{G}_n(t, \ddot{\mathcal{F}}_n) = \Phi(t) + n_r^{-\frac{1}{2}} c_0^{-\frac{3}{2}} \phi(t)\ p(t, \ddot{\mathcal{F}}_n) + O_p(n_r^{-1}), \tag{15}$$

where $p(t, \ddot{\mathcal{F}}_n) = -c_0 c_1 + (\frac{1}{6}\ddot{\kappa}_{3n}^2 T_4 + T_5)(1-t^2)$. Taking difference between (15) and (14), we obtain,

$$\mathcal{G}_n(t, \ddot{\mathcal{F}}_n) - \mathcal{G}_n(t, \mathcal{F}) = \tfrac{1}{6} n_r^{-\frac{1}{2}} c_0^{-\frac{3}{2}} T_4\ \phi(t)(1-t^2)(\ddot{\kappa}_{3n}^2 - \kappa_3^2) + O_p(n_r^{-1}).$$

By Lemma A8, when $\ddot{\mathcal{F}}_n = \hat{\mathcal{F}}_n$, $\hat{\kappa}_{3n}^2 - \kappa_3^2 = O_p(n_r^{-\frac{1}{2}})$, $\forall \mathcal{F}$. Thus, $\mathcal{G}_n(t, \hat{\mathcal{F}}_n) - \mathcal{G}_n(t, \mathcal{F}) = O_p(n_r^{-1})$ and $c_n(t, \hat{\mathcal{F}}_n) - c_n(t, \mathcal{F}) = O_p(n_r^{-1})$, showing the first part of Proposition 3.1. Now, when error distribution is symmetric, $\kappa_3 = 0$, and by Lemma A8 $\tilde{\kappa}_{3n}$ is $o_p(1)$, showing that $\hat{\mathcal{F}}_n$ can be replaced by $\tilde{\mathcal{F}}_n$with error becoming $o_p(n_r^{-\frac{1}{2}})$. Finally, the same set of results are obtained for the standardized LM statistic.

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Remark 3.1: When the error distribution is skewed, the bootstrap methods under $\texttt{RS}_{ur}$ and $\texttt{RS}_{rr}$, though not strictly valid, improve the asymptotic method as the main second-order terms involving c_0c_1 and T_5 are captured by the bootstrap methods, resulting partial asymptotic refinements.[9]

Remark 3.2: The detailed proof given in Appendix B shows the first three cumulants of $\text{LM}_{\texttt{SED}}|_{H_0}$: $\boldsymbol{\kappa}_{1,n} = n_r^{-\frac{1}{2}} c_0^{-\frac{1}{2}} c_1 + O(n_r^{-\frac{3}{2}})$, $\boldsymbol{\kappa}_{2,n} = 1 + O(n_r^{-1})$, and $\boldsymbol{\kappa}_{3,n} = n_r^{-\frac{1}{2}} c_0^{-\frac{3}{2}} (\kappa_3^2 T_4 + 6T_5) + O(n_r^{-\frac{3}{2}})$, from which we see precisely the reason why the finite sample distribution of a spatial LM test deviates more from its limiting distribution with a denser spatial weight matrix.

3.1.3 Monte Carlo Results

The Monte Carlo experiments are carried out based on the following DGP:

$$Y_n = \beta_0 1_n + X_{n1}\beta_1 + X_{n2}\beta_2 + u_n, \;\; u_n = \rho W_n u_n + \sigma e_n.$$

The parameter values are set at $\beta = \{5, 1, 1\}'$ and $\sigma = 1$ or 2. Four different sample sizes are considered, i.e., $n = 50, 100, 200$, and 500. All results are based on $M = 2,000$ Monte Carlo samples, and $B = 699$ bootstrap samples for each Monte Carlo sample. The methods for generating spatial layouts, error distributions, and fixed regressors' values are described in Appendix C.

For $\rho = \{-0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75\}$, two types of Monte Carlo results are recorded: (a) the means and standard deviations of the bootstrap critical values, and (b) the rejection frequencies of the LM and SLM tests. As the tests are invariant of the nuisance parameters, the results under $\texttt{RS}_{ur}$ coincide with those under $\texttt{RS}_{rr}$, and the results under $\texttt{RS}_{ru}$ are identical to those under $\texttt{RS}_{uu}$. Also, the results under $\texttt{RS}_{uf}$ are very close to those under $\texttt{RS}_{uu}$, and hence are not reported for brevity. Furthermore, the bootstrap results for $\texttt{SLM}_{\texttt{SED}}$ are also not reported as the rejection frequencies are almost identical to those for $\texttt{LM}_{\texttt{SED}}$, and the critical values, though different from those for $\texttt{LM}_{\texttt{SED}}$, show the same degree of stability and agreement with the 'true' finite sample critical values by Monte Carlo methods. Finally, a small sets of results are reported in Table 3.1a for the (average) bootstrap critical values and 3.1b for rejection frequencies. General observations are summarized as follows:

1. The (average) bootstrap critical values are all very close to the 'true' finite sample critical values (obtained by Monte Carlo simulation), but can all be far from their asymptotic critical values (ACR) which are ± 1.6449 and ± 1.96. The implication of this is clear: the use of asymptotic critical values may lead to large distortions on size and power of the tests. Working with SLM improves in this regard, but it is still not satisfactory if one sided tests are desired;
2. The bootstrap critical values for both LM and SLM under $\texttt{RS}_{uf}$, $\texttt{RS}_{uu}$ and $\texttt{RS}_{ru}$ are all very stable; those under $\texttt{RS}_{ur}$ and $\texttt{RS}_{rr}$ change with ρ slightly, confirming the Remark 3.1.
3. The standard deviations of the bootstrap critical values (not reported for brevity) are all small, in the magnitudes of (0.0425, 0.0376, 0.1042, 0.1363) for the four critical values of the $\texttt{LM}_{\texttt{SED}}$ test under normal errors. They increase a little bit when errors are nonnormal or when $\texttt{SLM}_{\texttt{SED}}$ is used; they don't change much with n but decrease when B increases (both are as expected). As far as the rejection frequency is concerned we found that using $B = 699$ is sufficient;
4. Use of the bootstrap critical values significantly improves the size of the LM tests, and the power of the left-tailed LM tests.

[9] Robinson and Rossi (2010) developed a finite sample correction for a simpler version of (10) without regressors and with normal errors, using Edgeworth expansion. Jin and Lee (2012b) presented first-order results for a test that can be approximated by a linear-quadratic form in the error vector, and gave a preliminary discussion of possible asymptotic refinements. The key issue on the type of estimates and residuals to be used in bootstrap DGP was not addressed.

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5. When the regressors are generated under the iid setting (XVAL-A), the finite sample distribution of LM_{SED} is more skewed to the right, making the left-tail rejection frequencies much lower than their nominal values. Use of a denser spatial weight matrix worsens this problem. However, in all these scenarios, standardizations method help and bootstrap methods work well.

<< Insert Table 3.1a Here >>

A note in passing to read Table 3.1b is that the values under the column of $|\rho|$ should read as negative if L2.5% and L5%, i.e., the left-tailed 2.5% and 5% tests, are concerned. All results in Table 3.1b correspond to LM_{SED}, except the rows labeled with ACR^* which correspond to SLM_{SED} referring to the asymptotic critical values.

<< Insert Table 3.1b Here >>

3.2 Linear Regression with Spatial Lag Dependence

We now present a case where the LM statistics depends on the nuisance parameters, the restricted estimates of nuisance parameters are inconsistent when the null hypothesis is false, but the LM statistic at the null is still robust against nonnormality. According to the general results presented in Section 2, only the bootstrap methods under RS_{uu} and RS_{uf} are valid. As this case is more involved, a more detailed study is given. This study contributes to the spatial econometrics literature by (i) providing theoretical justifications on the validity of various bootstrap methods with respect to the choice of bootstrap parameters and the choice of bootstrap error distribution, and (ii) providing detailed Monte Carlo results to support these theoretical arguments, in particular the results on the bootstrap critical values. Common Monte Carlo study on the performance of bootstrap tests typically reports the empirical rejection frequencies (size and power). This study reveals that judging a bootstrap test based only on size and power may be misleading as in reality one does not know whether or not the null hypothesis is true, and hence the seemingly 'correct' size and 'higher' power for certain tests may not be achievable. Some related works can be found in Lin et al. (2007, 2009).

3.2.1 The Model and the LM Tests.

The linear regression model with spatial lag dependence (SLD), also known as the spatial autoregressive (SAR) model, takes the following form:

$$Y_n = \lambda W_n Y_n + X_n\beta + \varepsilon_n, \;\; \varepsilon_n = \sigma e_n \tag{16}$$

where e_n, X_n and W_n satisfy Assumptions S1-S3, λ is the spatial parameter, and β is a $k \times 1$ vector of regression coefficient. Clearly, Model (16) fits into the general framework of Model (1) with $\theta = \{\beta', \sigma^2\}'$, $e_n = q(Y_n, X_n, W_n; \theta, \lambda) = [A_n(\lambda)Y_n - X_n\beta]/\sigma$, and its inverse $Y_n = h(X_n, W_n; \theta, \lambda; e_n) = A_n^{-1}(\lambda)(X_n\beta + \sigma e_n)$, where $A_n(\lambda) = I_n - \lambda W_n$. Given λ, the restricted QMLEs of β and σ^2 under Gaussian likelihood are, respectively, $\tilde{\beta}_n(\lambda) = (X_n'X_n)^{-1}X_n'A_n(\lambda)Y_n$, and $\tilde{\sigma}_n^2(\lambda) = \frac{1}{n}Y_n'A_n'(\lambda)M_nA_n(\lambda)Y_n$. Maximizing the concentrated Gaussian likelihood for λ gives the unrestricted QMLE $\hat{\lambda}_n$, and hence the unrestricted QMLEs $\hat{\beta}_n \equiv \tilde{\beta}_n(\hat{\lambda}_n)$, and $\hat{\sigma}_n^2 \equiv \tilde{\sigma}_n^2(\hat{\lambda}_n)$.

The LM test for testing $H_0: \lambda = 0$ vs $H_a: \lambda \neq 0$ $(< 0, > 0)$ is given in Anselin (1988a,b):

$$\text{LM}_{\text{SLD}} = \frac{\tilde{\varepsilon}_n' W_n Y_n}{\tilde{\sigma}_n^2\sqrt{\tilde{\eta}_n' M_n \tilde{\eta}_n + K_n}}, \tag{17}$$

where $\tilde{\varepsilon}_n = Y_n - X_n\tilde{\beta}_n$, $K_n = \text{tr}(W_n'W_n + W_nW_n)$, $\tilde{\eta}_n = \tilde{\sigma}_n^{-1}W_nX_n\tilde{\beta}_n$, $\tilde{\beta}_n = \tilde{\beta}_n(0)$ and $\tilde{\sigma}_n^2 = \tilde{\sigma}_n^2(0)$. A standardized version of LM_{SLD}, having better finite sample properties and more robust against the

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spatial layouts, is given in Yang and Shen (2011):

$$\mathrm{SLM}_{\mathtt{SLD}} = \frac{\tilde{\varepsilon}_n' W_n^\circ Y_n}{\tilde{\sigma}_n^2 \sqrt{\tilde{\eta}_n' M_n \tilde{\eta}_n + K_n^\dagger + \tilde{\kappa}_{4n} d_n' d_n + 2\tilde{\kappa}_{3n} \tilde{\eta}_n' M_n d_n}}, \tag{18}$$

where $W_n^\circ = W_n - \frac{1}{n-k}\mathrm{tr}(W_n M_n) I_n$, $K_n^\dagger = \mathrm{tr}[\mathcal{A}_n(\mathcal{A}_n + \mathcal{A}_n')]$, $a_n = \mathrm{diagv}(\mathcal{A}_n)$, $\mathcal{A}_n = M_n W_n^\circ$, and $\tilde{\kappa}_{3n}$ and $\tilde{\kappa}_{4n}$ are, respectively, the 3rd and 4th cumulants of $\tilde{e}_n = \tilde{\sigma}_n^{-1}\tilde{\varepsilon}_n$. Yang and Shen (2011) show that both $\mathrm{LM}_{\mathtt{SLD}}$ and $\mathrm{SLM}_{\mathtt{SLD}}$ have limiting null distribution $N(0,1)$, whether or not $\mathcal{F}$ is correctly specified, showing that both are asymptotically robust against distributional misspecification. To implement the bootstrap method under the resampling scheme $\mathtt{RS}_{uf}$, more general LM statistics for a nonzero λ, $\mathrm{LM}_{\mathtt{SLD}}(\lambda)$, and its standardized version, $\mathrm{SLM}_{\mathtt{SLD}}(\lambda)$, can be found in Yang and Shen (2011).

3.2.2 Validity of the Bootstrap Methods

To study the validity of various resampling schemes when bootstrapping the critical values of the LM and SLM tests of spatial lag dependence, we concentrate on the test $\mathrm{LM}_{\mathtt{SLD}}$. Under the real world null DGP: $Y_n = X_n\beta + \sigma e_n$, we have after some algebra,

$$\mathrm{LM}_{\mathtt{SLD}}|_{H_0} = \frac{\sqrt{n}(e_n' M_n W_n e_n + e_n' M_n \eta_n)}{(e_n' M_n e_n)^{\frac{1}{2}} \left\{\eta_n' M_n \eta_n + Q(e_n) + 2e_n' P_n W_n M_n \eta_n\right\}^{\frac{1}{2}}}, \tag{19}$$

where $Q(e_n) = n^{-1}K_n e_n' M_n e_n + e_n' P_n' W_n' M_n W_n P_n e_n$, $\eta_n = \sigma^{-1}W_n X_n \beta$, and $P_n = I_n - M_n$. This shows that $\mathrm{LM}_{\mathtt{SLD}}|_{H_0} = f(e_n, X_n, W_n, \beta, \sigma)$, meaning that $\mathrm{LM}_{\mathtt{SLD}}|_{H_0}$ is not an exact pivot whether or not $\mathcal{F}$ is known as its finite sample null distribution is governed by $\mathcal{F}$, the CDF of $\{e_{n,i}\}$, and the values of the nuisance parameters β and σ^2, given X_n and W_n. The dependence of $\mathrm{LM}_{\mathtt{SLD}}|_{H_0}$ on (β, σ^2) is expected to impose constraints on the choices of their estimates to be used as parameters in the bootstrap DGP. On the other hand, the limiting distribution of $\mathrm{LM}_{\mathtt{SLD}}|_{H_0}$ does not depend on (β, σ^2) and $\mathcal{F}$ (Kelejian and Prucha, 2001; Yang and Shen, 2011), suggesting (as in Section 3.1.2) that bootstrap methods can be applied to provide asymptotically refined critical values for $\mathrm{LM}_{\mathtt{SLD}}|_{H_0}$.

Under the bootstrap world, the bootstrap DGP that mimics the real world null DGP takes the form: $Y_n^* = X_n\ddot{\beta}_n + \ddot{\sigma}_n e_n^*$, where the elements of e_n^* are random draws from $\ddot{\mathcal{F}}_n$. Based on the bootstrap data (Y_n^*, X_n), estimating the bootstrap model and computing the test statistic (17) lead to the bootstrap analogue of $\mathrm{LM}_{\mathtt{SLD}}|_{H_0}$:

$$\mathrm{LM}^*_{\mathtt{SLD}} = \frac{\sqrt{n}(e_n^{*\prime} M_n W_n e_n^* + e_n^{*\prime} M_n \ddot{\eta}_n)}{(e_n^{*\prime} M_n e_n^*)^{\frac{1}{2}} \left\{\ddot{\eta}_n' M_n \ddot{\eta}_n + Q(e_n^*) + 2e_n^{*\prime} P_n W_n M_n \ddot{\eta}_n\right\}^{\frac{1}{2}}}, \tag{20}$$

where $\ddot{\eta}_n = \ddot{\sigma}_n^{-1} W_n X_n \ddot{\beta}_n$. Comparing (20) with (19), it is intuitively clear that for bootstrap to provide a higher-order approximation to the finite sample critical values of $\mathrm{LM}_{\mathtt{SLD}}|_{H_0}$, it is necessary that $\ddot{\beta}_n, \ddot{\sigma}_n^2$, and $\ddot{\mathcal{F}}_n$ are consistent whether or not H_0 is true. We have the following result.

Proposition 3.2. *Suppose Model (16) satisfies Assumptions S1-S3. If (i) $\hat{\lambda}_n$ is $\sqrt{n/h_n}$-consistent, and (ii) $|\mathrm{LM}_{\mathtt{SLD}}|_{H_0}| \leq U$ a.e., and $E(U^4)$ exists, then the bootstrap methods under $\mathtt{RS}_{uu}$ and $\mathtt{RS}_{uf}$ are valid for $\mathrm{LM}_{\mathtt{SLD}}$, $\forall\mathcal{F}$. If, in addition, $\kappa_3 = 0$ and the conditions in (A-3) hold, the bootstrap methods under $\mathtt{RS}_{ur}$ is valid as well. The same conclusions apply to $\mathrm{SLM}_{\mathtt{SLD}}$.*

Proof: For a quick appreciation of the results, we present the key arguments here, and give details in Appendix B (Lemma A8 and Proof of Proposition 3.2 (Cont'd)). First, $\mathrm{LM}_{\mathtt{SLD}}|_{H_0} \xrightarrow{D} N(0,1), \forall\mathcal{F}$. The finite sample CDF of $\mathrm{LM}_{\mathtt{SLD}}|_{H_0}$ admits the following Edgeworth expansion:

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$$\begin{aligned}\mathcal{G}_n(t,\theta,\mathcal{F}) &= \Phi(t)+n_r^{-\frac{1}{2}}c_0(\theta)^{-\frac{3}{2}}\phi(t)\ p(t,\theta,\mathcal{F})+O(n_r^{-1}), \quad (21)\\ p(t,\theta,\mathcal{F}) &= -c_0(\theta)c_1+[\tfrac{1}{6}\kappa_3^2T_4+T_5+\tfrac{1}{6}\kappa_3(S_3(\theta)+2S_5(\theta))+\tfrac{1}{3}S_4(\theta)](1-t^2),\end{aligned}$$

where $c_0(\theta)=\lim_{n\to\infty} n_r^{-1}(\eta_n' M_n\eta_n+K_n)$, $c_1=\lim_{n\to\infty}\mathrm{tr}(M_nW_n)$, and $T_i, i=4,5$ and $S_i(\theta), i=3,4,5$ are all $O(1)$ with their exact definitions given in Appendix B.

Similarly, $\mathrm{LM}^*_{\mathtt{SLD}} \xrightarrow{D^*} N(0,1)$. The bootstrap CDF of $\mathrm{LM}^*_{\mathtt{SLD}}$ admits the asymptotic expansion:

$$\begin{aligned}\mathcal{G}_n(t,\ddot\theta_n,\ddot{\mathcal{F}}_n) &= \Phi(t)+n_r^{-\frac{1}{2}}c_0(\ddot\theta_n)^{-\frac{3}{2}}\phi(t)\ p(t,\ddot\theta_n,\ddot{\mathcal{F}}_n)+O_p(n_r^{-1}), \quad (22)\\ p(t,\ddot\theta_n,\ddot{\mathcal{F}}_n) &= -c_0(\ddot\theta_n)c_1+[\tfrac{1}{6}\ddot\kappa_{3n}^2T_4+T_5+\tfrac{1}{6}\ddot\kappa_{3n}(S_3(\ddot\theta_n)+2S_5(\ddot\theta_n))+\tfrac{1}{3}S_4(\ddot\theta_n)](1-t^2).\end{aligned}$$

With these two expansions, the conclusions reached in Proposition 3.2 are clear. In particular, $\mathcal{G}_n(t,\ddot\theta_n,\ddot{\mathcal{F}}_n)-\mathcal{G}_n(t,\theta,\mathcal{F})=O_p(n_r^{-1})$ only when $\ddot\theta_n=\hat\theta_n$ and $\ddot{\mathcal{F}}_n=\hat{\mathcal{F}}_n$. Similar to the SED model, $p(t,\theta,\mathcal{F})$ depends on $\mathcal{F}$ only through κ_3, thus $\hat{\mathcal{F}}_n$ can be replaced by $\tilde{\mathcal{F}}_n$ when $\kappa_3=0$ and the conditions in (A-3) hold, leading to the validity of $\mathtt{RS}_{ur}$. Finally, the same set of results are obtained for $\mathrm{SLM}_{\mathtt{SLD}}$.

Remark 3.3: When the error distribution is skewed, the bootstrap method under $\mathtt{RS}_{ur}$, though not strictly valid, improves upon the asymptotic method as the main second-order terms involving T_5, $c_0(\theta)c_1$ and $S_4(\theta)$ are captured by bootstrap due to the consistency of $\hat\theta_n$, leading to the so-called partial asymptotic refinements. This explains why the Monte Carlo results (not reported for brevity) under $\mathtt{RS}_{ur}$ are very similar to these under $\mathtt{RS}_{uu}$ even when the errors are skewed.

Remark 3.4: Again, the cumulants of $\mathrm{LM}_{\mathtt{SLD}}|_{H_0}$ given in Appendix B show clearly the effect of spatial weight matrix on the finite sample distribution of $\mathrm{LM}_{\mathtt{SLD}}|_{H_0}$.

3.2.3 Monte Carlo Results.

The finite sample performance of $\mathrm{LM}_{\mathtt{SLD}}$ and $\mathrm{SLM}_{\mathtt{SLD}}$ for testing H_0: $\lambda=0$ vs H_a: $\lambda<0$ or H_a: $\lambda>0$, when referring to the asymptotic critical values and the bootstrap critical values under various resampling schemes, are investigated in terms of accuracy and stability of the bootstrap critical values with respect to the true value of λ, and the size and power of the tests. The Monte Carlo experiments are carried out based on the following data generating process:

$$Y_n=\lambda W_nY_n+\beta_0 1_n+X_{n1}\beta_1+X_{n2}\beta_2+\varepsilon_n$$

where the methods for generating W_n, X_n and ε_n are described in Appendix C. The regressors are treated as fixed in the experiments. The parameter values are set at $\beta=\{5,1,1\}'$ and $\sigma=1$ or 2, and sample sizes used are $n=(50,100,200,500)$. All results reported below are based on $M=2,000$ Monte Carlo samples, and $B=699$ bootstrap samples for each Monte Carlo sample generated. The bootstrap critical values are bench-marked against the Monte Carlo (MC) critical values obtained based on $M=30,000$ Monte Carlo samples.

Bootstrap critical values. We first report in Table 3.2a the averages of 2,000 bootstrap critical values of $\mathrm{LM}_{\mathtt{SLD}}$ and $\mathrm{SLM}_{\mathtt{SLD}}$ based on the restricted resampling scheme $\mathtt{RS}_{rr}$ and the unrestricted resampling scheme $\mathtt{RS}_{uu}$. The results with $\mathtt{RS}_{ru}$ are very similar to those with $\mathtt{RS}_{rr}$ and the results with $\mathtt{RS}_{ur}$ and $\mathtt{RS}_{uf}$ are very similar to those with $\mathtt{RS}_{uu}$, thus, are not reported for saving space. These unreported results show that whether to use the restricted or unrestricted residuals does not affect much the bootstrap critical values, which is consistent with Remark 3.3. Furthermore, Monte Carlo results clearly reveal the following:

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1. The bootstrap critical values can be quite different from the corresponding asymptotic critical values, showing the necessity of using finite sample critical values for testing the existence of spatial lag dependence in a linear regression model;
2. The bootstrap critical values based on $\texttt{RS}_{rr}$ (and $\texttt{RS}_{ru}$) vary significantly with λ. This suggests that, if when H_0 is true the bootstrap critical values and the resulted sizes of the tests are accurate (indeed they are), then when H_0 is false, the bootstrap critical values cannot be accurate and the resulted powers of the tests cannot be reliable;
3. The bootstrap critical values based on $\texttt{RS}_{uu}$ are very stable with respect to λ, and are very accurate as they agree very well with the corresponding Monte Carlo critical values obtained by imposing H_0 and using $M = 30,000$, and with the bootstrap critical values under $\texttt{RS}_{rr}$ and H_0 (considered as an ideal situation). The same holds when $|\lambda|$ further increases from 0.5.

The bootstrap critical values do not depend much on the error distributions due to the fact that the LM tests involved are asymptotically pivotal at the null under a general $\mathcal{F}$. As sample size n increases, the bootstrap critical values move closer to their limiting values, but the instability of those based on restricted estimates still exists. The above observations are consistent with the theoretical results: while the tests are asymptotically pivotal, their finite sample distributions depends on the nuisance parameter and the restricted estimates of the nuisance parameters are not consistent when null is false, which make the bootstrap methods based on the restricted estimates invalid.

`<< Insert Table 3.2a Here >>`

Size and power of the tests. We now report in Tables 3.2b and 3.2c the size and power of the one-sided LM tests based on the asymptotic critical values (`ACR`) and the bootstrap critical values with $\texttt{RS}_{rr}$ and $\texttt{RS}_{uu}$. Again the results based on other three resampling schemes $\texttt{RS}_{ru}$, $\texttt{RS}_{ur}$ and $\texttt{RS}_{uf}$ (not reported for brevity and clarity of presentation) are very close to those based on either $\texttt{RS}_{rr}$ or $\texttt{RS}_{uu}$, showing again the type of residuals to be used in resampling does not affect much the performance of the bootstrap methods. The results (reported and unreported) further reveal the following:

1. The tests referring to the asymptotic critical values can have severe size-distortion, and more so with heavier spatial dependence. Referring to bootstrap critical values effectively remove the size distortions under any resampling method, but one must bear in mind that this is unachievable with the restricted estimates as in practice whether H_0 is true or false remains unknown.
2. The bootstrap critical values of the LM statistic based on the restricted estimates tend to increase **in magnitude** as λ increases. As a result, the power tends to be lower for a right-tailed test, and higher for a left-tailed test, compared with the power of the tests based on the unrestricted estimates. The bootstrap critical values of the SLM statistic based on the restricted estimates decrease as λ increases. As a result, the power of both left- and right-tailed tests tends to be higher than that based on the unrestricted estimates. However, the former corresponds to a larger size due to smaller underlining bootstrap critical values.
3. As the original LM test is already asymptotically pivotal and robust, standardization does not provide further improvements on the bootstrap critical values in that the use of restricted estimates still lead to bootstrap critical values that vary with λ.

To summarize, using the restricted estimates of the nuisance parameters in the bootstrap DGP results in bootstrap critical values that can be either larger or smaller than the 'true' ones, leading to a test with either higher or lower power than it supposes to be. In contrast, using the unrestricted parameter estimates leads to test with 'realizable or achievable' power.

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<< Insert Table 3.2b and Table 3.2c Here >>

The bias of the restricted estimators of the regression coefficients and the error standard deviation when H_0 is false is investigated under the same setup as for the above results, as it is the major cause of instability of the bootstrap critical values. The results (not reported for brevity) show that the empirical means of the restricted estimators of $(\beta_0, \beta_1, \beta_2, \sigma) = (5, 1, 1, 1)$ range from $(3.35, 0.76, 0.70, 1.04)$ to $(9.96, 1.75, 1.92, 1.19)$ as λ changes from -0.5 to 0.5 with $n = 100$, and that the bias does not go away with larger sample sizes. In contrast, the unrestricted estimators are nearly unbiased.

3.3 Linear Regression with Spatial Error Components

In this section, we present a case where the usual LM test is not robust against the misspecification of the error distribution $\mathcal{F}$, but its finite sample distribution is invariant of the nuisance parameters. According to the general theories presented in Section 2, the bootstrap methods under $\texttt{RS}_{ru}$, $\texttt{RS}_{uu}$ and $\texttt{RS}_{uf}$ are valid. The results presented in this section contribute to the spatial econometrics literature by providing theoretical justifications and empirical evidence concerning the validity of various bootstrap methods applied to LM and SLM tests of spatial error components.

3.3.1 The Model and the LM Tests

Kelejian and Robinson (1995) proposed a spatial error components model which provides a useful alternative to the traditional spatial models with a spatial autoregressive (SAR) or a spatial moving average (SMA) error process, in particular in the situation where the range of spatial autocorrelation is constrained to close neighbors, e.g., spatial spillovers in the productivity of infrastructure investments (Kelejian and Robinson, 1997; Anselin and Moreno, 2003). The model takes the form:

$$Y_n = X_n\beta + u_n, \qquad \text{with } u_n = W_n v_n + \varepsilon_n, \text{and } \varepsilon_n = \sigma e_n \tag{23}$$

where v_n is an $n \times 1$ vector of errors that together with W_n incorporates the spatial dependence, and ε_n is an $n \times 1$ vector of location specific disturbance terms. The error components v_n and ε_n are assumed to be independent, with iid elements of mean zero and variances σ_v^2 and σ^2, respectively.

Let $\lambda = \sigma_v^2/\sigma^2$, and $\Omega_n(\lambda) = I_n + \lambda W_n W_n'$, we have $\text{Var}(u_n) = \sigma^2\Omega_n(\lambda)$. Maximizing the Gaussian likelihood for a given λ gives the restricted QMLEs $\tilde{\beta}_n(\lambda) = [X_n'\Omega_n^{-1}(\lambda)X_n]^{-1}X_n'\Omega_n^{-1}(\lambda)Y_n$ and $\tilde{\sigma}_n^2(\lambda) = \frac{1}{n}[Y_n - X_n\tilde{\beta}_n(\lambda)]'\Omega_n^{-1}(\lambda)[Y_n - X_n\tilde{\beta}_n(\lambda)]$ of β and σ^2, and maximizing the concentrated Gaussian likelihood of λ numerically gives the unrestricted QMLE $\hat{\lambda}_n$ of λ, which upon substitutions gives the unrestricted QMLEs for β and σ^2 as $\hat{\beta}_n \equiv \tilde{\beta}_n(\hat{\lambda}_n)$ and $\hat{\sigma}_n^2 \equiv \tilde{\sigma}_n^2(\hat{\lambda}_n)$. Although this model is not in the standard form used in Section 2, it can be 'turned' into that form as indicated in the footnote therein. In this case, simply write $u_n = \sigma\Omega_n^{\frac{1}{2}}(\lambda)e_n$, where $\Omega_n^{\frac{1}{2}}(\lambda)$ is the square-root matrix of $\Omega_n(\lambda)$, and $e_n \sim (0, I_n)$ though it is not exactly the same as the e_n in (23) in general. However, as far as bootstrap methods is concerned, all it is important is to be able to get a set of residuals whose EDF consistently estimates the true distribution of $e_{n,i}$.

For this model the null hypothesis of no spatial effect can be either $H_0 : \sigma_\nu^2 = 0$, or $\lambda = \sigma_\nu^2/\sigma^2 = 0$. The alternative hypothesis can only be one-sided as σ_ν^2 cannot be negative, i.e., $H_a : \sigma_\nu^2 > 0$, or $\lambda > 0$. Anselin (2001) derived an LM test based on the assumptions that errors are normally distributed, which can be rewritten in a simpler form

$$\text{LM}_{\text{SEC}} = \frac{n}{\sqrt{K_n}}\frac{\tilde{\varepsilon}_n' H_n \tilde{\varepsilon}_n}{\tilde{\varepsilon}_n'\tilde{\varepsilon}_n}, \tag{24}$$

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where $H_n = W_n W_n' - \frac{1}{n}\text{tr}(W_n W_n')I_n$, $K_n = 2\text{tr}(H_n^2)$, and $\tilde{\varepsilon}_n$ is the vector of OLS residuals. The limiting null distribution of LM_{SEC} is $N(0,1)$ when $\mathcal{F} = \Phi$.

Anselin and Moreno (2003) conducted Monte Carlo experiments to assess the finite sample behavior of Anselin's test, the GMM test of Kelejian and Robinson (1995) and Moran's (1950) I test, and found that none seems to perform satisfactorily in general. They recognized that the LM test for spatial error components of Anselin (2001) is sensitive to distributional misspecifications and the spatial layouts. Yang (2010) provided a robust/standardized LM test, which can be rewritten as:

$$\text{SLM}_{\text{SEC}} = \frac{n}{\sqrt{K_n^\dagger + \tilde{\kappa}_{4n} a_n' a_n}} \frac{\tilde{\varepsilon}_n' H_n^\dagger \tilde{\varepsilon}_n}{\tilde{\varepsilon}_n' \tilde{\varepsilon}_n}, \tag{25}$$

where $H_n^\dagger = W_n W_n' - \frac{1}{n-k}\text{tr}(W_n W_n' M_n)I_n$, $K_n^\dagger = 2\text{tr}(\mathcal{A}_n^2)$, $a_n = \text{diagv}(\mathcal{A}_n)$, $\mathcal{A}_n = M_n H_n^\dagger M_n$, and $\tilde{\kappa}_{4n}$ is the 4th cumulant of $\tilde{e}_n = \tilde{\sigma}_n^{-1}\tilde{\varepsilon}_n$. Yang (2010) showed that $\text{SLM}_{\text{SEC}}|_{H_0} \xrightarrow{D} N(0,1), \forall \mathcal{F}$.

Comparing (24) and (25) with (10) and (11), we see that they possess very similar structure. The major difference is that the diagonal elements of W_n in (10) are zero and as a result the quantity $a_n' a_n$ in (11) is of smaller order than $K_n^\dagger$, but the diagonal elements of $W_n W_n'$ in (24) are not zero and as a result the quantity $a_n' a_n$ in (25) can be of the same order as $K_n^\dagger$ therein. This gives the exact reason on why SLM_{SEC} is robust against the distributional misspecification and why LM_{SEC} is not.

3.3.2 Validity of the bootstrap methods

Note that under H_0, $\tilde{\varepsilon}_n' = M_n \varepsilon_n = \sigma M_n e_n$, and the statistics LM_{SEC} can be written as

$$\text{LM}_{\text{SEC}}|_{H_0} = \frac{n}{\sqrt{K_n}} \frac{e_n' M_n H_n M_n e_n}{e_n' M_n e_n} \tag{26}$$

which shows that $\text{LM}_{\text{SEC}}|_{H_0}$ is invariant of the nuisance parameters, and thus a pivot if $\mathcal{F}$ is known (to be Φ or some other CDF). In this case one can again, as for the SED model, simply use Monte Carlo method to find the finite sample critical values of $\text{LM}_{\text{SEC}}|_{H_0}$ to any level of accuracy. However, when $\mathcal{F}$ is unknown and possibly misspecified, $\text{LM}_{\text{SEC}}|_{H_0}$ is not even an asymptotic pivot as indicated above. Indeed, Lemma A2 leads to $(1+\kappa_4 c_0)^{-\frac{1}{2}}\text{LM}_{\text{SEC}}|_{H_0} \xrightarrow{D} N(0,1)$, $\forall \mathcal{F}$, where $c_0 = \lim_{n\to\infty} K_n^{-1} b_n' b_n$ and $b_n = \text{diagv}(M_n H_n M_n)$.[10] Then, what is the consequence of ignoring this when conducting bootstrap?

The bootstrap DGP that mimics the real world null DGP is again: $Y_n^* = X_n \ddot{\beta}_n + \ddot{\sigma}_n e_n^*$. Based on the bootstrap data (Y_n^*, X_n), compute the OLS estimate of $(\ddot{\beta}_n, \ddot{\sigma}_n)$, the corresponding OLS residuals, and the statistic (24). Some algebra leads to the bootstrap analogue of (26):

$$\text{LM}_{\text{SEC}}^* = \frac{n}{\sqrt{K_n}} \frac{e_n^{*\prime} M_n H_n M_n e_n^*}{e_n^{*\prime} M_n e_n^*}. \tag{27}$$

Similarly, Lemma A2 leads to $(1+\ddot{\kappa}_{4n} c_0)^{-\frac{1}{2}}\text{LM}_{\text{SEC}}^* \xrightarrow{D^*} N(0,1)$. This show that the leading terms in the asymptotic expansions of the finite sample CDF of $\text{LM}_{\text{SEC}}|_{H_0}$ and the bootstrap CDF of LM_{SEC}^* are, respectively, $\Phi(t/\sqrt{1+\kappa_4 c_0})$ and $\Phi(t/\sqrt{1+\ddot{\kappa}_{4n} c_0})$. Thus,

$$\Phi(t/\sqrt{1+\ddot{\kappa}_{4n} c_0}) - \Phi(t/\sqrt{1+\kappa_4 c_0}) = o_p(1), \text{ if } \ddot{\mathcal{F}}_n = \hat{\mathcal{F}}_n;\ O_p(1) \text{ if } \ddot{\mathcal{F}}_n = \tilde{\mathcal{F}}_n.$$

(See Lemma A8 in Appendix A.) This clearly shows that when $\mathcal{F}$ is misspecified the bootstrap method is not able to provide an improved approximation to the finite sample critical values of $\text{LM}_{\text{SEC}}|_{H_0}$

[10] Yang (2010) showed that when the group sizes are fixed and when there exist group size variations, c_0 is strictly positive, showing that in general $\text{LM}_{\text{SEC}}|_{H_0}$ is not an asymptotic pivot unless $\kappa_4 = 0$ or $\mathcal{F} = \Phi$.

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over the asymptotic critical values even if the unrestricted residuals are used, and that the use of the restricted residuals worsens the approximation.

The above arguments lead to the consideration of a standardized/robust LM statistic. Similar algebra as for $\text{LM}_{\text{SEC}}|_{H_0}$ and its bootstrap analogue LM^*_{SEC} gives

$$\text{SLM}_{\text{SEC}}|_{H_0} = \frac{n}{\sqrt{K_n^\dagger + \kappa_4(e_n)a_n'a_n}} \frac{e_n' M_n H_n^\dagger M_n e_n}{e_n' M_n e_n}, \tag{28}$$

where $\kappa_4(e_n)$ is the 4th cumulant of $M_n e_n / \sqrt{e_n' M_n e_n / n}$, and its bootstrap analogue

$$\text{SLM}^*_{\text{SEC}} = \frac{n}{\sqrt{K_n^\dagger + \kappa_4(e_n^*)a_n'a_n}} \frac{e_n^{*\prime} M_n H_n^\dagger M_n e_n^*}{e_n^{*\prime} M_n e_n^*}. \tag{29}$$

Similar to the fact that $\text{SLM}_{\text{SEC}}|_{H_0} \xrightarrow{D} N(0,1)$ for all $\mathcal{F}$, $\text{SLM}^*_{\text{SEC}} \xrightarrow{D^*} N(0,1)$ for both $\hat{\mathcal{F}}_n$ and $\tilde{\mathcal{F}}_n$. The implication of these results is that when bootstrapping the standardized LM test given in (25), using either unrestricted residuals or restricted residuals leads to bootstrap critical values that are correct asymptotically. However, as stated in the following proposition, only the use of unrestricted residuals leads to full asymptotic refinements. As the implementation of the resampling scheme $\texttt{RS}_{uf}$ is more complicated than $\texttt{RS}_{uu}$, it is excluded from this study for saving space.

Proposition 3.3. *Suppose Assumptions S1-S3 hold for Model (23) with* $u_n = \Omega_n^{\frac{1}{2}}(\lambda)\varepsilon_n$. *If (i)* $\hat{\lambda}_n$ *is* $\sqrt{n/h_n}$*-consistent, and (ii)* $|\text{SLM}_{\texttt{SEC}}|_{H_0}| \le U$ *a.e., and* $E(U^4)$ *exists, then the bootstrap methods under the resampling schemes* $\texttt{RS}_{uu}$*, and* $\texttt{RS}_{ru}$ *are valid for* $\text{SLM}_{\texttt{SEC}}$*. The results remain if instead* $u_n = W_n v_n + \varepsilon_n$ *such that the jth sample cumulant of* $\sigma^{-1}\Omega_n^{-\frac{1}{2}}(\lambda)u_n \xrightarrow{p} \kappa_j$, $j = 1, \ldots, 6$.

Proof: Again, we highlight the key arguments here for a quick understanding of the results, and put details in Appendix B (Lemma A8 and Proof of Proposition 3.3 (Cont'd)). We show that the finite sample CDF of $\text{SLM}_{\text{SEC}}|_{H_0}$ admits the following Edgeworth expansion:

$$\begin{aligned} \mathcal{G}_n(t, \mathcal{F}) &= \Phi(t) + n_r^{-\frac{1}{2}} c_1^{-\frac{3}{2}} \phi(t)\, p(t, \mathcal{F}) + O(n_r^{-1}), \text{ with} \\ p(t, \mathcal{F}) &= \{\tfrac{1}{3}(2T_3 - T_1 + 3T_5) + \tfrac{1}{6}[\kappa_6 T_1 + 2\kappa_4(6T_1 + T_3) + \kappa_3^2(10T_1 + T_4) + 2\kappa_3 T_2]\}(1 - t^2), \end{aligned} \tag{30}$$

where $c_1 = \lim_{n\to\infty} n_r^{-1}(K_n^\dagger + \kappa_4 a_n' a_n)$, $K_n^\dagger$ and a_n are defined in (25), $T_i = \lim_{n\to\infty} n_r^{-1} T_{i,n}$, and $T_{i,n}$ are defined in the detailed proof in Appendix B. Similarly, the bootstrap CDF of $\text{SLM}^*_{\texttt{SEC}}$ admits the following asymptotic expansion:

$$\begin{aligned} \mathcal{G}_n(t, \ddot{\mathcal{F}}_n) &= \Phi(t) + n_r^{-\frac{1}{2}} c_1^{-\frac{3}{2}} \phi(t)\, p(t, \ddot{\mathcal{F}}_n) + O_p(n_r^{-1}), \text{ with} \\ p(t, \ddot{\mathcal{F}}_n) &= \{\tfrac{1}{3}(2T_3 - T_1 + 3T_5) + \tfrac{1}{6}[\ddot{\kappa}_{6n} T_1 + 2\ddot{\kappa}_{4n}(6T_1 + T_3) + \ddot{\kappa}_{3n}^2(10T_1 + T_4) + 2\ddot{\kappa}_{3n} T_2]\}(1 - t^2). \end{aligned} \tag{31}$$

It is thus clear from (30) and (31) that $\mathcal{G}_n(t, \hat{\mathcal{F}}_n) - \mathcal{G}_n(t, \mathcal{F}) = o_p(n_r^{-\frac{1}{2}})$ due to the consistency of $\hat{\mathcal{F}}_n$, but $\mathcal{G}_n(t, \tilde{\mathcal{F}}_n) - \mathcal{G}_n(t, \mathcal{F}) = O_p(n_r^{-\frac{1}{2}})$ due to the inconsistency of $\tilde{\mathcal{F}}_n$.

Remark 3.5. Under $\text{SLM}_{\texttt{SEC}}$, use of $\hat{\mathcal{F}}_n$ leads to bootstrap critical values in error of order $o_p(n_r^{-\frac{1}{2}})$, whereas use of $\tilde{\mathcal{F}}_n$ leads to bootstrap critical values in error of order $O_p(n_r^{-\frac{1}{2}})$. This means that at least in theory the bootstrap critical values based on the restricted residuals offer no improvement over the asymptotic ones. However, a closer examination on the Edgeworth expansion shows that the bootstrap based on $\tilde{\mathcal{F}}_n$ can still do a better job as the main second-order effect, term involving $\frac{1}{3}(2T_3 - T_1 + 3T_5)$, is captured by the bootstrap. Our Monte Carlo results given below confirm this point.

Remark 3.6. The point that a denser weight matrix makes the finite sample null distribution of the test statistic deviate more from the limiting distribution is once again demonstrated by the first three cumulants of $\text{LM}_{\texttt{SEC}}|_{H_0}$, which are derived as those of $\text{SLM}_{\texttt{SEC}}|_{H_0}$ given in Appendix B.

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3.3.3 Monte Carlo results

The finite sample performance of $\text{LM}_{\texttt{SEC}}$ and $\text{SLM}_{\texttt{SEC}}$ for testing $H_0 : \lambda = 0$ vs $H_a : \lambda > 0$, when referring to the asymptotic critical values and the bootstrap critical values under various resampling schemes, are investigated in terms of the accuracy and stability of the bootstrap critical values with respect to the true value of λ, and the size and power of the tests. The Monte Carlo experiments are carried out based on the following data generating process:

$$Y_n = \beta_0 1_n + X_{n1}\beta_1 + X_{n2}\beta_2 + W_n v_n + \varepsilon_n$$

where $\{v_{n,i}\}$ are iid draws from $\sqrt{0.6}t_5$, and the methods for generating W_n, X_n and ε_n are described in Appendix C. The regressors are treated as fixed in the experiments. The parameter values are set at $\beta = \{5, 1, 1\}'$ and $\sigma = 1$, and sample sizes used are $n = (54, 108, 216, 513)$. All results reported below are based on $M = 2,000$ Monte Carlo samples, and $B = 699$ bootstrap samples for each Monte Carlo sample generated. The bootstrap critical values are bench-marked against the Monte Carlo (`MC`) critical values obtained based on $M = 50,000$ Monte Carlo samples under H_0.

Similar to the LM tests for SED model considered earlier, the LM tests for SEC model are also invariant of the nuisance parameters, thus the bootstrap methods with $\texttt{RS}_{ur}$ and $\texttt{RS}_{ru}$ are omitted as the former produces identical results as $\texttt{RS}_{rr}$ and the latter produces identical results as $\texttt{RS}_{uu}$. We also omit the $\texttt{RS}_{uf}$ method in this study as it requires the derivation of the test statistics for a general value of λ, and concentrate on $\texttt{RS}_{rr}$ and $\texttt{RS}_{uu}$.

Bootstrap critical values. We first report in Table 3.3a the bootstrap critical values for $\text{LM}_{\texttt{SEC}}$ and $\text{SLM}_{\texttt{SEC}}$. As discussed above, $\text{LM}_{\texttt{SEC}}$ is sensitive to the distributional misspecification, thus it is expected to produce bootstrap critical values that vary with λ when $\tilde{\mathcal{F}}_n$ is used, even if $\mathcal{F}$ is $N(0, 1)$. Indeed this is observed from the results under $\texttt{RS}_{rr}$ and `Normal Error` though the change is not big. In contrast, the bootstrap critical values based on $\hat{\mathcal{F}}_n$ with normal error are very stable.

When error distribution is not normal and is unknown, $\text{LM}_{\texttt{SEC}}|_{H_0}$ is no longer a pivot, and not even an asymptotic pivot as both its finite sample and limiting distributions depend on $\mathcal{F}$. It is thus expected that bootstrap critical values based on $\text{LM}_{\texttt{SEC}}$ would vary more with λ whether $\texttt{RS}_{rr}$ or $\texttt{RS}_{uu}$ is followed. Again, this is very much true and in fact the bootstrap critical values change (drop) much more significantly as λ increases. In contrast, if we bootstrap $\text{SLM}_{\texttt{SEC}}$, the bootstrap critical values become much more stable. In both cases, the method with $\texttt{RS}_{uu}$ performs better.

`<< Insert Table 3.3a Here >>`

Rejection Frequencies. Partial results corresponding to the rejection frequencies are reported in Table 3.3b. From the results reported and unreported, we observe the following.

1. When errors are normal, all other tests improve upon the $\text{LM}_{\texttt{SEC}}$ test referring to the asymptotic critical values, in particular when sample size is small;
2. When errors are nonnormal, $\text{LM}_{\texttt{SEC}}$ referring to the asymptotic critical values failed, but very interestingly $\text{LM}_{\texttt{SEC}}$ referring to the bootstrap critical values performs reasonably well although a clear sign of deterioration is observed for the cases of nonnormal errors;
3. $\text{SLM}_{\texttt{SEC}}$ performs well whether with asymptotic critical values or bootstrap critical values, but bootstrap shows clear improvements in particular when error distributions are skewed.

`<< Insert Table 3.3b Here >>`

We end the section with some important remarks. The bootstrap LM test seems to offer higher power than does the bootstrap SLM test. However, as cautioned earlier, such a higher power is built

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upon the 'hidden' lower critical values, thus is unachievable as in practice one does not know whether or not the null is true. Once again, we stress on that the performance of a bootstrap test should be judged based on whether it can offer critical values which are stable with respect to the change in the value of the parameters of interest. The Monte Carlo results reported in Table 3.3 correspond to a spatial layout (group interaction with fixed group sizes) that may not fully satisfy the condition stated in Proposition 3.3 (see the proof of Lemma A.8), which is why the bootstrap critical values are not as stable as those in the previous two models. However, the the results under group interaction with growing group sizes (unreported for brevity) show much more stable bootstrap critical values.

As discussed above, the SEC model is not the standard model considered in this paper. Proposition 3.3 clearly reveals the complications caused by the existence of error components $u_n = W_n v_n + \sigma e_n$: $\hat{u}_n = Y_n - X_n\hat{\beta}_n$ cannot be decomposed into $\hat{v}_n$ and $\hat{e}_n$ to give a consistent $\hat{\mathcal{F}}_n$ directly based $\hat{e}_n$, unless H_0 is imposed or v_n and ε_n are normal. In contrast, the same model but with $u_n = \sigma\Omega_n^{\frac{1}{2}}(\lambda)e_n$ leads to $\hat{e}_n$ and hence a consistent $\hat{\mathcal{F}}_n$ directly. This reveals an interesting issue: *bootstrap in models with error components*, such as the SEC model and the panel error components models with short panels, which merits a further study. See Lemma A8 and its proof for more detailed discussions.

4 Bootstrap LM Tests for More General Spatial Models

Section 3 proves/disproves the validity of the five bootstrap methods introduced in Section 2 in the context of three popular spatial regression models, and concludes that only the methods using unrestricted estimates of nuisance parameters and unrestricted residuals are generally valid. In this section, we further illustrate these methods using a more general model: the linear regression with both SLD and SED, also referred to as the SARAR model in the literature:

$$Y_n = \lambda W_{1n} Y_n + X_n\beta + u_n, \ \ u_n = \rho W_{2n} u_n + \varepsilon_n, \ \ \varepsilon_n = \sigma e_n, \tag{32}$$

where all quantities are defined as in (9) and (16). The spatial weight matrices W_{1n} and W_{2n} can be the same. Clearly, (32) has the form of the general model given in (1): $\sigma^{-1}B_n(\rho)[A_n(\lambda)Y_n - X_n\beta] = e_n$, where $A_n(\lambda) = I_n - \lambda W_{1n}$ and $B_n(\lambda) = I_n - \rho W_{2n}$. QMLEs (restricted or unrestricted) of model parameters can be obtained in a similar manner.

Several interesting tests arise from this model: (*i*) joint or marginal LM tests, (*ii*) LM tests of spatial dependence under local misspecification, and (*iii*) LM tests of spatial dependence under unknown heteroskedasticity. We apply the proposed bootstrap methods to each of these tests. Monte Carlo results show strong support of the main point of the paper: in bootstrapping the finite sample distribution of an LM test, the unrestricted estimates and residuals should be used in setting up the bootstrap DGP.

4.1 Bootstrap LM tests for SARAR effects

We are interested in testing three hypothesis: $H_0^a : \lambda = 0, \rho = 0$, $H_0^b : \rho = 0$ allowing the presence of λ, and $H_0^c : \lambda = 0$ allowing the presence of ρ. The corresponding LM tests can be found in Anselin et al. (1996) and can be written as (assuming $W_{1n} = W_{2n} = W_n$): for testing H_0^a,

$$\text{LM}_{\texttt{SARAR}} = \frac{(\tilde{\varepsilon}_n' W_n Y_n - \tilde{\varepsilon}_n' W_n \tilde{\varepsilon}_n)^2}{\tilde{\sigma}_n^4 \tilde{\eta}_n' M_n \tilde{\eta}_n} + \frac{(\tilde{\varepsilon}_n' W_n \tilde{\varepsilon}_n)^2}{\tilde{\sigma}_n^4 K_n}, \tag{33}$$

where all quantities are defined in (10) and (17); for testing H_0^b,

$$\text{LM}_{\texttt{SED|SLD}} = \frac{\tilde{\varepsilon}_n' W_n \tilde{\varepsilon}_n}{\tilde{\sigma}_n^2 [K_n - \tilde{S}_{1n}^2/(\tilde{\eta}_n' M_n \tilde{\eta}_n + \tilde{S}_{2n})]^{1/2}}, \tag{34}$$

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where $\tilde{S}_{1n} = \text{tr}[(W_n + W_n')\tilde{G}_n]$, $\tilde{S}_{2n} = \text{tr}[(\tilde{G}_n^\circ + \tilde{G}_n^{\circ\prime})\tilde{G}_n^\circ]$, $\tilde{G}_n = W_n A_n^{-1}(\tilde{\lambda}_n)$, and $\tilde{G}_n^\circ = \tilde{G}_n - \frac{1}{n}\text{tr}(\tilde{G}_n)I_n$; and for testing H_0^c,

$$\text{LM}_{\texttt{SLD|SED}} = \frac{\tilde{\varepsilon}_n' \tilde{B}_n W_n Y_n}{\tilde{\sigma}_n^2[\tilde{S}_{3n} + \tilde{\eta}_n' \tilde{B}_n' \tilde{B}_n \tilde{\eta}_n + \tilde{h}_n' \tilde{J}_n^{-1} \tilde{h}_n]^{1/2}}, \tag{35}$$

where $\tilde{S}_{3n} = \text{tr}(W_n^2 + \tilde{Q}_n' \tilde{B}_n' \tilde{B}_n \tilde{Q}_n)$, $\tilde{h}_n = \{\tilde{\sigma}_n^{-1} X_n' \tilde{B}_n' \tilde{B}_n \tilde{\eta}_n,\ 0,\ \text{tr}((\tilde{Q}_n' \tilde{B}_n + W_n)\tilde{Q}_n)\}'$, $\tilde{J}_n = J_n(\tilde{\theta}_n)$ given in (B-1) of Appendix B, $\tilde{B}_n = \tilde{B}_n(\tilde{\rho}_n)$, and $\tilde{Q}_n = W_n \tilde{B}_n^{-1}$. Recall that $\tilde{\theta}_n$ and $\tilde{\varepsilon}_n$ denote generically the restricted QML estimates of the nuisance parameters θ and the errors ε_n under the null hypothesis.

The bootstrap methods can be implemented in the same manner as in Section 3. The bootstrap DGPs that mimic the real world null DGPs are, $Y_n^* = X_n \ddot{\beta}_n + \ddot{\sigma}_n e_n^*$, $Y_n^* = A_n^{-1}(\ddot{\lambda}_n)(X_n \ddot{\beta}_n + \ddot{\sigma}_n e_n^*)$, and $Y_n^* = X_n \ddot{\beta}_n + \ddot{\sigma}_n B_n^{-1}(\ddot{\rho}_n) e_n^*$, respectively, for testing H_0^a, H_0^b and H_0^c, where e_n^* are the iid draws from the EDF $\ddot{\mathcal{F}}$ of $\hat{e}_n$ or $\tilde{e}_n$, standardized to have mean zero and standard deviation one. For example, to bootstrap the α-quantile of $\text{LM}_{\texttt{SLD|SED}}|_{H_0^c}$, based on the unrestricted estimates/residuals,

(a) Compute the unrestricted QMLEs $(\hat{\beta}_n, \hat{\sigma}_n^2, \hat{\lambda}_n, \hat{\rho}_n)$ based on Model (32);
(b) Compute $\hat{e}_n = \hat{\sigma}_n^{-1} B_n(\hat{\rho}_n)[A_n(\hat{\lambda}_n)Y_n - X_n \hat{\beta}_n]$, and standardize, to give $\hat{\mathcal{F}}_n$;
(c) Draw a bootstrap sample e_n^* from $\hat{\mathcal{F}}_n$, and compute $Y_n^* = X_n \hat{\beta}_n + \hat{\sigma}_n B_n^{-1}(\hat{\rho}_n) e_n^*$;
(d) Estimate the null model $Y_n = X_n \beta + u_n, u_n = \rho W_n u_n + \varepsilon_n$, based on the bootstrap data (Y_n^*, X_n, W_n), and then compute a bootstrap value $\text{LM}^b_{\texttt{SLD|SED}}$ of $\text{LM}_{\texttt{SLD|SED}}$;
(e) Repeat (*c*) and (*d*) B times to obtain the EDF of $\{\text{LM}^b_{\texttt{SLD|SED}}\}_{b=1}^B$ and its α-quantile. The latter gives a bootstrap estimate of the true finite sample α-quantile of $\text{LM}_{\texttt{SLD|SED}}|_{H_0^c}$.

We state without proofs of the following results: *(i) the limiting null distributions of the three test statistics are χ_2^2 for $\text{LM}_{\texttt{SARAR}}$, and $N(0,1)$ for the other two, for a general $\mathcal{F}$ satisfying Assumption S1; and (ii) the bootstrap methods with $\texttt{RS}_{uu}$ and $\texttt{RS}_{uf}$ resampling schemes are generally valid.*

Monte Carlo results. Extensive Monte Carlo experiments are performed for assessing the finite sample performance of the bootstrap methods, based on a DGP that adds a spatial lag term onto the DGP used in Section 3.1.3. Due to space limitation, only partial Monte Carlo results are reported in Table 4.1. The results lead to a general conclusion: the bootstrap method with $\texttt{RS}_{uu}$ performs very well in general. In contrast, the bootstrap methods with either restricted estimates or restricted residuals or both may give critical values quite different from the 'true' ones (the ones under H_0), thus leading to unreliable power (either unduly too low or unduly too high), in particular in the case of H_0^a.

`<< Insert Tables 4.1a, 4.1b and 4.1c Here >>`

4.2 Bootstrap spatial LM tests under local misspecification

Anselin et al. (1996), following Bera and Yoon (1993), obtained a modified LM test for testing $H_0^a : \rho = 0$, robust against the presence of local misspecification involving a spatial lag with $\lambda = \delta/\sqrt{n}$:

$$\text{LM}_{\texttt{SED}|\lambda} = \frac{\tilde{\varepsilon}_n' W_n \tilde{\varepsilon}_n - \tilde{H}_n \tilde{\varepsilon}_n' W_n Y_n}{\tilde{\sigma}_n^2[K_n(1 - \tilde{H}_n)]^{1/2}}, \tag{36}$$

and a modified LM test for testing $H_0^b : \lambda = 0$, robust against local misspecification involving a spatial error process with $\rho = \delta/\sqrt{n}$:

$$\text{LM}_{\texttt{SLD}|\rho} = \frac{\tilde{\varepsilon}_n' W_n \tilde{\varepsilon}_n - \tilde{\varepsilon}_n' W_n Y_n}{\tilde{\sigma}_n^2(\tilde{\eta}_n' M_n \tilde{\eta}_n)^{1/2}}, \tag{37}$$

where $\tilde{H}_n = K_n(\tilde{\eta}_n' M_n \tilde{\eta}_n + K_n)^{-1}$, δ is a constant, and all other quantities are defined in (10) and (17).

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We again state the following conclusions without formal proofs: *Both tests statistics have limiting null distributions being standard normal, for a general $\mathcal{F}$ satisfying Assumption S1; (ii) the bootstrap methods with* $\texttt{RS}_{uu}$ *and* $\texttt{RS}_{uf}$ *resampling schemes are generally valid.*

Monte Carlo results. The same DGP as in Section 4.1 is followed. The local misspecification parameter is taken as $\lambda = 0.1/\sqrt{n}$ for (36) and $\rho = 0$ for (37). Partial Monte Carlo results are give in Table 4.2. The results show that the bootstrap method under $\texttt{RS}_{uu}$ is the most reliable one, leading to significant improvements on the finite sample performance of the LM tests. Comparing the results in Table 4.2b with the corresponding results in Table 3.2b, we see that $\text{LM}_{\texttt{SLD}|\rho}$ is less powerful than $\text{LM}_{\texttt{SLD}}$, consistent with the observations made by Anselin et al. (1996). Detailed comparisons of $(\text{LM}_{\texttt{SED}}, \text{LM}_{\texttt{SED|SLD}}, \text{LM}_{\texttt{SED}|\lambda})$; and of $(\text{LM}_{\texttt{SLD}}, \text{LM}_{\texttt{SLD|SED}}, \text{LM}_{\texttt{SLD}|\rho})$ are interesting. Formal justifications on the validity of the bootstrap methods applied to (36) and (37) need to be given. These studies are clearly beyond the scope of this paper, and will be pursued in a future research.

`<< Insert Tables 4.2a and 4.2b Here >>`

4.3 Bootstrap spatial LM tests with unknown heteroskedasticity

When the errors in the spatial models are heteroskedastic, none of the tests considered above are generally valid. Born and Breitung (2011) proposed OPG (outer product of gradients) variants of the three LM tests given in (10), (17) and (33), which are shown to be robust against unknown heteroskedasticity and non-normality. The three tests can be written more compactly as:

$$\text{LM}^{\texttt{OPG}}_{\texttt{SED}} = \frac{\tilde{\varepsilon}_n' W_{2n}\tilde{\varepsilon}_n}{(\tilde{\varepsilon}_n^{2\,\prime}\, \tilde{\xi}_{2n}^2)^{\frac{1}{2}}}, \tag{38}$$

$$\text{LM}^{\texttt{OPG}}_{\texttt{SLD}} = \frac{\tilde{\varepsilon}_n' W_{1n} Y_n}{(\tilde{\varepsilon}_n^{2\,\prime}\, \tilde{\xi}_{1n}^2)^{\frac{1}{2}}}, \text{ and} \tag{39}$$

$$\text{LM}^{\texttt{OPG}}_{\texttt{SARAR}} = \begin{pmatrix} \tilde{\varepsilon}_n' W_{1n} Y_n \\ \tilde{\varepsilon}_n' W_{2n}\tilde{\varepsilon}_n \end{pmatrix}' \begin{pmatrix} \tilde{\varepsilon}_n^{2\,\prime}\tilde{\xi}_{1n}^2 & \tilde{\varepsilon}_n^{2\,\prime}(\tilde{\xi}_{1n}\odot\tilde{\xi}_{2n}) \\ \tilde{\varepsilon}_n^{2\,\prime}(\tilde{\xi}_{1n}\odot\tilde{\xi}_{2n}) & \tilde{\varepsilon}_n^{2\,\prime}\tilde{\xi}_{2n}^2 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\varepsilon}_n' W_{1n} Y_n \\ \tilde{\varepsilon}_n' W_{2n}\tilde{\varepsilon}_n \end{pmatrix}, \tag{40}$$

where $\tilde{\xi}_{1n} = (W_{1n}^{u\prime} + W_{1n}^{l})\tilde{\varepsilon}_n + M_n\tilde{\eta}_n$, $\tilde{\xi}_{2n} = (W_{2n}^{u\prime} + W_{2n}^{l})\tilde{\varepsilon}_n$, W_{rn}^u and W_{rn}^l are the upper and lower triangular matrices of $W_{rn}, r = 1, 2$, '$\odot$' denotes the Hadamard product, and $a^2 = a\odot a$ for a vector a.

Like the original tests, the OPG variants do not take into account the estimation of β, and hence may suffer from the problems of size distortion due mainly to the lack of centering and rescaling (Baltagi and Yang, 2013). It is interesting to see how the bootstrap can help in this regard. The three tests have the same null DGP: $Y_n = X_n\beta + \sigma e_n$ where the errors $e_{n,i}$ are independent but heteroskedastic. As indicated by Davidson and Flachaire (2008), heteroskedasticity of unknown form cannot be mimicked in the bootstrap distribution. The wild bootstrap gets round this problem by using a DGP:

$$Y_n^* = X_n\ddot{\beta}_n + \ddot{\sigma}_n e_n^*, \;\; e_{n,i}^* = f_i(\ddot{e}_{n,i})v_i, \tag{41}$$

where f_i is a transformation, and the v_i are mutually independent draws, completely independent of original data, from an auxiliary distribution with mean 0 and variance 1. We follow Davidson and Flachaire (2008) and consider an identity function for f_i and a two points (-1,1) distribution with equal probability for v_i. More detailed discussions on this can be found in Godfrey (2009, Ch. 5).

Monte Carlo results. The same set of DGPs as in Sections 3.1, 3.2 and 4.1 are used. Error variances are made proportional to the group sizes. Partial results are reported in Table 4.3., from which

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we see (i) the OPG variants of LM tests can have large finite sample size distortion when referred to the asymptotic critical values, which are largely removed when referred to the bootstrap critical values; (ii) bootstrap critical values show noticeable variations for all four resampling schemes considered. This is because both restricted and unrestricted estimates used in the bootstrap DGP ignore the unknown heteroskedasticity, and hence are inconsistent in general (Lin and Lee, 2010).

The above observations are in fact consistent with our theoretical findings: use of consistent (fully unrestricted) estimates leads to full asymptotic refinements, whereas use of inconsistent (restricted somehow) estimates may still lead to partial asymptotic refinements, provided the underlining test statistic is an asymptotic pivot. The latter finding is also interesting as in certain LM testing situations fully unrestricted (or generally consistent) estimates may not be available, such as QML estimation with unknown heteroskedasticity. The robust GMM estimators of Lin and Lee (2010) or Kelejian and Prucha (2010) may be used instead, and formal justification on the validity of the bootstrap method described above should be given. However, these studies are clearly beyond the scope of this paper, and will be pursued in a future research.

<< Insert Tables 4.3a, 4.3b and 4.3c Here >>

5 Conclusions and Discussions

In bootstrapping the critical values of an LM test, one faces two important issues: one is the choice of the type of estimates of nuisance parameters to be used as parameters in the bootstrap data generating process, and the other is the choice of the type of residuals to be used to construct the bootstrap error distribution. We argue in general and show through three popular spatial regression models that the choice that is correct in general is the one which uses the unrestricted estimates and the unrestricted residuals. However, if the test statistic is invariant of the nuisance parameters or the restricted estimates of the nuisance parameters are consistent in general, the restricted estimates can be used in place of the unrestricted estimates; if the test statistic at the null is robust against the distributional misspecification, then use of restricted residuals leads to full asymptotic refinements if the error distribution is symmetric, otherwise it leads to partial asymptotic refinements.

It is emphasized that comparison on the performance of various bootstrap methods should not be made based on the size and power of the tests, instead it should be made based on the stability of the bootstrap critical values with respect to the change in the value of the parameters of interest. The main reason is that in reality, one does not know whether or not the null hypothesis is true, thus the size of the bootstrap tests based on restricted estimates and/or residuals may not be achievable if the null hypothesis is false, and the resulting power would be unreliable. The power in this situation tends to be higher (than that based on unrestricted resampling) if the underlining bootstrap critical values are smaller than the true ones, or lower if the underlining bootstrap critical values are larger. Furthermore, the evaluation of the performance of various bootstrap methods should also be based on how close the bootstrap critical values are to the Monte Carlo critical values.

While the theories and Monte Carlo results presented the paper clearly suggest that the bootstrap with $\texttt{RS}_{uu}$ scheme be followed in practice for its ability to achieve full asymptotic refinements on the finite sample critical values of LM tests and for its simplicity when compared with $\texttt{RS}_{uf}$,[11] we do not

[11] The computational cost of the five resampling schemes is the same in the process of bootstrapping the test statistics. Except $\texttt{RS}_{rr}$, all other four require the estimation of the parameter(s) being tested based on the original data. The $\texttt{RS}_{uf}$ is equivalent to $\texttt{RS}_{uu}$, at least in theory, but it involves more complicated expressions of LM statistics.

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rule out the other three schemes as they may be able to achieve partial asymptotic refinements for cases where the fully unrestricted estimates are not available, such as the LM tests of spatial dependence under unknown heteroskedasticity considered in Section 4.3.

With the general principles laid out in this paper, it would be interesting to proceed to study the properties of the bootstrap LM tests discussed in Section 4. While the validity of the bootstrap methods applied to the LM tests in Sections 4.1 and 4.2 can largely be inferred from the results presented in Section 3, formal theoretical justifications needs to be given. The LM tests in Section 4.3 deviate from the main set up of the paper, traditional bootstrap resampling methods fail, but the *wild* bootstrap methods are shown to be very promising. Hence, further theoretical and empirical investigations would be highly desirable. Nonetheless, the results presented in this section are very supportive to the general theoretical findings of this paper, and encouraging for further research.

Appendix A: Some Fundamental Results

Following lemmas are essential for the theoretical discussions in Sections 2 and 3.

Lemma A1: (Kelejian and Prucha, 2001; Lee, 2004a) *Let A_n and B_n be $n \times n$ matrices, c_n be an $n \times 1$ vector, $a_n = \text{diagv}(A_n)$ and $b_n = \text{diagv}(B_n)$. Let ε_n be an $n \times 1$ vector of iid elements with mean zero, variance σ^2, and jth cumulant $\kappa_j, j = 3, 4$. Define $P_n = \varepsilon_n' A_n \varepsilon_n + c_n' \varepsilon_n$ and $Q_n = \varepsilon_n' B_n \varepsilon_n$. Then,*

(i) $\text{E}(P_n) = \sigma^2 \text{tr}(A_n)$, *and* $\text{E}(Q_n) = \sigma^2 \text{tr}(B_n)$,

(ii) $\text{Var}(P_n) = \sigma^4 \text{tr}(A_n' A_n + A_n^2) + \kappa_4 a_n' a_n + \sigma^2 c_n' c_n + 2\kappa_3 a_n' c_n$,

(iii) $\text{Var}(Q_n) = \sigma^4 \text{tr}(B_n' B_n + B_n^2) + \kappa_4 b_n' b_n$,

(iv) $\text{Cov}(P_n, Q_n) = \sigma^4 \text{tr}[(A_n' + A_n) B_n] + \kappa_4 a_n' b_n + \kappa_3 b_n' c_n$.

Lemma A2: (CLT for Linear-Quadratic Forms, Kelejian and Prucha, 2001) *Let A_n, a_n, c_n and ε_n be defined in Lemma A1. Assume (i) A_n is bounded uniformly in row and column sums, (ii) $n^{-1} \sum_{i=1}^n |c_{n,i}^{2+\eta_1}| < \infty, \eta_1 > 0$, and (iii) $E|\varepsilon_{n,i}^{4+\eta_2}| < \infty$, $\eta_2 > 0$. Then,*

$$\frac{\varepsilon_n' A_n \varepsilon_n + c_n' \varepsilon_n - \sigma^2 \text{tr}(A_n)}{\{\sigma^4 \text{tr}(A_n' A_n + A_n^2) + \kappa_4 a_n' a_n + \sigma^2 c_n' c_n + 2\kappa_3 a_n' c_n\}^{\frac{1}{2}}} \xrightarrow{D} N(0, 1).$$

Lemma A3: *Let $P_n = \varepsilon_n' A_n \varepsilon_n + c_n' \varepsilon_n$ be defined in Lemma A1. Let $A_n = \{a_{ij}\}$, $c_n = \{c_i\}$ and $\varepsilon_n = \{\varepsilon_i\}$ where ε_i has cumulants $\kappa_j, j = 1 \cdots, 6$, $\kappa_1 = 0$, and $\kappa_2 = \sigma^2$. Then, we have,*

$$\begin{aligned} \text{E}[(P_n - \text{E}P_n)^3] = \;& \kappa_6 T_{1n} + 3\kappa_5 S_{1n} + \kappa_4 (12\sigma^2 T_{1n} + 2\sigma^2 T_{3n} + 3S_{2n}) \\ & + \kappa_3^2 (10 T_{1n} + T_{4n}) + \kappa_3 (24\sigma^2 S_{1n} + 2\sigma^2 S_{5n} + S_{3n} + 2\sigma^2 T_{2n}) \\ & + 2\sigma^6 (2T_{3n} - T_{1n} + 3T_{5n}) + 2\sigma^4 (3S_{2n} + S_{4n}), \end{aligned}$$

where $T_{1n} = \sum_{i=1}^n a_{ii}^3$, $T_{2n} = \sum_{i=1}^n a_{ii} d_{1i}$, $T_{3n} = \sum_{i=1}^n a_{ii} d_{2i}$, $T_{4n} = \sum_{i=1}^n \sum_{j=1}^{i-1} \bar{a}_{ij}^3$, $\bar{a}_{ij} = a_{ij} + a_{ji}$, $T_{5n} = \sum_{i=1}^n \sum_{j=1}^{i-1} \sum_{k=1}^{j-1} \bar{a}_{ij} \bar{a}_{ik} \bar{a}_{jk}$; $S_{1n} = \sum_{i=1}^n a_{ii}^2 c_i$, $S_{2n} = \sum_{i=1}^n a_{ii} c_i^2$, $S_{3n} = \sum_{i=1}^n c_i^3$, $S_{4n} = \sum_{i=1}^n c_i d_{1i}$, $S_{5n} = \sum_{i=1}^n c_i d_{2i}$, $d_{1i} = \sum_{j=1}^{i-1} \bar{a}_{ij}$, and $d_{2i} = \sum_{j=1}^{i-1} \bar{a}_{ij}^2$.

Proof: Decompose $A_n = A_n^u + A_n^\ell + A_n^d$, the sum of upper triangular, lower triangular, and diagonal matrices, and define $\zeta_n = (A_n^{u\prime} + A_n^\ell)\varepsilon_n$. Then, $P_n - \text{E}P_n = \sum_{i=1}^n u_i + \sum_{i=1}^n v_i$ where $u_i = \varepsilon_i \zeta_i$ and $v_i = a_{ii}(\varepsilon_i^2 - \sigma^2) + c_i \varepsilon_i$. Taking use of the facts that $u_i's$ are uncorrelated due to the independence between ε_i and ζ_i, $v_i's$ are independent, and u_n and v_n are uncorrelated, the rest of the proof is straightforward though tedious.

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Lemma A4: (Lee, 2004b, Lemma A.8) *Let $\{A_n\}$ be a sequence of $n \times n$ matrices of which the elements $\{a_{n,ij}\}$ are $O(h_n^{-1})$ uniformly in all i and j. Let $\{B_n\}$ be a sequence of conformable $n \times n$ matrices, uniformly bounded in column sums, or uniformly bounded in row sums. Then,*

(i) for the former case, the elements of A_nB_n have uniform order $O(h_n^{-1})$,
(ii) for the latter case, the elements of B_nA_n have uniform order $O(h_n^{-1})$, and
(iii) for both cases, $\mathrm{tr}(A_nB_n) = \mathrm{tr}(B_nA_n) = O(nh_n^{-1})$.

Lemma A5: (Lee, 2004b, Lemma A.9) *Let X_n be defined at the beginning of Section 3 and satisfy Assumption S2. Let $M_n = I_n - X_n(X_nX_n')^{-1}X_n'$, and A_n be an $n \times n$ matrix uniformly bounded in both row and column sums. Then, $\mathrm{tr}(M_nA_n) = \mathrm{tr}(A_n) + O(1)$.*

Lemma A6: *For X_n and W_n defined at the beginning of Section 3, satisfying, respectively, Assumption S2 and Assumption S3 therein, let $M_n = I_n - X_n(X_nX_n')^{-1}X_n'$, $A_n = M_nW_n$ or $M_nW_nM_n$, and write $A_n = \{a_{ij}\}$. Then, we have,*

(i) $\sum_{i=1}^n a_{ii}^r = O(n^{-(r-1)}), r = 1, 2, 3, \cdots,$
(ii) $\sum_{i=1}^n \sum_{j=1}^n a_{ii}a_{ij}a_{ji} = O(h_n^{-1}),$
(iii) $\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ij}a_{ik}a_{jk} = O(nh_n^{-1}).$

If W_n is replaced by a general $n \times n$ matrix $\mathcal{W}_n$, which shares all the properties of W_n except that the diagonal elements are not zero but rather $O(h_n^{-1})$ uniformly. Then, we have,

(iv) $\sum_{i=1}^n a_{ii} = O(nh_n^{-1})$; $\sum_{i=1}^n a_{ii}^r = O(nh_n^{-r}), r = 2, 3,$
(v) $\sum_{i=1}^n \sum_{j=1}^n a_{ii}a_{ij}a_{ji} = O(nh_n^{-2}),$
(vi) $\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ij}a_{ik}a_{jk} = O(nh_n^{-1}).$

Proof: Following the arguments in the proof of Lemma A.9 in Lee (2004b).

Lemma A7: (Hall, 1992, p.46-48) *Let T_n denote a statistic with a limiting standard normal distribution, and $\boldsymbol{\kappa}_{j,n}$ be the jth cumulant of T_n. If $\boldsymbol{\kappa}_{4,n}$ exists, and $\boldsymbol{\kappa}_{j,n}$ can be expanded as a power series in n^{-1}: $\boldsymbol{\kappa}_{j,n} = n^{-\frac{(j-2)}{2}}(k_{j,1} + n^{-1}k_{j,2} + n^{-2}k_{j,3} + \cdots)$, $j = 1, 2, 3$, where $k_{1,1} = 0$ and $k_{2,1} = 1$, then,*

$$P(T_n \le t) = \Phi(t) + n^{-\frac{1}{2}}\phi(t)\; p(t) + O(n^{-1}), \tag{A-1}$$

where $p(t) = -k_{1,2} + \frac{1}{6}k_{3,1}(1 - t^2)$, and ϕ and Φ are, respectively, the pdf and CDF of $N(0, 1)$.

Note: From the expansion for $\boldsymbol{\kappa}_{j,n}$, we see $k_{1,2} = \lim_{n\to\infty} n^{\frac{1}{2}}\boldsymbol{\kappa}_{1,n}$, and $k_{3,1} = \lim_{n\to\infty} n^{\frac{1}{2}}\boldsymbol{\kappa}_{3,n}$. That $k_{1,1} = 0$ and $k_{2,1} = 1$ correspond to $\boldsymbol{\kappa}_{1,n} = \mathrm{E}(T_n) \to 0$ and $\boldsymbol{\kappa}_{2,n} = \mathrm{Var}(T_n) \to 1$.

Lemma A8: *For models specified by (9), (16) and (23) with $u_n = \Omega_n^{\frac{1}{2}}(\lambda)\varepsilon_n$, assume* (*a*) *Assumptions S1-S3 hold,* (*b*) *the unrestricted QMLEs of the parameters that the tests concern are $\sqrt{n_r}$-consistent,*[12] *and* (*c*) *the matrices $B_n^{-1}(\rho)$, $A_n^{-1}(\lambda)$ and $\Omega_n^{-\frac{1}{2}}(\lambda)$ defined therein are uniformly bounded in both row and column sums. Then,* (*i*) $\hat{\kappa}_{3n} = \kappa_3 + O_p(n_r^{-\frac{1}{2}})$ *and* $\hat{\kappa}_{jn} = \kappa_j + o_p(1), j = 4, 5, 6,$ (*ii*) $\tilde{\kappa}_{jn} = O_p(1), j = 3, 4, 5, 6,$ *and* (*iii*) *if $\kappa_3 = 0$ and conditions in* (A-3) *hold for model (16), then both $\hat{\kappa}_{3n}$ and $\tilde{\kappa}_{3n}$ are $o_p(1)$. Finally, the results remain for Model (23) if instead $u_n = W_nv_n + \varepsilon_n$ such that the jth sample cumulant of $\sigma^{-1}\Omega_n^{-\frac{1}{2}}(\lambda)u_n \xrightarrow{p} \kappa_j$, $j = 1, \ldots, 6$.*

Proof: Note that $\ddot{\kappa}_{jn}$ is the jth cumulant of $\ddot{\sigma}_n^{-1}\ddot{\varepsilon}_n$ where $\ddot{}$ denotes either $\hat{}$ or $\tilde{}$, and that $\ddot{\kappa}_{1n} = \kappa_1 = 0$ and $\ddot{\kappa}_{2n} = \kappa_2 = 1$ by construction. Detailed proofs for the three models are tedious, and are put in

[12] The $\sqrt{n_r}$-consistency of $\hat{\lambda}_n$ for the SLD model is proved by Lee (2004a). Similarly, one can prove the $\sqrt{n_r}$-consistency of $\hat{\rho}_n$ for the SED model and that of $\hat{\lambda}_n$ for the SEC model. Following Lee (2004a), it can be proved that $\hat{\sigma}_n^2$ is always $\sqrt{n}$-consistent, but $\hat{\beta}_n$ is $\sqrt{n_r}$-consistent in general for the SLD model and $\sqrt{n}$-consistent for the other two models.

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a *Supplementary Appendix* made available at `http://www.mysmu.edu/faculty/zlyang/`. A sketch is given here. For the SED model, we have using $M_n(\rho)$ defined below (9),

$$\hat{\varepsilon}_n = M_n(\hat{\rho}_n)B_n(\hat{\rho}_n)Y_n = M_n(\hat{\rho}_n)\varepsilon_n - (\hat{\rho}_n - \rho)M_n(\hat{\rho}_n)W_nB_n^{-1}(\rho)\varepsilon_n = \varepsilon_n + O_p(n_r^{-\frac{1}{2}}), \tag{A-2}$$

Thus, $\hat{\kappa}_{3n} = \frac{1}{n\hat{\sigma}_n^3}\sum_{i=1}^n \hat{\varepsilon}_{n,i}^3 = \frac{1}{n\sigma^3}\sum_{i=1}^n \varepsilon_{n,i}^3 + O_p(n_r^{-\frac{1}{2}}) = \kappa_3 + O_p(n_r^{-\frac{1}{2}})$ where the last step follows by the generalized Chebyshev inequality and Assumption S1; $\hat{\kappa}_{4n} = \frac{1}{n\hat{\sigma}_n^4}\sum_{i=1}^n \hat{\varepsilon}_{n,i}^4 - 3 = \frac{1}{n}\sum_{i=1}^n e_{n,i}^4 - 3 + O_p(n_r^{-\frac{1}{2}}) = \kappa_4 + o_p(1)$ by Kolmogorov law of large numbers; and similarly $\hat{\kappa}_{jn} = \kappa_j + o_p(1), j = 5, 6$. To prove (*ii*), note that $\tilde{\varepsilon}_n = M_nY_n = M_nB_n^{-1}(\rho)\varepsilon_n \equiv G_n\varepsilon_n$ where $M_n = M_n(0)$. Let $g'_{n,i}$ denote the ith row of G_n. We have $\tilde{\kappa}_{3n} = \frac{1}{n\tilde{\sigma}_n^3}\sum_{i=1}^n (g'_{n,i}\varepsilon_n)^3$. Decomposing $\frac{1}{n}\sum_{i=1}^n (g'_{n,i}\varepsilon_n)^3$ into three sums of martingale difference sequences and using $\tilde{\sigma}_n^2 = \frac{1}{n}\tilde{\varepsilon}'_n\tilde{\varepsilon}_n = \frac{\sigma^2}{n}\text{tr}(G'_nG_n) + O_p(n^{-\frac{1}{2}}) \equiv \bar{\sigma}_n^2 + O_p(n^{-\frac{1}{2}})$, one shows that $\tilde{\kappa}_{3n} = \kappa_3\left(\frac{\sigma^3}{n\bar{\sigma}^3}\sum_{i=1}^n\sum_{j=1}^n g_{n,ij}^3\right) + o_p(1) = O_p(1)$. The remaining elements in (*ii*) follow similarly. The results in (*iii*) follow directly from the results in (*i*) and (*ii*) by setting $\kappa_3 = 0$.

The proof for the SLD model is similar to that for the SED model, except for (*iii*): $\tilde{\kappa}_{3n} = \frac{1}{n\bar{\sigma}_n^3}\sum_{i=1}^n \mu_{n,i}^3 + \frac{\sigma^2}{n\bar{\sigma}_n^3}\sum_{j=1}^n \zeta_{n,j} + \kappa_3\big(\frac{\sigma^3}{n\bar{\sigma}_n^3}\sum_{i=1}^n\sum_{j=1}^n g_{n,ij}^3\big) + o_p(1)$, where $\mu_n = G_nX_n\beta$, $G_n = M_nA_n^{-1}(\lambda)$, and $\zeta_{n,j} = \sum_{i=1}^n \mu_{n,i}g_{n,ij}^2$. Thus, $\tilde{\kappa}_{3n} = o_p(1)$ if $\kappa_3 = 0$ and

$$\lim_{n\to\infty}\tfrac{1}{n}\textstyle\sum_{i=1}^n \mu_{n,i}^3 = 0 \text{ and } \lim_{n\to\infty}\tfrac{1}{n}\sum_{i=1}^n \zeta_{n,i} = 0. \tag{A-3}$$

The proof for the SEC model with $u_n = \Omega_n^{\frac{1}{2}}(\lambda)\varepsilon_n$ is similar to that for the SED model. For the SEC model with $u_n = W_nv_n + \varepsilon_n$, it is easy to see that $\hat{\varepsilon}_n = \Omega_n^{-\frac{1}{2}}(\hat{\lambda}_n)\hat{u}_n = \Omega_n^{-\frac{1}{2}}(\lambda)u_n + O_p(n_r^{-\frac{1}{2}})$. Thus, if the sample cumulants κ_{jn} of $\sigma^{-1}\Omega_n^{-\frac{1}{2}}(\lambda)u_n$ converges to κ_j, the sample cumulants of $\hat{\sigma}_n^{-1}\hat{\varepsilon}_n$, $\hat{\kappa}_{jn}$, follows. To see the plausibility of this condition, we have proved the following useful result,

$$\kappa_{jn} = \kappa_{vj}\frac{\lambda^{j/2}}{n}\sum_{i=1}^n\sum_{t=1}^n h_{n,it}^j + \kappa_j\frac{1}{n}\sum_{i=1}^n\sum_{t=1}^n g_{n,it}^j + o_p(1), \tag{A-4}$$

where κ_{vj} is the jth cumulant of $v_{n,i}$, $\{h_{n,it}\}$ are the elements of $\Omega_n^{-\frac{1}{2}}(\lambda)W_n$ and $g_{n,it}$ the elements of $\Omega_n^{-\frac{1}{2}}(\lambda)$. Under certain conditions, e.g., h_n in Assumption S3 is unbounded, $\frac{\lambda^{j/2}}{n}\sum_{i=1}^n\sum_{t=1}^n h_{n,it}^j \to 0$ and $\frac{1}{n}\sum_{i=1}^n\sum_{t=1}^n g_{n,it}^j \to 1$, giving $\kappa_{jn} \xrightarrow{p} \kappa_j$, for $j \geq 3$. See the *Supplementary Appendix* for details.

Appendix B: Derivations and Proofs for Section 3

Derivations of $\text{LM}_{\texttt{SED}}(\rho)$ **and** $\text{SLM}_{\texttt{SED}}(\rho)$ **in Section 3.1:** To implement the $\texttt{RS}_{uf}$ scheme, the LM statistics under $\rho \neq 0$ are needed. The loglikelihood function is: $\ell_n(\beta, \sigma^2, \rho) = -\frac{n}{2}\log(2\pi\sigma^2) + \log|B_n(\rho)| - \frac{1}{2\sigma^2}u'_n(\beta)B'_n(\rho)B_n(\rho)u_n(\beta)$, where $u_n(\beta) = Y_n - X_n\beta$. The score functions are: $\frac{\partial\ell_n}{\partial\beta} = \frac{1}{\sigma^2}X'_nB'_n(\rho)B_n(\rho)u_n(\beta)$, $\frac{\partial\ell_n}{\partial\sigma^2} = \frac{1}{2\sigma^4}u'_n(\beta)B'_n(\rho)B_n(\rho)u_n(\beta) - \frac{n}{2\sigma^2}$, and $\frac{\partial\ell_n}{\partial\rho} = \frac{1}{\sigma^2}u'_n(\beta)B'_n(\rho)W_nu_n(\beta) - \text{tr}[W_nB_n^{-1}(\rho)]$. Plugging $\tilde{\beta}_n(\rho)$ and $\tilde{\sigma}_n^2(\rho)$ into the last expression gives the concentrated score for ρ:

$$S_n^c(\rho) = \tilde{\sigma}_n^{-2}(\rho)\tilde{\varepsilon}_n(\rho)'Q_n^\circ(\rho)(\rho)\tilde{\varepsilon}_n(\rho),$$

where $\tilde{\varepsilon}_n(\rho) = B_n(\rho)(Y_n - X_n\tilde{\beta}_n(\rho))$, $Q_n^\circ(\rho) = Q_n(\rho) - \frac{1}{n}\text{tr}[Q_n(\rho)]I_n$ and $Q_n(\rho) = W_nB_n^{-1}(\rho)$. The expected information matrix is:

$$J_n(\beta, \sigma^2, \rho) = \begin{pmatrix} \frac{1}{\sigma^2}X'_nB'_n(\rho)B_n(\rho)X_n, & 0, & 0 \\ 0, & \frac{n}{2\sigma^4}, & \frac{1}{\sigma^2}\text{tr}[Q_n(\rho)] \\ 0, & \frac{1}{\sigma^2}\text{tr}[Q_n(\rho)], & \text{tr}[Q'_n(\rho)Q_n(\rho) + Q_n^2(\rho)] \end{pmatrix}. \tag{B-1}$$

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Thus, $\text{AVar}[S_n^c(\rho)] = J_{n,22} - J_{n,21}J_{n,11}^{-1}J_{n,12} = \text{tr}[Q_n^\circ(\rho)'Q_n^\circ(\rho) + Q_n^\circ(\rho)^2] \equiv K_n(\rho)$, where $\{J_{n,ij}, i, j = 1, 2\}$ partitions J_n according to (β, σ^2) and ρ. Putting the two together gives:

$$\text{LM}_{\texttt{SED}}(\rho) = \frac{n}{\sqrt{K_n(\rho)}} \frac{\tilde{\varepsilon}_n(\rho)'Q_n^\circ(\rho)\tilde{\varepsilon}_n(\rho)}{\tilde{\varepsilon}_n(\rho)'\tilde{\varepsilon}_n(\rho)}. \tag{B-2}$$

To improve the finite sample performance and to enhance the robustness of $\text{LM}_{\texttt{SED}}(\rho)$, we derive a standardized version of it by centering and rescaling (without assuming normality) its numerator $\tilde{\varepsilon}_n(\rho)'Q_n^\circ(\rho)\tilde{\varepsilon}_n(\rho)$. The resulted statistic takes the form:

$$\text{SLM}_{\texttt{SED}}(\rho) = \frac{\tilde{\varepsilon}_n(\rho)'Q_n^\dagger(\rho)\tilde{\varepsilon}_n(\rho)}{\tilde{\sigma}_n^2(\rho)[K_n^\dagger(\rho) + \tilde{\kappa}_{4n}(\rho)a_n'(\rho)a_n(\rho)]^{\frac{1}{2}}}, \tag{B-3}$$

where $Q_n^\dagger(\rho) = Q_n(\rho) - \frac{1}{n-k}\text{tr}[M_n(\rho)Q_n(\rho)]M_n(\rho)$, $K_n^\dagger(\rho) = \text{tr}[\mathcal{A}_n(\rho)(\mathcal{A}_n(\rho) + \mathcal{A}_n(\rho)')]$, $a_n(\rho) = \text{diagv}[\mathcal{A}_n(\rho)]$, $\mathcal{A}_n(\rho) = M_n(\rho)Q_n^\dagger(\rho)M_n(\rho)$, and $\tilde{\kappa}_{4n}(\rho)$ is the 4th cumulant of $\tilde{\sigma}_n^{-1}(\rho)\tilde{\varepsilon}_n(\rho)$. These two statistics can also be used to construct a confidence interval for ρ without having to estimate it.

Proof of Proposition 3.1 (Cont'd): Additional details needed are the proofs of

(a) $\text{LM}_{\texttt{SED}}|_{H_0} \xrightarrow{D} N(0,1), \forall \mathcal{F}$, $\qquad (c)$ $p(t, \mathcal{F}) = -c_0c_1 + (\frac{1}{6}\kappa_3^2 T_4 + T_5)(1 - t^2)$,

(b) $\text{LM}^*_{\texttt{SED}} \xrightarrow{D^*} N(0,1)$, $\forall \ddot{\mathcal{F}}_n$, $\qquad (d)$ $p(t, \ddot{\mathcal{F}}_n) = -c_0c_1 + (\frac{1}{6}\ddot{\kappa}_{3n}^2 T_4 + T_5)(1 - t^2)$,

where $T_i = \lim_{n\to\infty} n_r^{-1}T_{in}, i = 4, 5$, and T_{in} are defined in Lemma A3 under $A_n = M_nW_nM_n$.

First, (a) follows from Kelejian and Prucha (2001) or Baltagi and Yang (2013). We prove (b) by directly applying Lemmas A1 and A2 to (13) under $A_n = M_nW_nM_n$ and $c_n = 0$. Lemma A1 gives,

$$\text{E}^*(e_n^{*\prime}A_ne_n^*) = \text{tr}(A_n), \text{ and } \text{Var}^*(e_n^{*\prime}A_ne_n^*) = \text{tr}(A_n^2 + A_n'A_n) + \ddot{\kappa}_{4n}a_n'a_n.$$

By Assumption S3 and Lemma A4, $K_n = O(n_r)$; by Lemmas A5 and A6, $\lim_{n\to\infty} K_n^{-\frac{1}{2}}\text{tr}(A_n) = 0$, $\lim_{n\to\infty} K_n^{-1}a_n'a_n = 0$, and $\lim_{n\to\infty} K_n^{-1}\text{tr}(A_n'A_n + A_n^2) = 1$; by lemma A2, $K_n^{-\frac{1}{2}}e_n^{*\prime}A_ne_n^* \xrightarrow{D^*} N(0,1)$; by Kolmogorov's law of large numbers, $\frac{1}{n}e_n^{*\prime}M_ne_n^* = \frac{1}{n}e_n^{*\prime}e_n^* + o_p(1) \xrightarrow{p^*} 1$; and by Slutsky's theorem,

$$\text{LM}^*_{\texttt{SED}} \xrightarrow{D^*} N(0,1), \text{ for } \ddot{\mathcal{F}}_n = \hat{\mathcal{F}}_n \text{ or } \tilde{\mathcal{F}}_n.$$

With (a) and (b), and the existence of the 4th moment of $\text{LM}_{\texttt{SED}}|_{H_0}$, Lemma A7 is applicable to $\text{LM}_{\texttt{SED}}|_{H_0}$ and $\text{LM}^*_{\texttt{SED}}$, leading to (14) and (15). For these it suffices to show (c) and (d). Applying Lemma A3 with $A_n = M_nW_nM_n$, $c_n = 0$, $\varepsilon_n = e_n(\sigma^2 = 1)$ and the quantities $T_{in}, i = 1, \cdots, 5$, defined therein, we obtain, $\text{E}[(e_n'A_ne_n - \text{tr}(A_n))^3] = \kappa_6T_{1n} + 2\kappa_4(6T_{1n} + T_{3n}) + \kappa_3^2(10T_{1n} + T_{4n}) + 2\kappa_3T_{2n} + 2(2T_{3n} - T_{1n} + 3T_{5n}))$. By Lemma A6, $T_{1n} = \sum_{i=1}^n a_{ii}^3 = O(n^{-2})$, $T_{2n} = \sum_{i=1}^n a_{ii}\sum_{j=1}^{i-1}(a_{ij} + a_{ji}) = O(1)$, and $T_{3n} = \sum_{i=1}^n a_{ii}\sum_{j=1}^{i-1}(a_{ij} + a_{ji})^2 = O(h_n^{-1})$. It follows that

$$\text{E}[(e_n'A_ne_n - \text{tr}(A_n))^3] = \kappa_3^2T_{4n} + 6T_{5n} + O(1). \tag{B-4}$$

It left to show that the first three cumulants of $\text{LM}_{\texttt{SED}}|_{H_0}$ have the following asymptotic expansions:

$$\boldsymbol{\kappa}_{1,n} = n_r^{-\frac{1}{2}}c_0^{-\frac{1}{2}}c_1 + O(n_r^{-\frac{3}{2}}), \;\; \boldsymbol{\kappa}_{2,n} = 1 + O(n_r^{-1}), \text{ and } \; \boldsymbol{\kappa}_{3,n} = n_r^{-\frac{1}{2}}c_0^{-\frac{3}{2}}(\kappa_3^2T_4 + 6T_5) + O(n_r^{-\frac{3}{2}}).$$

By (a) and the conditions given in Proposition 3.1, we have by the dominated convergence theorem (DCT), (see, e.g., Chung, 1974, p. 42), $\boldsymbol{\kappa}_{1,n} = o(1)$, $\boldsymbol{\kappa}_{2,n} = 1 + o(1)$ and $\boldsymbol{\kappa}_{3,n} = o(1)$. To derive the higher-order terms for $\boldsymbol{\kappa}_{j,n}$, let $Z_n = K_n^{-\frac{1}{2}}e_n'A_ne_n$ and $q_n = \frac{1}{n}e_n'M_ne_n$, so that $\text{LM}_{\texttt{SED}}|_{H_0} = Z_n/q_n$. As $q_n = 1 + O_p(n^{-\frac{1}{2}})$, it is easy to show that $q_n^{-j} = 1 + O_p(n^{-\frac{1}{2}}), j = 1, 2, 3, 4$. By Taylor series expansion,

$$q_n^{-j} = 1 - j(q_n - 1) + j(j+1)(q_n - 1)^2 + O_p(n^{-\frac{3}{2}}), \; j = 1, 2, 3. \tag{B-5}$$

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Let $c_{n0} = n_r^{-1} K_n$ and $c_{n1} = \text{tr}(A_n)$. Using (B-5) with $j = 1$, we have

$$\boldsymbol{\kappa}_{1,n} = \text{E}(Z_n/q_n) = \text{E}(Z_n) - \text{E}[Z_n(q_n - 1)] + 2\text{E}[Z_n(q_n - 1)^2] + E[O_p(n^{-\frac{3}{2}})],$$

where $\text{E}(Z_n) = K_n^{-\frac{1}{2}} c_{n1}$ and $\text{E}[Z_n(q_n - 1)] = O(n_r^{-\frac{1}{2}})O(n^{-1})$ by Lemma A1(iv). The existence of the 4th moment of $\text{LM}_{\texttt{SED}}|_{H_0}$ implies the existence of the 8th moment of $e_{n,i}$. It follows by Chebyshev's inequality that $(q_n - 1)^2 = E[(q_n - 1)^2] + O(n^{-\frac{3}{2}})$. Thus, $\text{E}[Z_n(q_n - 1)^2] = O(n_r^{-\frac{1}{2}})O(n^{-1})$. Finally, the remainder term follows from a simplifying assumption, e.g., $E[O_p(n^{-\frac{3}{2}})] = O(n^{-\frac{3}{2}})$.[13] Thus, $k_{1,1} = \lim_{n\to\infty} \boldsymbol{\kappa}_{1,n} = 0$ and $k_{1,2} = \lim_{n\to\infty} n_r^{\frac{1}{2}} \boldsymbol{\kappa}_{1,n} = c_0^{-\frac{1}{2}} c_1$, giving $\boldsymbol{\kappa}_{1,n} = n_r^{-\frac{1}{2}} c_0^{-\frac{1}{2}} c_1 + O(n_r^{-\frac{3}{2}})$.

For $\boldsymbol{\kappa}_{2,n}$, noting that the diagonal elements of A_n are $O(n^{-1})$ by Assumption S2, we have, by Lemmas $A1(iii)$ and A5, $\text{Var}(Z_n) = 1 + O(n_r^{-1})$. By (B-5) with $j = 2$, we obtain

$$\boldsymbol{\kappa}_{2,n} = \text{Var}(\text{LM}_{\texttt{SED}}|_{H_0}) = \text{Var}(Z_n) - 2\text{E}[Z_n^2(q_n - 1)] + O(n^{-1}),$$

giving $k_{2,1} = 1$, $k_{2,2} = O(n_r^{-1})$, and thus $\boldsymbol{\kappa}_{2,n} = 1 + O(n_r^{-1})$, provided that $\text{E}[Z_n^2(q_n - 1)]$ is $O(n_r^{-1})$ or smaller. It is easy to see that $q_n - 1 = \frac{1}{n}\sum_{i=1}^n \nu_i + O_p(n^{-1})$ and $Z_n - \text{E}(Z_n) = K_n^{-\frac{1}{2}} \sum_{i=1}^n u_i + O_p(n_r^{-1})$, where $\nu_i = e_{n,i}^2 - 1$ and u_i is defined as in the proof of Lemma A3, which lead to $\text{E}[Z_n^2(q_n - 1)] = O(n^{-1})$.

For $\boldsymbol{\kappa}_{3,n}$, noting that $\boldsymbol{\kappa}_{3,n} = \text{E}[(\text{LM}_{\texttt{SED}}|_{H_0} - E(\text{LM}_{\texttt{SED}}|_{H_0}))^3] = \text{E}[(\text{LM}_{\texttt{SED}}|_{H_0})^3] - 3E(\text{LM}_{\texttt{SED}}|_{H_0}) + O(n_r^{-\frac{3}{2}})$, we obtain by (B-5) with $j = 3$ and the fact that $E(\text{LM}_{\texttt{SED}}|_{H_0}) = \text{E}(Z_n) + O(n_r^{-\frac{3}{2}})$,

$$\boldsymbol{\kappa}_{3,n} = \text{E}[(Z_n - \text{E}(Z_n))^3] - 3\text{E}[Z_n^3(q_n - 1)] + 12\text{E}[Z_n^3(q_n - 1)^2] + O(n_r^{-\frac{3}{2}}),$$

where $\text{E}[(Z_n - \text{E}(Z_n))^3] = K_n^{-\frac{3}{2}}(\kappa_3^2 T_{4n} + 6T_{5n})$ by (B-4), $\text{E}[Z_n^3(q_n - 1)] = O(n_r^{-\frac{1}{2}})O(n^{-1})$ by $q_n - 1 = \frac{1}{n}\sum_{i=1}^n \nu_i + O_p(n^{-1})$ and $Z_n - \text{E}(Z_n) = K_n^{-\frac{1}{2}}\sum_{i=1}^n u_i + O_p(n_r^{-1})$, and $[Z_n^3(q_n - 1)^2] = O(n_r^{-\frac{1}{2}})O(n^{-1})$ by (B-4) and $(q_n - 1)^2 = \text{E}[(q_n - 1)^2] + O(n^{-\frac{3}{2}})$. Thus, $k_{3,1} = \lim_{n\to\infty} n_r^{\frac{1}{2}} \boldsymbol{\kappa}_{3,n} = c_0^{-\frac{3}{2}}(\kappa_3^2 T_4 + 6T_5)$, which gives $\boldsymbol{\kappa}_{3,n} = n_r^{-\frac{1}{2}} c_0^{-\frac{3}{2}}(\kappa_3^2 T_4 + 6T_5) + O(n_r^{-\frac{3}{2}})$. These give the function in Lemma A7: $p(t) = -k_{1,2} + \frac{1}{6}k_{3,1}(1 - t^2) = -c_0^{-\frac{1}{2}} c_1 + c_0^{-\frac{3}{2}}(\frac{1}{6}\kappa_3^2 T_4 + T_5)(1 - t^2)$, and thus (c) and hence (14).[14] Similarly, one proves (d) and hence (15). The rest follows from Lemma A8.

Proof of Proposition 3.2 (Cont'd): Similar to the proof of Proposition 3.1, the necessary details for the proof of Proposition 3.2 amount to show that

(a) $\text{LM}_{\texttt{SLD}}|_{H_0} \xrightarrow{D} N(0,1), \forall \mathcal{F}$,
(b) $\text{LM}^*_{\texttt{SLD}} \xrightarrow{D^*} N(0,1), \forall(\ddot{\theta}_n, \ddot{\mathcal{F}}_n)$,
(c) $p(t, \theta, \mathcal{F}) = -c_0(\theta)c_1 + [\frac{1}{6}\kappa_3^2 T_4 + T_5 + \frac{1}{6}\kappa_3(S_3(\theta) + 2S_5(\theta)) + \frac{1}{3}S_4(\theta)](1 - t^2)$, and
(d) $p(t, \ddot{\theta}_n, \ddot{\mathcal{F}}_n) = -c_0(\ddot{\theta}_n)c_1 + [\frac{1}{6}\ddot{\kappa}_{3n}^2 T_4 + T_5 + \frac{1}{6}\ddot{\kappa}_{3n}(S_3(\ddot{\theta}_n) + 2S_5(\ddot{\theta}_n)) + \frac{1}{3}S_4(\ddot{\theta}_n)](1 - t^2)$,

where $T_i = \lim_{n\to\infty} n_r^{-1} T_{in}, i = 4, 5$, and $S_i(\theta) = \lim_{n\to\infty} n_r^{-1} S_{in}, i = 3, 4, 5$, with T_{in} and S_{in} being defined in Lemma A3 under $A_n = M_n W_n$ and $c_n = M_n \eta_n$.

First, (a) is proved in Yang and Shen (2011). We prove (b) by directly applying Lemmas A1 and A2 to (20) under $A_n = M_n W_n$ and $c_n = M_n \ddot{\eta}_n$. In particular, by Lemma A1, we have,

$$\begin{aligned} \text{E}^*(e_n^{*\prime} A_n e_n^* + e_n^{*\prime} M_n \ddot{\eta}_n) &= \text{tr}(A_n); \text{ and} \\ \text{Var}^*(e_n^{*\prime} A_n e_n^* + e_n^{*\prime} M_n \ddot{\eta}_n) &= \ddot{\eta}_n' M_n \ddot{\eta}_n + \text{tr}(A_n^2 + A_n A_n') + \ddot{\kappa}_{4n} a_n' a_n + 2\ddot{\kappa}_{3n} a_n' M_n \ddot{\eta}_n, \end{aligned}$$

By Lemmas A5 and A6, we have as $n \to \infty$, $K_n^{-\frac{1}{2}}\text{tr}(A_n) \to 0$, $K_n^{-1} a_n' a_n \to 0$, and $K_n^{-1}\text{tr}(A_n^2 + A_n A_n') \to 1$. By law of large numbers, $\frac{1}{n} e_n^{*\prime} M_n e_n^* \xrightarrow{p^*} 1$. Thus, $K_n^{-1} Q(e_n^*) \xrightarrow{p^*} 1$ and $K_n^{-1} e_n^{*\prime} P_n W_n M_n \ddot{\eta}_n \xrightarrow{p^*} 0$.

[13] This is a slight simplification as in Hall (1992, p. 54-55), which is also followed in the derivations for $\boldsymbol{\kappa}_{2,n}$ and $\boldsymbol{\kappa}_{3,n}$.
[14] The remainder of (14) is $O(n_r^{-1})$ as $k_{2,2} = O(n_r^{-1})$, the key term in the Edgeworth expansion; see Hall (1992, p.46-28).

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Furthermore, it is easy to show that $\text{plim}_{n\to\infty} K_n^{-1} a_n' M_n \ddot{\eta}_n = 0$. It follows that

$$\frac{\ddot{\eta}_n' M_n \ddot{\eta}_n + Q(e_n^*) + 2e_n^{*\prime} P_n W_n M_n \ddot{\eta}_n}{\ddot{\eta}_n' M_n \ddot{\eta}_n + \text{tr}(A_n^2 + A_n A_n') + \ddot{\kappa}_{4n} a_n' a_n + 2\ddot{\kappa}_{3n} a_n' M_n \ddot{\eta}_n} \xrightarrow{p^*} 1.$$

Thus, by Lemma A2 and Slutsky's theorem, we have $\text{LM}_{\texttt{SLD}}^* \xrightarrow{D^*} N(0,1)$, $\forall(\ddot{\theta}_n, \ddot{\mathcal{F}}_n)$.

With (a) and (b), and the existence of the 4th moment of $\text{LM}_{\texttt{SLD}}|_{H_0}$, Lemma A7 is applicable to $\text{LM}_{\texttt{SLD}}|_{H_0}$ and $\text{LM}_{\texttt{SLD}}^*$, leading to (21) and (22). For these it suffices to show (c) and (d). Applying Lemma A3 with $A_n = M_n W_n$, $c_n = M_n \eta_n$ and $\varepsilon_n = e_n(\sigma^2 = 1)$, and using Lemma A6 to show that $T_{in}, i = 1, 2, 3$, and $S_{in}, i = 1, 2$, are all of order $O(1)$ or smaller, we obtain,

$$\text{E}[(P_n - \text{E}P_n)^3] = \kappa_3^2 T_{4n} + 6T_{5n} + \kappa_3[S_{3n}(\theta) + 2S_{5n}(\theta)] + 2S_{4n}(\theta).$$

Similar to the proof of Proposition 3.1, we show by DCT and some tedious algebra that the first three cumulants of $\text{LM}_{\texttt{SLD}}|_{H_0}$ are: $\boldsymbol{\kappa}_{1,n} = n_r^{-\frac{1}{2}} c_0(\theta)^{-\frac{1}{2}} c_1 + O(n_r^{-\frac{3}{2}})$, $\boldsymbol{\kappa}_{2,n} = 1 + O(n_r^{-1})$, and $\boldsymbol{\kappa}_{3,n} = n_r^{-\frac{1}{2}} c_0(\theta)^{-\frac{3}{2}}[\kappa_3^2 T_4 + 6T_5 + \kappa_3(S_3(\theta) + 2S_5(\theta)) + 2S_4(\theta)] + O(n_r^{-\frac{3}{2}})$, leading to (c) and hence the Edgeworth expansion (21). Similar arguments lead to (d) and (22). The rest follows from Lemma A8.

Proof of Proposition 3.3 (Cont'd): It suffices to prove the Edgeworth expansion (30) and the asymptotic expansion (31). For the former, we have, $\text{SLM}_{\texttt{SEC}}|_{H_0} \xrightarrow{D} N(0,1)$ by Lemma A2, and the first three cumulants of $\text{SLM}_{\texttt{SEC}}|_{H_0}$: $\boldsymbol{\kappa}_{1,n} = O(n_r^{-\frac{3}{2}})$, $\boldsymbol{\kappa}_{2,n} = 1 + O(n_r^{-2})$,[15] and $\boldsymbol{\kappa}_{3,n} = n_r^{-\frac{1}{2}} c_1^{-\frac{3}{2}}[4T_3 - 2T_1 + 6T_5 + \kappa_6 T_1 + 2\kappa_4(6T_1 + T_3) + \kappa_3^2(10T_1 + T_4) + 2\kappa_3 T_2] + O(n_r^{-\frac{3}{2}})$, by applying Lemma A1, Lemma A3 (with $A_n = M_n H_n^\dagger M_n$ and $c_n = 0$), and DCT. Now, applying Lemma A7 to $\text{LM}_{\texttt{SEC}}|_{H_0}$ gives (30), where $c_1 = \lim_{n\to\infty} n_r^{-1} K_n$, $T_i = \lim_{n\to\infty} n_r^{-1} T_{in}$, and T_{in} are given in Lemma A3 with $A_n = M_n H_n^\dagger M_n$. The asymptotic expansion (31) can be proved in an similar manner. The rest follows from Lemma A8.

Appendix C: Settings for Monte Carlo Experiments

We now describe the methods for generating the regressors values, the spatial weight matrices, and the errors, to be used in the Monte Carlo experiments. All the DGPs used in our Monte Carlo experiments contain two regressors.

Regressors Values. The simplest method for generating the values for the regressors is to make random draws from a certain distribution, i.e., the values $\{x_{1i}\}$ of X_{n1} and the values $\{x_{2i}\}$ of X_{n2} in the Monte Carlo experiments are generated according to:

`XVal-A`: $\{x_{1i}\} \overset{iid}{\sim} N(0,1)$, and $\{x_{2i}\} \overset{iid}{\sim} N(0,1)$,

where X_{n1} and X_{n2} are independent. Alternatively, to allow for the possibility that there might be systematic differences in X values across the different sets of spatial units, e.g., spatial groups, spatial clusters, etc., the ith value in the rth 'group' $\{x_{1,ir}\}$ of X_{n1}, and the ith value in the rth group $\{x_{2,ir}\}$ of X_{n2} are generated as follows:

`XVal-B`: $\{x_{1,ir}\} = (2z_r + z_{ir})/\sqrt{5}$, and $\{x_{2,ir}\} = (2v_r + v_{ir})/\sqrt{5}$,

where $\{z_r, z_{ir}, v_r, v_{ir}\} \overset{iid}{\sim} N(0,1)$, across all i and r. Apparently, unlike the `XVal-A` scheme that gives iid X values, the `XVal-B` scheme gives non-iid X values, or different group means in terms of group interaction (Lee 2004a).

[15] Note that the standardization brings the first two moments of the test statistic closer to their asymptotic values, even when the errors are normal. The same issue applies to the earlier tests.

Spatial Weight Matrix. The spatial weight matrices used in the Monte Carlo experiments are generated according to `Rook Contiguity`, `Queen Contiguity` and `Group Interaction`, using the same methods as in Baltagi and Yang (2013), except that in the group interaction scheme, the group sizes $(n_1, n_2, \cdots, n_g)$ are generated according to a discrete uniform distribution from 2 to $m-2$ where $g = \text{Round}(n^\delta)$, and δ is chosen to be $0.3, 0.5$ and 0.7. In the first two cases, the number of neighbors for each spatial unit does not change with n, whereas in the last case, the number of neighbors for each spatial unit increases with n but at a slower rate, and changes from group to group. A special group interaction scheme is also considered, where a set of fixed group sizes, e.g., $\{2, 3, 4, 5, 6, 7\}$, is repeated m times. In this case, the group sizes and their variance are both fixed fixed with respect to n, leading to a case where the LM_{SEC} test in Section 3.3 is non-robust against nonnormality. See Case (1991), Lee (2007) and Yang (2010) for more discussions on the group interaction scheme.

Error Distributions. The reported Monte Carlo results correspond to the following three error distributions: (i) `standard normal`, (ii) `mixture normal`, $e_i = ((1-\xi_i)Z_i + \xi_i \tau Z_i)/(1-p+p*\tau^2)^{0.5}$, where ξ_i is Bernoulli with parameter $p = .05$ or $.1$, Z_i is $N(0,1)$ independent of ξ_i, and $\tau = 4$; and (iii) `log-normal`, $e_i = [\exp(Z_i) - \exp(0.5)]/[\exp(2) - \exp(1)]^{0.5}$. See Baltagi and Yang (2013) for details.

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Table 3.1a. Bootstrap and MC Critical Values for Burridge's LM Test of SED
spatial Layout: Group Interaction with $g = n^{0.5}$; $H_0 : \rho = 0; \sigma = 2$

		$n = 50$				$n = 200$			
Method	ρ	2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
					Normal Error				
$\mathtt{RS}_{rr}$	-0.5	-1.9437	-1.7281	0.9379	1.3402	-1.8809	-1.7281	1.1689	1.5818
	0.0	-1.9466	-1.7318	0.9379	1.3417	-1.8827	-1.7292	1.1658	1.5802
	0.5	-1.9426	-1.7276	0.9395	1.3446	-1.8833	-1.7292	1.1655	1.5820
$\mathtt{RS}_{uu}$	-0.5	-1.9450	-1.7292	0.9385	1.3428	-1.8802	-1.7277	1.1682	1.5836
	0.0	-1.9476	-1.7316	0.9390	1.3438	-1.8827	-1.7297	1.1654	1.5801
	0.5	-1.9420	-1.7276	0.9400	1.3448	-1.8819	-1.7285	1.1659	1.5795
MC		-1.9615	-1.7236	0.9375	1.3204	-1.8818	-1.7294	1.1664	1.6138
					Normal Mixture, $p = .05$				
$\mathtt{RS}_{rr}$	-0.5	-1.8721	-1.6584	0.8608	1.2739	-1.8771	-1.7134	1.1291	1.5341
	0.0	-1.8758	-1.6607	0.8629	1.2705	-1.8770	-1.7141	1.1295	1.5382
	0.5	-1.8815	-1.6686	0.8704	1.2787	-1.8788	-1.7166	1.1317	1.5391
$\mathtt{RS}_{uu}$	-0.5	-1.8690	-1.6579	0.8608	1.2730	-1.8765	-1.7132	1.1295	1.5337
	0.0	-1.8773	-1.6616	0.8642	1.2714	-1.8769	-1.7146	1.1296	1.5393
	0.5	-1.8728	-1.6625	0.8618	1.2685	-1.8781	-1.7146	1.1271	1.5367
MC		-1.8653	-1.6217	0.9279	1.2777	-1.8830	-1.7173	1.1046	1.5283
					Log-Normal Error				
$\mathtt{RS}_{rr}$	-0.5	-1.8161	-1.6256	0.8157	1.2596	-1.8165	-1.6603	1.0887	1.5252
	0.0	-1.8079	-1.6237	0.8100	1.2597	-1.8157	-1.6589	1.0872	1.5218
	0.5	-1.8345	-1.6389	0.8324	1.2697	-1.8279	-1.6686	1.0959	1.5267
$\mathtt{RS}_{uu}$	-0.5	-1.8133	-1.6263	0.8164	1.2601	-1.8156	-1.6589	1.0853	1.5272
	0.0	-1.8120	-1.6263	0.8115	1.2613	-1.8160	-1.6597	1.0872	1.5212
	0.5	-1.8138	-1.6263	0.8163	1.2603	-1.8177	-1.6596	1.0875	1.5266
MC		-1.8184	-1.5850	0.8618	1.2742	-1.8248	-1.6603	1.0932	1.5195

$\mathtt{RS}_{rr}$ and $\mathtt{RS}_{uu}$: Average bootstrap critical values based on $M = 2,000$ and $B = 699$;
MC: Monte Carlo critical values based on $M = 30,000$; Regressors generated according to XVal-B.
Note: The same pattern holds when $|\rho|$ further increases from 0.5.

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Table 3.1b. Rejection Frequencies for One-Sided LM Test of SED, $H_0 : \rho = 0$
spatial Layout: Group Interaction with $g = n^{0.5}$; $\sigma = 2$

Method	$\lvert\rho\rvert$	$n = 100$				$n = 200$			
		L2.5%	L5%	R5%	R2.5%	L2.5%	L5%	R5%	R2.5%
					Normal Error				
ACR	0.00	0.0155	0.0690	0.0175	0.0110	0.0200	0.0740	0.0200	0.0115
	0.25	0.0700	0.2440	0.2295	0.1760	0.0930	0.2740	0.3245	0.2625
	0.50	0.1925	0.4735	0.7865	0.7400	0.3005	0.6000	0.8860	0.8585
RS_{rr}	0.00	0.0270	0.0530	0.0485	0.0215	0.0290	0.0565	0.0445	0.0235
	0.25	0.1165	0.2000	0.3550	0.2520	0.1370	0.2270	0.4450	0.3410
	0.50	0.2715	0.4105	0.8570	0.8050	0.3875	0.5245	0.9280	0.8915
RS_{uu}	0.00	0.0265	0.0530	0.0470	0.0210	0.0300	0.0555	0.0435	0.0235
	0.25	0.1170	0.2020	0.3595	0.2560	0.1375	0.2235	0.4430	0.3390
	0.50	0.2740	0.4060	0.8565	0.8030	0.3845	0.5275	0.9280	0.8915
ACR^*	0.00	0.0015	0.0170	0.0705	0.0420	0.0300	0.0555	0.0435	0.0235
	0.25	0.0145	0.0825	0.4035	0.3440	0.1375	0.2235	0.4430	0.3390
	0.50	0.0555	0.2180	0.8785	0.8465	0.3845	0.5275	0.9280	0.8915
					Normal Mixture, $p = .05$				
ACR	0.00	0.0165	0.0605	0.0150	0.0090	0.0155	0.0540	0.0205	0.0135
	0.25	0.0710	0.2110	0.2305	0.1735	0.0945	0.2635	0.3325	0.2625
	0.50	0.2045	0.4460	0.7815	0.7390	0.2940	0.5850	0.9020	0.8705
RS_{rr}	0.00	0.0250	0.0530	0.0480	0.0230	0.0215	0.0415	0.0520	0.0245
	0.25	0.0930	0.1730	0.3705	0.2710	0.1355	0.2255	0.4575	0.3600
	0.50	0.2580	0.3915	0.8690	0.8110	0.3595	0.5290	0.9410	0.9120
RS_{uu}	0.00	0.0245	0.0510	0.0475	0.0235	0.0200	0.0415	0.0515	0.0245
	0.25	0.0925	0.1780	0.3700	0.2735	0.1365	0.2235	0.4570	0.3575
	0.50	0.2550	0.3935	0.8675	0.8105	0.3665	0.5335	0.9410	0.9105
ACR^*	0.00	0.0045	0.0170	0.0600	0.0370	0.0065	0.0165	0.0690	0.0425
	0.25	0.0325	0.0790	0.4070	0.3410	0.0260	0.0995	0.5050	0.4270
	0.50	0.0960	0.2145	0.8855	0.8535	0.0860	0.3045	0.9480	0.9325
					Log-Normal Error				
ACR	0.00	0.0125	0.0490	0.0180	0.0090	0.0150	0.0530	0.0210	0.0110
	0.25	0.0735	0.1975	0.2190	0.1630	0.0820	0.2605	0.3115	0.2395
	0.50	0.2120	0.4350	0.7910	0.7340	0.2805	0.5605	0.9180	0.8900
RS_{rr}	0.00	0.0295	0.0440	0.0485	0.0240	0.0250	0.0495	0.0660	0.0285
	0.25	0.1155	0.1950	0.3600	0.2605	0.1525	0.2485	0.4540	0.3460
	0.50	0.2860	0.4235	0.8870	0.8165	0.4090	0.5460	0.9525	0.9255
RS_{uu}	0.00	0.0290	0.0445	0.0490	0.0250	0.0255	0.0495	0.0635	0.0290
	0.25	0.1155	0.1965	0.3650	0.2580	0.1560	0.2520	0.4550	0.3470
	0.50	0.2905	0.4255	0.8865	0.8170	0.4110	0.5525	0.9530	0.9230
ACR^*	0.00	0.0045	0.0140	0.0535	0.0375	0.0015	0.0165	0.0720	0.0470
	0.25	0.0310	0.0760	0.3915	0.3140	0.0255	0.0870	0.4825	0.4025
	0.50	0.1170	0.2185	0.9015	0.8630	0.0985	0.2925	0.9570	0.9430

L = Left tail ($\rho < 0$), R = Right tail ($\rho > 0$); Regressors generated according to XVal-B

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Table 3.2a. Bootstrap Critical Values for LM and SLM Tests of SLD, $H_0 : \lambda = 0$
Spatial Layout: Group Interaction with $g = n^{0.5}$; $n = 100$; $\sigma = 1$

		LM Test				SLM Test			
Method	λ	L2.5%	L5%	U5%	U2.5%	L2.5%	L5%	U5%	U2.5%
				Normal Error					
$\mathtt{RS}_{rr}$	-0.5	-2.0718	-1.8294	1.2718	1.6270	-1.8282	-1.5691	1.7465	2.1265
	-0.3	-2.0872	-1.8313	1.2960	1.6438	-1.8529	-1.5813	1.7331	2.1033
	0.0	-2.1064	-1.8372	1.3469	1.6844	-1.8904	-1.6090	1.7195	2.0722
	0.3	-2.1144	-1.8318	1.4030	1.7303	-1.9238	-1.6322	1.7031	2.0407
	0.5	-2.1135	-1.8245	1.4375	1.7608	-1.9383	-1.6417	1.6994	2.0307
$\mathtt{RS}_{uu}$	-0.5	-2.1034	-1.8378	1.3510	1.6849	-1.8908	-1.6133	1.7145	2.0635
	-0.3	-2.1030	-1.8312	1.3507	1.6870	-1.8905	-1.6072	1.7121	2.0638
	0.0	-2.1064	-1.8363	1.3559	1.6924	-1.8949	-1.6127	1.7163	2.0682
	0.3	-2.1099	-1.8376	1.3563	1.6908	-1.8982	-1.6139	1.7183	2.0667
	0.5	-2.1049	-1.8366	1.3578	1.6898	-1.8929	-1.6132	1.7184	2.0655
MC	0.0	-2.1190	-1.8415	1.3262	1.6512	-1.9018	-1.6117	1.7002	2.0447
				Normal Mixture, $p = .05$					
$\mathtt{RS}_{rr}$	-0.5	-2.0640	-1.8098	1.2502	1.6027	-1.8228	-1.5513	1.7074	2.0825
	-0.3	-2.0809	-1.8167	1.2730	1.6198	-1.8494	-1.5695	1.6954	2.0620
	0.0	-2.0941	-1.8170	1.3308	1.6675	-1.8818	-1.5923	1.6900	2.0411
	0.3	-2.1066	-1.8191	1.3962	1.7254	-1.9197	-1.6235	1.6859	2.0250
	0.5	-2.1095	-1.8175	1.4302	1.7542	-1.9361	-1.6367	1.6885	2.0196
$\mathtt{RS}_{uu}$	-0.5	-2.0972	-1.8206	1.3424	1.6743	-1.8888	-1.6003	1.6899	2.0362
	-0.3	-2.1001	-1.8210	1.3401	1.6761	-1.8918	-1.6008	1.6887	2.0385
	0.0	-2.0959	-1.8175	1.3414	1.6763	-1.8872	-1.5971	1.6898	2.0389
	0.3	-2.0978	-1.8204	1.3428	1.6777	-1.8900	-1.6009	1.6899	2.0368
	0.5	-2.0975	-1.8229	1.3425	1.6761	-1.8886	-1.6023	1.6913	2.0389
MC	0.0	-2.1175	-1.8320	1.3125	1.6077	-1.9059	-1.6033	1.6781	1.9927
				Log-Normal Error					
$\mathtt{RS}_{rr}$	-0.5	-2.0232	-1.7734	1.2626	1.6337	-1.7806	-1.5159	1.6860	2.0766
	-0.3	-2.0374	-1.7797	1.2960	1.6574	-1.8064	-1.5353	1.6806	2.0586
	0.0	-2.0556	-1.7869	1.3500	1.6995	-1.8455	-1.5663	1.6759	2.0381
	0.3	-2.0807	-1.7979	1.4160	1.7513	-1.8982	-1.6079	1.6794	2.0233
	0.5	-2.0947	-1.8026	1.4362	1.7671	-1.9235	-1.6251	1.6797	2.0169
$\mathtt{RS}_{uu}$	-0.5	-2.0612	-1.7899	1.3612	1.7118	-1.8549	-1.5735	1.6780	2.0391
	-0.3	-2.0592	-1.7883	1.3631	1.7083	-1.8530	-1.5722	1.6782	2.0348
	0.0	-2.0608	-1.7884	1.3581	1.7057	-1.8545	-1.5721	1.6764	2.0344
	0.3	-2.0667	-1.7921	1.3664	1.7162	-1.8626	-1.5780	1.6790	2.0388
	0.5	-2.0614	-1.7901	1.3601	1.7104	-1.8553	-1.5743	1.6762	2.0373
MC	0.0	-2.0276	-1.7597	1.3454	1.6944	-1.8154	-1.5290	1.6663	2.0354

$\mathtt{RS}_{rr}$ and $\mathtt{RS}_{uu}$: Average bootstrap critical values based on $M = 2,000$ and $B = 699$;
MC: Monte Carlo critical values based on $M = 30,000$; Regressors generated according to XVal-B;
Note: As $|\rho|$ further increases from .5, the values diverge further under $\mathtt{RS}_{rr}$, but stable under $\mathtt{RS}_{uu}$.

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Table 3.2b. Rejection Frequencies for **LM** Tests of SLD, $H_0 : \lambda = 0$
Spatial Layout: Group Interaction with $g = n^{0.5}$; $\sigma = 1$

Method	$\lvert\lambda\rvert$	$n = 50$ L2.5%	L5%	U5%	U2.5%	$n = 100$ L2.5%	L5%	U5%	U2.5%
					Normal Error				
ACR	0.0	0.0435	0.0970	0.0190	0.0085	0.0430	0.0875	0.0235	0.0095
	0.1	0.1010	0.1905	0.0905	0.0550	0.1405	0.2300	0.1240	0.0805
	0.2	0.2150	0.3510	0.2885	0.1985	0.2955	0.4400	0.4510	0.3430
	0.3	0.3585	0.5420	0.6110	0.4990	0.4705	0.6410	0.8535	0.7690
RS_{rr}	0.0	0.0285	0.0565	0.0485	0.0260	0.0305	0.0540	0.0445	0.0235
	0.1	0.0655	0.1220	0.1640	0.0975	0.1045	0.1725	0.1960	0.1190
	0.2	0.1555	0.2455	0.3975	0.2890	0.2405	0.3505	0.5495	0.4310
	0.3	0.2870	0.4175	0.7135	0.6055	0.4075	0.5390	0.8920	0.8340
RS_{uu}	0.0	0.0270	0.0575	0.0555	0.0280	0.0290	0.0555	0.0475	0.0245
	0.1	0.0605	0.1195	0.1715	0.1030	0.0995	0.1755	0.2015	0.1255
	0.2	0.1415	0.2440	0.4070	0.3020	0.2325	0.3500	0.5590	0.4420
	0.3	0.2610	0.4025	0.7260	0.6220	0.3955	0.5350	0.8935	0.8410
					Normal Mixture, $p = .05$				
ACR	0.0	0.0445	0.0860	0.0160	0.0075	0.0335	0.0765	0.0250	0.0125
	0.1	0.1045	0.1925	0.0975	0.0520	0.1265	0.2285	0.1355	0.0800
	0.2	0.2290	0.3795	0.3070	0.2160	0.2995	0.4355	0.4630	0.3400
	0.3	0.3800	0.5505	0.6380	0.5335	0.5035	0.6625	0.8415	0.7780
RS_{rr}	0.0	0.0295	0.0545	0.0470	0.0215	0.0245	0.0520	0.0485	0.0255
	0.1	0.0745	0.1335	0.1730	0.1015	0.0995	0.1705	0.2070	0.1300
	0.2	0.1860	0.2705	0.4200	0.3105	0.2515	0.3610	0.5520	0.4400
	0.3	0.3140	0.4440	0.7375	0.6350	0.4480	0.5790	0.8850	0.8285
RS_{uu}	0.0	0.0280	0.0525	0.0515	0.0235	0.0240	0.0510	0.0495	0.0275
	0.1	0.0675	0.1325	0.1820	0.1055	0.0985	0.1680	0.2180	0.1325
	0.2	0.1720	0.2685	0.4360	0.3215	0.2425	0.3535	0.5660	0.4500
	0.3	0.2935	0.4390	0.7460	0.6485	0.4260	0.5755	0.8890	0.8345
					Log-Normal Error				
ACR	0.0	0.0275	0.0715	0.0140	0.0070	0.0310	0.0800	0.0270	0.0130
	0.1	0.1165	0.2020	0.1175	0.0655	0.1355	0.2375	0.1630	0.1030
	0.2	0.2725	0.4065	0.3895	0.2795	0.3275	0.4620	0.5045	0.4020
	0.3	0.4380	0.5925	0.7130	0.6260	0.5360	0.6880	0.8675	0.8060
RS_{rr}	0.0	0.0175	0.0480	0.0375	0.0150	0.0255	0.0550	0.0420	0.0225
	0.1	0.0880	0.1540	0.1860	0.1110	0.1145	0.1800	0.2290	0.1515
	0.2	0.2255	0.3260	0.4865	0.3825	0.3020	0.4035	0.5980	0.4725
	0.3	0.3945	0.5045	0.8000	0.7055	0.4970	0.6185	0.9030	0.8520
RS_{uu}	0.0	0.0165	0.0415	0.0450	0.0185	0.0235	0.0495	0.0460	0.0230
	0.1	0.0745	0.1420	0.1935	0.1155	0.1115	0.1815	0.2390	0.1565
	0.2	0.2050	0.3160	0.5010	0.3935	0.2895	0.3990	0.6075	0.4860
	0.3	0.3750	0.4920	0.8080	0.7160	0.4810	0.6165	0.9080	0.8570

L = Left tail ($\lambda < 0$), R = Right tail ($\lambda > 0$); Regressors generated according to XVal-B

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Table 3.2c. Rejection Frequencies for **SLM** Tests of SLD, $H_0 : \lambda = 0$
Spatial Layout: Group Interaction with $g = n^{0.5}$; $\sigma = 1$

		$n = 50$				$n = 100$			
Method	$\lvert\lambda\rvert$	L2.5%	L5%	U5%	U2.5%	L2.5%	L5%	U5%	U2.5%
		Normal Error							
ACR	0.0	0.0230	0.0475	0.0635	0.0355	0.0235	0.0495	0.0510	0.0280
	0.1	0.0520	0.1100	0.1865	0.1160	0.0915	0.1610	0.2065	0.1350
	0.2	0.1265	0.2290	0.4345	0.3325	0.2050	0.3340	0.5740	0.4575
	0.3	0.2275	0.3755	0.7500	0.6505	0.3500	0.5055	0.9050	0.8510
$\mathtt{RS}_{rr}$	0.0	0.0280	0.0565	0.0520	0.0280	0.0300	0.0535	0.0450	0.0240
	0.1	0.0655	0.1215	0.1695	0.0980	0.1050	0.1715	0.1955	0.1190
	0.2	0.1540	0.2440	0.4005	0.2950	0.2380	0.3505	0.5525	0.4355
	0.3	0.2865	0.4145	0.7190	0.6120	0.4075	0.5350	0.8925	0.8340
$\mathtt{RS}_{uu}$	0.0	0.0235	0.0520	0.0515	0.0235	0.0250	0.0525	0.0450	0.0220
	0.1	0.0575	0.1105	0.1650	0.0925	0.0960	0.1690	0.1930	0.1175
	0.2	0.1340	0.2270	0.4005	0.2920	0.2240	0.3395	0.5485	0.4220
	0.3	0.2485	0.3910	0.7165	0.6060	0.3820	0.5230	0.8905	0.8320
		Normal Mixture, $p = .05$							
ACR	0.0	0.0230	0.0475	0.0565	0.0285	0.0185	0.0460	0.0540	0.0290
	0.1	0.0540	0.1145	0.1960	0.1245	0.0825	0.1535	0.2220	0.1470
	0.2	0.1450	0.2395	0.4525	0.3520	0.2080	0.3240	0.5760	0.4705
	0.3	0.2475	0.3945	0.7610	0.6750	0.3925	0.5400	0.8940	0.8430
$\mathtt{RS}_{rr}$	0.0	0.0290	0.0530	0.0505	0.0235	0.0245	0.0515	0.0495	0.0275
	0.1	0.0730	0.1280	0.1780	0.1075	0.0985	0.1680	0.2110	0.1295
	0.2	0.1845	0.2690	0.4305	0.3190	0.2520	0.3580	0.5530	0.4445
	0.3	0.3085	0.4415	0.7445	0.6440	0.4460	0.5775	0.8875	0.8305
$\mathtt{RS}_{uu}$	0.0	0.0265	0.0505	0.0505	0.0220	0.0215	0.0490	0.0475	0.0265
	0.1	0.0640	0.1200	0.1765	0.1050	0.0920	0.1560	0.2115	0.1255
	0.2	0.1595	0.2550	0.4275	0.3135	0.2320	0.3395	0.5580	0.4390
	0.3	0.2775	0.4180	0.7430	0.6395	0.4160	0.5610	0.8855	0.8290
		Log-Normal Error							
ACR	0.0	0.0120	0.0350	0.0480	0.0235	0.0165	0.0415	0.0485	0.0285
	0.1	0.0605	0.1295	0.2055	0.1315	0.0910	0.1570	0.2395	0.1670
	0.2	0.1800	0.2890	0.5125	0.4175	0.2455	0.3605	0.6145	0.5025
	0.3	0.3250	0.4535	0.8160	0.7400	0.4180	0.5670	0.9090	0.8655
$\mathtt{RS}_{rr}$	0.0	0.0160	0.0460	0.0440	0.0185	0.0235	0.0525	0.0450	0.0230
	0.1	0.0830	0.1505	0.1935	0.1175	0.1125	0.1770	0.2330	0.1555
	0.2	0.2185	0.3200	0.4990	0.3985	0.2990	0.3995	0.6040	0.4815
	0.3	0.3875	0.4975	0.8110	0.7160	0.4925	0.6135	0.9045	0.8550
$\mathtt{RS}_{uu}$	0.0	0.0135	0.0355	0.0445	0.0185	0.0195	0.0465	0.0430	0.0215
	0.1	0.0685	0.1345	0.1920	0.1135	0.1050	0.1700	0.2340	0.1520
	0.2	0.1960	0.3010	0.5010	0.3935	0.2785	0.3845	0.6045	0.4770
	0.3	0.3565	0.4730	0.8115	0.7170	0.4675	0.5970	0.9065	0.8530

L = Left tail ($\lambda < 0$), R = Right tail ($\lambda > 0$); Regressors generated according to XVal-B

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Table 3.3a. Bootstrap Critical Values for LM and SLM Tests of SEC, $H_0 : \lambda = 0$

		Normal Error			Normal Mixture p=.05			Lognormal		
Method	λ	10%	5%	1%	10%	5%	1%	10%	5%	1%
LM Test: Group Sizes $\{2,3,4,5,6,7\}$, $m=8$										
$\mathtt{RS}_{rr}$	0.0	1.0763	1.4706	2.2198	1.6682	2.3600	3.6943	2.1365	3.2755	5.5770
	0.5	1.0766	1.4684	2.2162	1.5400	2.1660	3.3711	1.9092	2.9030	4.8534
	1.0	1.0784	1.4699	2.2376	1.4653	2.0475	3.1833	1.8069	2.7365	4.5543
	1.5	1.0836	1.4811	2.2416	1.4126	1.9668	3.0347	1.6942	2.5609	4.2301
	2.0	1.0935	1.4932	2.2571	1.3744	1.9066	2.9449	1.6207	2.4350	4.0063
$\mathtt{RS}_{uu}$	0.0	1.0754	1.4690	2.2184	1.6453	2.3256	3.6383	2.0866	3.1835	5.3784
	0.5	1.0738	1.4649	2.2097	1.5392	2.1640	3.3659	1.9024	2.8723	4.7672
	1.0	1.0709	1.4609	2.2217	1.4829	2.0749	3.2285	1.8312	2.7751	4.5934
	1.5	1.0710	1.4632	2.2140	1.4439	2.0192	3.1225	1.7438	2.6375	4.3598
	2.0	1.0732	1.4657	2.2190	1.4137	1.9705	3.0440	1.6968	2.5611	4.2373
MC	0.0	1.0772	1.4737	2.2308	1.7310	2.4793	4.0564	2.2162	3.4827	7.4663
SLM Test: Group Sizes $\{2,3,4,5,6,7\}$, $m=8$										
$\mathtt{RS}_{rr}$	0.0	1.3219	1.7255	2.4923	1.3693	1.8443	2.7315	1.4028	1.9818	2.9503
	0.5	1.3204	1.7213	2.4860	1.3578	1.8204	2.6939	1.3953	1.9451	2.8880
	1.0	1.3181	1.7185	2.4993	1.3520	1.8043	2.6625	1.3877	1.9264	2.8542
	1.5	1.3175	1.7208	2.4910	1.3498	1.7939	2.6297	1.3729	1.8944	2.8019
	2.0	1.3218	1.7272	2.4974	1.3463	1.7834	2.6192	1.3654	1.8749	2.7717
$\mathtt{RS}_{uu}$	0.0	1.3215	1.7251	2.4921	1.3675	1.8399	2.7248	1.3998	1.9700	2.9357
	0.5	1.3202	1.7212	2.4856	1.3581	1.8205	2.6921	1.3954	1.9418	2.8843
	1.0	1.3176	1.7182	2.4977	1.3543	1.8077	2.6717	1.3900	1.9348	2.8701
	1.5	1.3169	1.7186	2.4882	1.3529	1.8049	2.6488	1.3783	1.9076	2.8291
	2.0	1.3197	1.7224	2.4938	1.3505	1.7988	2.6390	1.3748	1.8983	2.8169
MC	0.0	1.3189	1.7238	2.5153	1.3714	1.8843	2.8192	1.3823	2.0921	3.1531
LM Test: Group Sizes $\{2,3,4,5,6,7\}$, $m=19$										
$\mathtt{RS}_{rr}$	0.0	1.1502	1.5338	2.2687	1.8199	2.4772	3.7743	2.6510	4.0188	6.6391
	0.5	1.1526	1.5372	2.2731	1.6764	2.2758	3.4514	2.3728	3.6083	5.9152
	1.0	1.1558	1.5418	2.2794	1.5888	2.1559	3.2507	2.1695	3.2701	5.2968
	1.5	1.1612	1.5485	2.2905	1.5333	2.0724	3.1247	2.0451	3.0671	4.9338
	2.0	1.1710	1.5607	2.3023	1.4866	2.0145	3.0414	1.9375	2.8884	4.6049
$\mathtt{RS}_{uu}$	0.0	1.1499	1.5333	2.2678	1.8084	2.4606	3.7472	2.6015	3.9204	6.4371
	0.5	1.1495	1.5341	2.2673	1.6880	2.2929	3.4773	2.3740	3.5833	5.8370
	1.0	1.1489	1.5332	2.2617	1.6219	2.2022	3.3299	2.2332	3.3549	5.4274
	1.5	1.1473	1.5295	2.2640	1.5833	2.1467	3.2393	2.1538	3.2382	5.2166
	2.0	1.1525	1.5362	2.2629	1.5435	2.0858	3.1581	2.0716	3.1130	4.9923
MC	0.0	1.1569	1.5445	2.2472	1.8325	2.5278	3.9093	2.6464	4.1103	8.5357
SLM Test: Group Sizes $\{2,3,4,5,6,7\}$, $m=19$										
$\mathtt{RS}_{rr}$	0.0	1.3026	1.6901	2.4312	1.3369	1.7769	2.6263	1.3836	1.9416	2.9125
	0.5	1.3029	1.6909	2.4326	1.3286	1.7583	2.5835	1.3722	1.9220	2.8692
	1.0	1.3003	1.6882	2.4274	1.3259	1.7512	2.5635	1.3661	1.8927	2.8153
	1.5	1.3011	1.6880	2.4293	1.3239	1.7435	2.5493	1.3579	1.8737	2.7870
	2.0	1.3048	1.6925	2.4299	1.3194	1.7361	2.5346	1.3549	1.8583	2.7535
$\mathtt{RS}_{uu}$	0.0	1.3024	1.6899	2.4311	1.3360	1.7745	2.6227	1.3820	1.9352	2.9025
	0.5	1.3026	1.6911	2.4319	1.3287	1.7610	2.5867	1.3742	1.9187	2.8649
	1.0	1.3010	1.6895	2.4243	1.3274	1.7571	2.5783	1.3696	1.9025	2.8324
	1.5	1.3000	1.6862	2.4279	1.3280	1.7526	2.5666	1.3657	1.8932	2.8177
	2.0	1.3045	1.6926	2.4266	1.3238	1.7442	2.5579	1.3643	1.8821	2.8027
MC	0.0	1.3033	1.6967	2.4031	1.3209	1.7774	2.6576	1.3432	2.0206	3.0694

MC: Monte Carlo critical values based on $M = 50,000$, $\sigma = 1$, XVAL-B.

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Table 3.3b. Rejection Frequencies for LM and SLM Tests of SEC, $H_0 : \lambda = 0$

		Normal Error			Normal Mixture p=.05			Lognormal		
Method	λ	10%	5%	1%	10%	5%	1%	10%	5%	1%
		LM Test: Group Sizes $\{2,3,4,5,6,7\}$, $m=8$								
ACR	0.0	0.0690	0.0330	0.0070	0.1480	0.1075	0.0575	0.1790	0.1420	0.0960
	0.5	0.5845	0.4640	0.2490	0.5550	0.4590	0.2980	0.5795	0.5015	0.3665
	1.0	0.9005	0.8460	0.6780	0.8540	0.7870	0.6470	0.8110	0.7635	0.6525
	2.0	0.9960	0.9910	0.9665	0.9850	0.9750	0.9340	0.9530	0.9375	0.9010
RS$_{rr}$	0.0	0.1010	0.0465	0.0120	0.1045	0.0555	0.0135	0.1180	0.0625	0.0180
	0.5	0.6560	0.5215	0.2760	0.4890	0.3505	0.1505	0.4735	0.3275	0.1610
	1.0	0.9330	0.8720	0.7045	0.8140	0.6980	0.4520	0.7190	0.5945	0.3850
	2.0	0.9960	0.9935	0.9720	0.9805	0.9545	0.8570	0.9215	0.8560	0.7130
RS$_{uu}$	0.0	0.1010	0.0480	0.0115	0.1065	0.0605	0.0205	0.1215	0.0685	0.0395
	0.5	0.6570	0.5230	0.2840	0.4835	0.3490	0.1540	0.4690	0.3245	0.1580
	1.0	0.9330	0.8740	0.7055	0.8090	0.6850	0.4320	0.7145	0.5820	0.3670
	2.0	0.9960	0.9930	0.9715	0.9795	0.9490	0.8295	0.9160	0.8395	0.6735
		SLM Test: Group Sizes $\{2,3,4,5,6,7\}$, $m=8$								
ACR	0.0	0.1025	0.0525	0.0130	0.1090	0.0660	0.0255	0.1160	0.0795	0.0440
	0.5	0.6640	0.5465	0.3210	0.4985	0.3875	0.2085	0.4685	0.3735	0.2225
	1.0	0.9340	0.8845	0.7485	0.8210	0.7320	0.5420	0.7160	0.6320	0.4745
	2.0	0.9965	0.9950	0.9805	0.9810	0.9590	0.9010	0.9045	0.8725	0.7900
RS$_{rr}$	0.0	0.1015	0.0465	0.0120	0.0970	0.0515	0.0105	0.1040	0.0590	0.0170
	0.5	0.6535	0.5205	0.2755	0.4730	0.3380	0.1435	0.4400	0.3105	0.1585
	1.0	0.9330	0.8715	0.7050	0.8045	0.6825	0.4510	0.6870	0.5705	0.3765
	2.0	0.9960	0.9935	0.9715	0.9785	0.9510	0.8530	0.8955	0.8335	0.7080
RS$_{uu}$	0.0	0.1000	0.0485	0.0110	0.0975	0.0525	0.0110	0.1035	0.0595	0.0210
	0.5	0.6550	0.5220	0.2780	0.4750	0.3405	0.1460	0.4420	0.3105	0.1560
	1.0	0.9320	0.8730	0.7020	0.8050	0.6795	0.4470	0.6875	0.5670	0.3715
	2.0	0.9960	0.9935	0.9715	0.9780	0.9485	0.8475	0.8940	0.8330	0.7010
		LM Test: Group Sizes $\{2,3,4,5,6,7\}$, $m=19$								
ACR	0.0	0.0850	0.0425	0.0080	0.1680	0.1205	0.0630	0.2025	0.1705	0.1200
	0.5	0.8835	0.8175	0.6270	0.8205	0.7510	0.5775	0.7595	0.6995	0.5670
	1.0	0.9980	0.9945	0.9795	0.9900	0.9805	0.9340	0.9520	0.9330	0.8785
RS$_{rr}$	0.0	0.1070	0.0560	0.0085	0.0955	0.0525	0.0100	0.1105	0.0615	0.0140
	0.5	0.9125	0.8420	0.6400	0.7415	0.5895	0.3340	0.5735	0.4145	0.2165
	1.0	0.9985	0.9960	0.9810	0.9790	0.9405	0.8095	0.8590	0.7565	0.5485
RS$_{uu}$	0.0	0.1085	0.0550	0.0085	0.0975	0.0535	0.0145	0.1130	0.0695	0.0295
	0.5	0.9115	0.8450	0.6400	0.7365	0.5825	0.3255	0.5685	0.4105	0.2110
	1.0	0.9985	0.9960	0.9820	0.9765	0.9375	0.7895	0.8530	0.7440	0.5185
		SLM Test: Group Sizes $\{2,3,4,5,6,7\}$, $m=19$								
ACR	0.0	0.1105	0.0585	0.0125	0.1010	0.0610	0.0205	0.1055	0.0765	0.0360
	0.5	0.9135	0.8510	0.6760	0.7520	0.6220	0.3980	0.5650	0.4600	0.2965
	1.0	0.9985	0.9970	0.9845	0.9805	0.9510	0.8555	0.8465	0.7870	0.6485
RS$_{rr}$	0.0	0.1075	0.0565	0.0080	0.0935	0.0490	0.0105	0.0975	0.0560	0.0120
	0.5	0.9120	0.8425	0.6385	0.7310	0.5845	0.3295	0.5360	0.3985	0.2105
	1.0	0.9985	0.9960	0.9820	0.9765	0.9375	0.8120	0.8265	0.7330	0.5480
RS$_{uu}$	0.0	0.1065	0.0555	0.0080	0.0920	0.0515	0.0125	0.0980	0.0570	0.0180
	0.5	0.9115	0.8450	0.6385	0.7300	0.5820	0.3240	0.5345	0.3965	0.2095
	1.0	0.9985	0.9960	0.9815	0.9760	0.9385	0.8065	0.8275	0.7305	0.5385

Note: $\sigma = 1$, XVAL-B.

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Table 4.1a. Bootstrap Critical Values and Rejection Frequencies for LM_{SARAR}, H_0: $\lambda = \rho = 0$
Group Interaction with $g = n^{0.5}$, $\sigma = 1$, $n = 50$, XVal-B

			Normal Error			Normal Mixture p=.1			Lognormal Error		
Method	λ	ρ	10%	5%	1%	10%	5%	1%	10%	5%	1%
			Bootstrap Critical Values								
RS_{rr}	-0.50	-0.2	4.649	5.938	9.399	4.927	6.343	10.132	5.429	7.384	15.622
	-0.25	-0.1	4.568	5.790	9.016	4.796	6.116	9.525	5.164	6.869	12.081
	0.00	0.0	4.526	5.711	8.792	4.676	5.917	9.082	4.981	6.504	10.876
	0.25	0.1	4.499	5.650	8.594	4.594	5.776	8.752	4.764	6.113	9.603
	0.50	0.2	4.530	5.692	8.580	4.577	5.738	8.639	4.675	5.934	9.173
RS_{ur}	-0.50	-0.2	4.626	5.867	9.080	4.954	6.378	10.573	5.668	7.755	16.345
	-0.25	-0.1	4.609	5.843	9.059	5.045	6.557	11.428	5.616	7.740	17.464
	0.00	0.0	4.604	5.841	9.054	5.015	6.510	11.137	5.657	7.816	16.494
	0.25	0.1	4.616	5.857	9.098	5.173	6.831	13.028	6.490	9.328	22.458
	0.50	0.2	4.624	5.874	9.175	5.393	7.342	15.868	8.227	12.890	48.097
RS_{ru}	-0.50	-0.2	4.446	5.537	8.267	4.460	5.539	8.183	4.403	5.485	8.195
	-0.25	-0.1	4.452	5.551	8.292	4.473	5.546	8.189	4.408	5.489	8.196
	0.00	0.0	4.447	5.546	8.280	4.464	5.544	8.216	4.410	5.489	8.218
	0.25	0.1	4.449	5.543	8.283	4.466	5.550	8.199	4.421	5.508	8.235
	0.50	0.2	4.455	5.550	8.286	4.470	5.556	8.205	4.423	5.502	8.234
RS_{uu}	-0.50	-0.2	4.445	5.539	8.276	4.461	5.544	8.194	4.413	5.504	8.223
	-0.25	-0.1	4.454	5.553	8.296	4.471	5.547	8.191	4.414	5.496	8.213
	0.00	0.0	4.446	5.544	8.282	4.464	5.544	8.219	4.411	5.489	8.217
	0.25	0.1	4.450	5.543	8.281	4.466	5.548	8.197	4.416	5.501	8.218
	0.50	0.2	4.452	5.545	8.285	4.469	5.551	8.198	4.415	5.491	8.219
MC	0.00	0.00	4.450	5.542	8.356	4.428	5.517	8.145	4.315	5.385	8.048
			Rejection Frequencies								
ACR	-0.50	-0.2	.5600	.3520	.1000	.6140	.4035	.1225	.6805	.5135	.2140
	-0.25	-0.1	.2525	.1305	.0275	.3065	.1685	.0355	.3680	.2010	.0495
	0.00	0.0	.0985	.0360	.0070	.0910	.0320	.0055	.0910	.0350	.0070
	0.25	0.1	.4455	.3465	.1755	.4605	.3470	.1640	.5170	.4075	.2180
	0.50	0.2	.9830	.9730	.9275	.9890	.9770	.9440	.9845	.9745	.9405
RS_{rr}	-0.50	-0.2	.5525	.3580	.1005	.5790	.3795	.1165	.6065	.4400	.1510
	-0.25	-0.1	.2595	.1430	.0335	.2870	.1655	.0365	.3175	.1745	.0335
	0.00	0.0	.0990	.0450	.0110	.0880	.0385	.0060	.0800	.0295	.0045
	0.25	0.1	.4520	.3630	.1970	.4640	.3680	.1820	.5090	.4095	.2185
	0.50	0.2	.9840	.9755	.9415	.9895	.9780	.9530	.9810	.9705	.9385
RS_{uu}	-0.50	-0.2	.5975	.4140	.1515	.6300	.4655	.1870	.7060	.5715	.2965
	-0.25	-0.1	.2720	.1580	.0490	.3220	.1980	.0625	.3885	.2590	.0795
	0.00	0.0	.1030	.0540	.0130	.1020	.0500	.0090	.1010	.0495	.0100
	0.25	0.1	.4565	.3755	.2080	.4765	.3855	.2050	.5305	.4435	.2665
	0.50	0.2	.9840	.9755	.9475	.9905	.9780	.9565	.9850	.9790	.9560

MC: Monte Carlo critical values based on $M = 30,000$;
Note: Larger values of $|\lambda|$ and $|\rho|$ are considered, and the patterns on bootstrap critical values remain.

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Table 4.1b. Bootstrap Critical Values and Rejection Frequencies for $LM_{SED|SLD}$, H_0: $\rho = 0$
Group Interaction with $g = n^{0.5}$, $\sigma = 1$, $n = 100$, $\lambda = 0.25$, XVal-B

		Normal Error				Lognormal Error			
Method	ρ	L2.5%	L5%	U5%	U2.5%	L2.5%	L5%	U5%	U2.5%
		Bootstrap Critical Values							
RS_{rr}	-0.50	-2.1615	-1.9302	1.1107	1.4812	-1.9531	-1.7562	1.0984	1.5787
	-0.25	-2.1735	-1.9390	1.1115	1.4823	-1.9577	-1.7561	1.0847	1.5536
	0.00	-2.1857	-1.9459	1.1178	1.4897	-1.9903	-1.7667	1.0758	1.5207
	0.25	-2.2003	-1.9557	1.1326	1.4991	-2.0973	-1.7993	1.0647	1.4808
	0.50	-2.2453	-1.9832	1.1639	1.5186	-2.3318	-1.8836	1.0706	1.4511
RS_{ur}	-0.50	-2.1766	-1.9397	1.1182	1.4870	-2.0037	-1.7737	1.0731	1.5107
	-0.25	-2.1856	-1.9463	1.1174	1.4863	-2.0271	-1.7780	1.0706	1.5140
	0.00	-2.1912	-1.9489	1.1212	1.4916	-2.0460	-1.7848	1.0741	1.5158
	0.25	-2.1881	-1.9474	1.1236	1.4931	-2.0702	-1.7895	1.0752	1.5193
	0.50	-2.1914	-1.9508	1.1256	1.4921	-2.0858	-1.7998	1.0786	1.5218
RS_{ru}	-0.50	-2.1485	-1.9194	1.1051	1.4742	-1.9382	-1.7440	1.0948	1.5791
	-0.25	-2.1633	-1.9297	1.1059	1.4727	-1.9514	-1.7511	1.0795	1.5523
	0.00	-2.1816	-1.9425	1.1140	1.4866	-1.9825	-1.7638	1.0727	1.5146
	0.25	-2.2076	-1.9619	1.1365	1.5033	-2.0891	-1.8010	1.0653	1.4805
	0.50	-2.2990	-2.0294	1.1910	1.5552	-2.3591	-1.9098	1.0844	1.4731
RS_{uu}	-0.50	-2.1756	-1.9394	1.1190	1.4897	-1.9881	-1.7672	1.0728	1.5149
	-0.25	-2.1850	-1.9457	1.1173	1.4841	-2.0160	-1.7759	1.0696	1.5167
	0.00	-2.1913	-1.9499	1.1204	1.4922	-2.0265	-1.7810	1.0733	1.5110
	0.25	-2.1887	-1.9476	1.1244	1.4928	-2.0675	-1.7890	1.0740	1.5138
	0.50	-2.1928	-1.9509	1.1271	1.4915	-2.0794	-1.7905	1.0732	1.5178
MC	0.00	-2.1641	-1.9271	1.1382	1.4957	-2.1120	-1.8134	1.0815	1.4698
	$\|\rho\|$	Rejection Frequencies							
ACR	0.00	0.0415	0.1030	0.0180	0.0080	0.0325	0.0750	0.0145	0.0090
	0.25	0.1815	0.3080	0.1370	0.0920	0.1095	0.2360	0.1225	0.0770
	0.50	0.4105	0.5720	0.5275	0.4300	0.2965	0.4920	0.5235	0.4285
RS_{rr}	0.00	0.0225	0.0450	0.0520	0.0235	0.0310	0.0490	0.0420	0.0185
	0.25	0.3215	0.4275	0.2550	0.1645	0.1110	0.1780	0.2510	0.1540
	0.50	0.3215	0.4275	0.6765	0.5745	0.3050	0.4175	0.6980	0.5855
RS_{ur}	0.00	0.0225	0.0425	0.0515	0.0230	0.0255	0.0470	0.0415	0.0170
	0.25	0.1155	0.1880	0.2590	0.1680	0.1010	0.1745	0.2505	0.1385
	0.50	0.3140	0.4250	0.6815	0.5815	0.2910	0.4115	0.7010	0.5640
RS_{ru}	0.00	0.0245	0.0470	0.0500	0.0225	0.0325	0.0525	0.0405	0.0180
	0.25	0.3295	0.4260	0.2505	0.1650	0.1160	0.1815	0.2555	0.1520
	0.50	0.3140	0.4250	0.6690	0.5605	0.3120	0.4300	0.6965	0.5855
RS_{uu}	0.00	0.0230	0.0440	0.0530	0.0240	0.0290	0.0465	0.0400	0.0175
	0.25	0.1195	0.1890	0.2585	0.1715	0.1005	0.1725	0.2495	0.1405
	0.50	0.3140	0.4250	0.6820	0.5840	0.2920	0.4170	0.7025	0.5710

L = Left tail ($\rho = -0.25, -0.5$, in the rejection frequencies), R = Right tail ($\rho = 0.25, 0.5$);
MC: Monte Carlo critical values based on $M = 30,000$;
Note: As $|\rho|$ further increases from 0.5, the patterns on bootstrap critical values remain.

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Table 4.1c. Bootstrap Critical Values and Rejection Frequencies for $LM_{SLD|SED}$, H_0: $\lambda = 0$
Group Interaction with $g = n^{0.5}$, $\sigma = 1$, $n = 100$, $\rho = 0.25$, XVal-B

		Normal Error				Lognormal Error			
Method	λ	L2.5%	L5%	U5%	U2.5%	L2.5%	L5%	U5%	U2.5%
		Bootstrap Critical Values							
RS_{rr}	-0.50	-2.0881	-1.7779	1.6071	1.9263	-2.1305	-1.8197	1.8749	2.3047
	-0.25	-2.0652	-1.7471	1.6662	1.9839	-2.0594	-1.7505	1.8153	2.2001
	0.00	-2.0469	-1.7311	1.7164	2.0320	-2.0311	-1.7230	1.7878	2.1438
	0.25	-2.0498	-1.7350	1.7511	2.0605	-2.0498	-1.7377	1.7944	2.1315
	0.50	-2.0333	-1.7279	1.7506	2.0366	-2.3410	-1.9435	1.8272	2.1340
RS_{ur}	-0.50	-2.0533	-1.7361	1.7028	2.0204	-2.1716	-1.8266	1.8291	2.2021
	-0.25	-2.0556	-1.7395	1.7010	2.0168	-2.1903	-1.8348	1.8271	2.2009
	0.00	-2.0521	-1.7364	1.7063	2.0202	-2.1448	-1.8087	1.8282	2.2073
	0.25	-2.0564	-1.7398	1.7019	2.0174	-2.1021	-1.7839	1.8244	2.2137
	0.50	-2.0474	-1.7354	1.7081	2.0216	-2.0979	-1.7810	1.8177	2.2215
RS_{ru}	-0.50	-2.0924	-1.7780	1.6048	1.9253	-2.1283	-1.8176	1.8750	2.2999
	-0.25	-2.0642	-1.7472	1.6655	1.9830	-2.0583	-1.7498	1.8129	2.1954
	0.00	-2.0457	-1.7299	1.7155	2.0297	-2.0276	-1.7203	1.7868	2.1407
	0.25	-2.0510	-1.7351	1.7501	2.0589	-2.0390	-1.7295	1.7935	2.1308
	0.50	-2.0377	-1.7296	1.7515	2.0356	-2.3268	-1.9328	1.8290	2.1334
RS_{uu}	-0.50	-2.0546	-1.7375	1.7025	2.0188	-2.0941	-1.7745	1.8133	2.1834
	-0.25	-2.0547	-1.7382	1.7004	2.0157	-2.0913	-1.7693	1.8057	2.1723
	0.00	-2.0517	-1.7355	1.7058	2.0195	-2.0865	-1.7674	1.8047	2.1709
	0.25	-2.0561	-1.7402	1.7017	2.0169	-2.0682	-1.7569	1.8018	2.1763
	0.50	-2.0502	-1.7354	1.7066	2.0214	-2.0616	-1.7545	1.8002	2.1772
MC	0.00	-2.0219	-1.6982	1.7281	2.0400	-2.0399	-1.7167	1.7464	2.0859
	$\lvert\lambda\rvert$	Rejection Frequencies							
ACR	0.00	0.0330	0.0625	0.0645	0.0320	0.0325	0.0570	0.0605	0.0315
	0.25	0.1995	0.3050	0.4120	0.2795	0.2745	0.3840	0.4850	0.3650
	0.50	0.5025	0.6295	0.8615	0.7280	0.6040	0.7055	0.8280	0.7475
RS_{rr}	0.00	0.0265	0.0565	0.0590	0.0250	0.0265	0.0500	0.0495	0.0230
	0.25	0.1755	0.2665	0.3585	0.2380	0.2515	0.3450	0.4350	0.3005
	0.50	0.4435	0.5765	0.8200	0.6850	0.5415	0.6565	0.7905	0.6945
RS_{ur}	0.00	0.0265	0.0540	0.0665	0.0345	0.0250	0.0465	0.0430	0.0200
	0.25	0.1810	0.2715	0.3870	0.2610	0.2530	0.3475	0.4255	0.2820
	0.50	0.4620	0.5955	0.8235	0.6840	0.5725	0.6730	0.7920	0.6865
RS_{ru}	0.00	0.0260	0.0560	0.0575	0.0270	0.0260	0.0510	0.0485	0.0230
	0.25	0.1765	0.2665	0.3585	0.2355	0.2515	0.3445	0.4300	0.2975
	0.50	0.4445	0.5810	0.8190	0.6875	0.5450	0.6530	0.7895	0.6970
RS_{uu}	0.00	0.0260	0.0545	0.0645	0.0340	0.0275	0.0490	0.0450	0.0205
	0.25	0.1785	0.2730	0.3840	0.2605	0.2580	0.3525	0.4315	0.2780
	0.50	0.4580	0.5960	0.8270	0.6845	0.5765	0.6820	0.7930	0.6835

L = Left tail ($\lambda = -0.25, -0.5$, in the rejection frequencies), R = Right tail ($\lambda = 0.25, 0.5$);
MC: Monte Carlo critical values based on $M = 30,000$;
Note: As $|\lambda|$ further increases from 0.5, the patterns on bootstrap critical values remain.

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Table 4.2a. Bootstrap Critical Values and Rejection Frequencies for $LM_{SED|\lambda}$, H_0: $\rho = 0$
Group Interaction with $g = n^{0.5}$, $\sigma = 1$, $n = 100$, $\lambda = 0.1/\sqrt{n}$, XVal-B

Method	ρ	Normal Mixture p=.1				Lognormal Error			
		L2.5%	L5%	U5%	U2.5%	L2.5%	L5%	U5%	U2.5%
		Bootstrap Critical Values							
RS_{rr}	-0.50	-2.0659	-1.8374	1.1455	1.5160	-2.0565	-1.8214	1.0966	1.4772
	-0.25	-2.0616	-1.8340	1.1401	1.5053	-2.0547	-1.8215	1.0888	1.4752
	0.00	-2.0568	-1.8332	1.1356	1.5028	-2.0519	-1.8186	1.0909	1.4712
	0.25	-2.0589	-1.8337	1.1405	1.5077	-2.0554	-1.8216	1.0909	1.4762
	0.50	-2.0721	-1.8414	1.1629	1.5257	-2.0662	-1.8306	1.1108	1.4835
RS_{ur}	-0.50	-2.0582	-1.8329	1.1383	1.5102	-2.0479	-1.8153	1.0911	1.4743
	-0.25	-2.0588	-1.8317	1.1377	1.5041	-2.0509	-1.8177	1.0892	1.4763
	0.00	-2.0547	-1.8309	1.1369	1.5059	-2.0514	-1.8176	1.0952	1.4752
	0.25	-2.0578	-1.8329	1.1400	1.5086	-2.0540	-1.8203	1.0919	1.4763
	0.50	-2.0635	-1.8376	1.1512	1.5173	-2.0559	-1.8239	1.0989	1.4758
RS_{ru}	-0.50	-2.0650	-1.8372	1.1468	1.5135	-2.0585	-1.8231	1.0973	1.4791
	-0.25	-2.0609	-1.8344	1.1405	1.5071	-2.0571	-1.8238	1.0924	1.4755
	0.00	-2.0566	-1.8336	1.1374	1.5066	-2.0543	-1.8209	1.0933	1.4746
	0.25	-2.0593	-1.8334	1.1397	1.5089	-2.0575	-1.8239	1.0917	1.4775
	0.50	-2.0707	-1.8400	1.1597	1.5258	-2.0643	-1.8276	1.1066	1.4821
RS_{uu}	-0.50	-2.0583	-1.8323	1.1395	1.5082	-2.0490	-1.8160	1.0903	1.4761
	-0.25	-2.0575	-1.8321	1.1373	1.5038	-2.0524	-1.8195	1.0925	1.4759
	0.00	-2.0548	-1.8315	1.1389	1.5078	-2.0529	-1.8188	1.0963	1.4775
	0.25	-2.0584	-1.8317	1.1397	1.5090	-2.0554	-1.8220	1.0925	1.4776
	0.50	-2.0631	-1.8364	1.1481	1.5172	-2.0552	-1.8206	1.0950	1.4746
MC	0.00	-2.0844	-1.8431	1.1522	1.5052	-2.0330	-1.8251	1.1176	1.4931
	$\|\rho\|$	Rejection Frequencies							
ACR	0.00	0.0325	0.0800	0.0160	0.0085	0.0345	0.0875	0.0105	0.0065
	0.25	0.0950	0.1960	0.1765	0.1225	0.1170	0.2070	0.1735	0.1255
	0.50	0.1835	0.3220	0.7190	0.6600	0.2015	0.3355	0.7155	0.6475
RS_{rr}	0.00	0.0225	0.0510	0.0495	0.0245	0.0295	0.0530	0.0505	0.0190
	0.25	0.0735	0.1300	0.3050	0.2045	0.0935	0.1500	0.2810	0.2040
	0.50	0.1500	0.2290	0.8060	0.7405	0.1710	0.2565	0.8120	0.7425
RS_{ur}	0.00	0.0210	0.0500	0.0530	0.0265	0.0280	0.0505	0.0525	0.0210
	0.25	0.0710	0.1295	0.3090	0.2095	0.0915	0.1490	0.2830	0.2075
	0.50	0.1505	0.2305	0.8060	0.7410	0.1705	0.2565	0.8080	0.7475
RS_{ru}	0.00	0.0230	0.0510	0.0490	0.0235	0.0285	0.0535	0.0495	0.0195
	0.25	0.0750	0.1335	0.3065	0.2030	0.0905	0.1530	0.2810	0.2015
	0.50	0.1535	0.2275	0.8045	0.7455	0.1700	0.2570	0.8145	0.7440
RS_{uu}	0.00	0.0215	0.0510	0.0525	0.0245	0.0275	0.0520	0.0505	0.0200
	0.25	0.0725	0.1325	0.3120	0.2065	0.0875	0.1500	0.2830	0.2030
	0.50	0.1525	0.2305	0.8050	0.7425	0.1655	0.2595	0.8120	0.7440

L = Left tail ($\rho = -0.25, -0.5$, in the rejection frequencies), R = Right tail ($\rho = 0.25, 0.5$);
MC: Monte Carlo critical values based on $M = 30,000$;
Note: With larger values of $|\lambda|$ and $|\rho|$, the patterns on bootstrap critical values remain.

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Table 4.2b. Bootstrap Critical Values and Rejection Frequencies for $\text{LM}_{\texttt{SLD}|\rho}$, H_0: $\lambda = 0$
`Group Interaction with` $g = n^{0.35}$`,` $\sigma = 1$, `Lognormal Error`, $\rho = 0$, `XVal-B`

		$n = 50$				$n = 100$			
Method	λ	L2.5%	L5%	U5%	U2.5%	L2.5%	L5%	U5%	U2.5%
		`Bootstrap Critical Values`							
$\texttt{RS}_{rr}$	-0.50	-2.2163	-1.8053	1.8734	2.2132	-2.0149	-1.6857	1.7067	2.0194
	-0.25	-2.1720	-1.7780	1.8583	2.1851	-1.9925	-1.6679	1.7084	2.0136
	0.00	-2.1445	-1.7667	1.8305	2.1475	-1.9791	-1.6602	1.7078	2.0114
	0.25	-2.0934	-1.7339	1.7962	2.1058	-1.9751	-1.6636	1.6908	1.9934
	0.50	-2.0616	-1.7289	1.7302	2.0399	-2.0075	-1.6982	1.6491	1.9585
$\texttt{RS}_{ur}$	-0.50	-2.0887	-1.7021	1.9351	2.2538	-1.9643	-1.6396	1.7435	2.0543
	-0.25	-2.0783	-1.6972	1.9318	2.2485	-1.9644	-1.6397	1.7368	2.0452
	0.00	-2.0826	-1.7047	1.9229	2.2365	-1.9632	-1.6372	1.7578	2.0766
	0.25	-2.0823	-1.6967	1.9384	2.2539	-1.9646	-1.6363	1.7552	2.0709
	0.50	-2.0851	-1.6959	1.9537	2.2734	-1.9714	-1.6404	1.7557	2.0743
$\texttt{RS}_{ru}$	-0.50	-2.0170	-1.6865	1.6840	1.9875	-1.9852	-1.6759	1.6733	1.9784
	-0.25	-2.0275	-1.6957	1.6855	1.9906	-1.9825	-1.6773	1.6698	1.9754
	0.00	-2.0394	-1.7041	1.6870	1.9892	-1.9806	-1.6752	1.6753	1.9798
	0.25	-2.0462	-1.7107	1.6864	1.9880	-1.9831	-1.6738	1.6706	1.9734
	0.50	-2.0501	-1.7168	1.6862	1.9886	-1.9782	-1.6719	1.6676	1.9765
$\texttt{RS}_{uu}$	-0.50	-2.0310	-1.7000	1.6856	1.9858	-1.9823	-1.6745	1.6720	1.9774
	-0.25	-2.0307	-1.6995	1.6879	1.9928	-1.9807	-1.6760	1.6695	1.9747
	0.00	-2.0332	-1.7009	1.6908	1.9972	-1.9807	-1.6747	1.6755	1.9796
	0.25	-2.0337	-1.6999	1.6901	1.9927	-1.9832	-1.6753	1.6711	1.9754
	0.50	-2.0360	-1.7015	1.6857	1.9900	-1.9803	-1.6742	1.6679	1.9781
`MC`	0.00	-2.0130	-1.6810	1.6749	1.9615	-2.0049	-1.6920	1.6561	1.9679
	$\lvert\lambda\rvert$	`Rejection Frequencies`							
`ACR`	0.00	0.0310	0.0605	0.0590	0.0295	0.0210	0.0450	0.0535	0.0265
	0.25	0.0985	0.1595	0.2880	0.1940	0.2810	0.4000	0.6480	0.5540
	0.50	0.1535	0.2420	0.7805	0.7015	0.5860	0.6980	0.9905	0.9865
$\texttt{RS}_{rr}$	0.00	0.0275	0.0595	0.0400	0.0240	0.0205	0.0490	0.0465	0.0225
	0.25	0.0950	0.1605	0.2420	0.1620	0.2805	0.4065	0.6375	0.5440
	0.50	0.1400	0.2490	0.7625	0.6785	0.5645	0.6755	0.9925	0.9885
$\texttt{RS}_{ur}$	0.00	0.0300	0.0615	0.0235	0.0120	0.0215	0.0510	0.0390	0.0190
	0.25	0.1010	0.1645	0.1755	0.0955	0.2905	0.4130	0.6235	0.5235
	0.50	0.1510	0.2585	0.7360	0.6415	0.5770	0.6990	0.9895	0.9835
$\texttt{RS}_{ru}$	0.00	0.0270	0.0525	0.0530	0.0280	0.0195	0.0435	0.0510	0.0260
	0.25	0.0910	0.1455	0.2760	0.1860	0.2730	0.3880	0.6395	0.5475
	0.50	0.1425	0.2270	0.7705	0.6930	0.5705	0.6925	0.9900	0.9850
$\texttt{RS}_{uu}$	0.00	0.0280	0.0500	0.0540	0.0255	0.0195	0.0450	0.0510	0.0250
	0.25	0.0895	0.1465	0.2735	0.1850	0.2735	0.3890	0.6385	0.5485
	0.50	0.1330	0.2235	0.7700	0.6925	0.5735	0.6920	0.9900	0.9850

L = Left tail ($\lambda = -0.25, -0.5$, in the rejection frequencies), R = Right tail ($\lambda = 0.25, 0.5$);
`MC`: Monte Carlo critical values based on $M = 30,000$;
Note: With larger values of $|\lambda|$ and $|\rho|$, the patterns on bootstrap critical values remain.

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Table 4.3a. Bootstrap Critical Values and Rejection Frequencies for $(\text{LM}_{\texttt{SED}}^{\texttt{OPG}})^2$, H_0: $\rho = 0$
Group Interaction with $g = n^{0.5}$, $\sigma = 1$, $n = 100$, XVal-B

Method	ρ	Normal Error 10%	Normal Error 5%	Normal Error 1%	Normal Mixture p=.1 10%	Normal Mixture p=.1 5%	Normal Mixture p=.1 1%	Lognormal Error 10%	Lognormal Error 5%	Lognormal Error 1%
		Bootstrap Critical Values								
RS_{rr}	-0.50	4.2298	5.4966	8.2242	3.8352	4.9103	7.1814	3.6285	4.6342	6.7905
	-0.25	4.2322	5.4930	8.2321	3.8212	4.8931	7.1986	3.6597	4.6578	6.7998
	0.00	4.2132	5.4697	8.1889	3.8456	4.9286	7.2311	3.6447	4.6357	6.7333
	0.25	4.2126	5.4794	8.2101	3.8145	4.8912	7.1815	3.6526	4.6255	6.6607
	0.50	4.1596	5.4268	8.1718	3.7960	4.8901	7.2334	3.6738	4.6882	6.8030
RS_{ur}	-0.50	4.2298	5.4966	8.2242	3.8352	4.9103	7.1814	3.6285	4.6342	6.7905
	-0.25	4.2322	5.4930	8.2321	3.8212	4.8931	7.1986	3.6597	4.6578	6.7998
	0.00	4.2132	5.4697	8.1889	3.8456	4.9286	7.2311	3.6447	4.6357	6.7333
	0.25	4.2126	5.4794	8.2101	3.8145	4.8912	7.1815	3.6526	4.6255	6.6607
	0.50	4.1596	5.4268	8.1718	3.7960	4.8901	7.2334	3.6738	4.6882	6.8030
RS_{ru}	-0.50	4.2415	5.5112	8.2479	3.8346	4.9157	7.2090	3.6582	4.6795	6.8856
	-0.25	4.2399	5.5112	8.2411	3.8242	4.9045	7.2117	3.6904	4.7142	6.9129
	0.00	4.2195	5.4733	8.1938	3.8518	4.9357	7.2513	3.6732	4.6837	6.8291
	0.25	4.1761	5.4447	8.1687	3.8068	4.8989	7.2295	3.7025	4.7252	6.8875
	0.50	3.9215	5.1613	7.8709	3.6826	4.8243	7.3023	3.6584	4.7710	7.1278
RS_{uu}	-0.50	4.2415	5.5112	8.2479	3.8346	4.9157	7.2090	3.6582	4.6795	6.8856
	-0.25	4.2399	5.5112	8.2411	3.8242	4.9045	7.2117	3.6904	4.7142	6.9129
	0.00	4.2195	5.4733	8.1938	3.8518	4.9357	7.2513	3.6732	4.6837	6.8291
	0.25	4.1761	5.4447	8.1687	3.8068	4.8989	7.2295	3.7025	4.7252	6.8875
	0.50	3.9215	5.1613	7.8709	3.6826	4.8243	7.3023	3.6584	4.7710	7.1278
MC	0.00	4.1710	5.4494	8.1923	3.8039	4.8905	7.2895	3.6219	4.8850	7.6153
		Rejection Frequencies								
ACR	-0.50	0.6815	0.4775	0.1525	0.6275	0.4050	0.1325	0.5725	0.3725	0.1390
	-0.25	0.4735	0.2890	0.0795	0.4365	0.2385	0.0595	0.4130	0.2500	0.0765
	0.00	0.2130	0.1060	0.0245	0.1940	0.0970	0.0175	0.2045	0.1055	0.0210
	0.25	0.1490	0.0785	0.0185	0.1715	0.0940	0.0225	0.1820	0.1030	0.0310
	0.50	0.5670	0.4640	0.3000	0.6220	0.5290	0.3525	0.6495	0.5500	0.3545
RS_{rr}	-0.50	0.4320	0.2590	0.0715	0.4300	0.2720	0.0970	0.4055	0.2575	0.1040
	-0.25	0.2580	0.1425	0.0315	0.2595	0.1490	0.0350	0.2655	0.1645	0.0505
	0.00	0.0900	0.0460	0.0125	0.1045	0.0585	0.0105	0.1090	0.0575	0.0115
	0.25	0.0705	0.0325	0.0085	0.1010	0.0635	0.0190	0.1295	0.0885	0.0400
	0.50	0.4520	0.3655	0.2170	0.5395	0.4625	0.3175	0.5775	0.5125	0.3705
RS_{uu}	-0.50	0.4285	0.2590	0.0730	0.4340	0.2715	0.0925	0.4015	0.2520	0.1000
	-0.25	0.2530	0.1400	0.0320	0.2570	0.1495	0.0340	0.2655	0.1590	0.0490
	0.00	0.0880	0.0460	0.0125	0.1030	0.0575	0.0090	0.1070	0.0570	0.0115
	0.25	0.0745	0.0395	0.0100	0.1015	0.0615	0.0155	0.1270	0.0780	0.0305
	0.50	0.4660	0.3890	0.2510	0.5485	0.4670	0.3150	0.5755	0.5025	0.3415

Heteroskedasticity = Group Size/Mean Group Size; MC: Monte Carlo critical values, $M = 30,000$.
Note: With larger values of $|\rho|$, the patterns on bootstrap critical values remain.

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Table 4.3b. Bootstrap Critical Values and Rejection Frequencies for $(\mathrm{LM}_{\mathtt{SLD}}^{\mathtt{OPG}})^2$, H_0: $\lambda = 0$
Group Interaction with $g = n^{0.5}$, $\sigma = 1$, $n = 100$, XVal-B

		Normal Error			Normal Mixture p=.1			Lognormal Error		
Method	λ	10%	5%	1%	10%	5%	1%	10%	5%	1%
		Bootstrap Critical Values								
$\mathtt{RS}_{rr}$	-0.50	3.3218	4.4780	6.9787	3.1414	4.1841	6.4259	3.0121	4.0061	6.1958
	-0.25	3.2023	4.3345	6.8421	3.0670	4.0927	6.3188	2.9522	3.9366	6.1067
	0.00	3.0961	4.2206	6.6973	2.9921	4.0159	6.2409	2.8685	3.8226	5.9170
	0.25	3.0072	4.1264	6.6152	2.9156	3.9301	6.1695	2.8350	3.7905	5.9133
	0.50	2.9384	4.0394	6.5423	2.9125	3.9795	6.3447	2.8590	3.8980	6.1988
$\mathtt{RS}_{ur}$	-0.50	3.1150	4.2507	6.7507	2.9970	4.0315	6.2846	2.8973	3.8830	6.0654
	-0.25	3.0802	4.2068	6.7125	2.9864	4.0090	6.2393	2.8846	3.8718	6.0433
	0.00	3.0740	4.1948	6.6778	2.9739	3.9993	6.2305	2.8532	3.8100	5.8966
	0.25	3.0755	4.2036	6.6860	2.9667	3.9842	6.2187	2.8712	3.8270	5.9550
	0.50	3.0790	4.2066	6.7140	3.0239	4.1018	6.4674	2.9509	3.9930	6.3094
$\mathtt{RS}_{ru}$	-0.50	3.3128	4.4618	6.9600	3.1317	4.1573	6.3814	2.9872	3.9463	6.0390
	-0.25	3.2066	4.3479	6.8424	3.0666	4.0924	6.3055	2.9465	3.9117	6.0485
	0.00	3.1008	4.2264	6.7197	2.9970	4.0243	6.2634	2.8766	3.8417	5.9520
	0.25	3.0091	4.1269	6.6262	2.9092	3.9255	6.1272	2.8268	3.7726	5.8845
	0.50	2.9215	4.0301	6.5383	2.8586	3.8790	6.1324	2.7740	3.7272	5.8283
$\mathtt{RS}_{uu}$	-0.50	3.1053	4.2360	6.7423	2.9834	4.0072	6.2420	2.8679	3.8246	5.9156
	-0.25	3.0839	4.2135	6.7113	2.9869	4.0053	6.2358	2.8798	3.8472	5.9867
	0.00	3.0764	4.2003	6.6942	2.9807	4.0097	6.2482	2.8612	3.8226	5.9321
	0.25	3.0801	4.2091	6.6968	2.9626	3.9806	6.1864	2.8615	3.8109	5.9199
	0.50	3.0703	4.1932	6.6813	2.9636	3.9823	6.2193	2.8505	3.8088	5.9028
MC	0.00	3.0778	4.1540	6.6321	2.9792	4.0013	6.3022	2.8502	3.8316	6.0753
		Rejection Frequencies								
ACR	-0.50	0.8550	0.7325	0.3905	0.8580	0.7345	0.4065	0.8695	0.7565	0.4815
	-0.25	0.5505	0.3945	0.1290	0.5475	0.3800	0.1215	0.5995	0.4395	0.1740
	0.00	0.1330	0.0600	0.0080	0.1220	0.0585	0.0070	0.1105	0.0445	0.0080
	0.25	0.4495	0.3215	0.1415	0.5070	0.3840	0.1850	0.5895	0.4745	0.2560
	0.50	0.9975	0.9920	0.9710	0.9930	0.9835	0.9560	0.9980	0.9940	0.9730
$\mathtt{RS}_{rr}$	-0.50	0.7935	0.6730	0.3575	0.8275	0.7215	0.4260	0.8605	0.7625	0.5205
	-0.25	0.4875	0.3445	0.1195	0.4960	0.3590	0.1450	0.5705	0.4395	0.2000
	0.00	0.1045	0.0500	0.0085	0.1040	0.0525	0.0100	0.1020	0.0460	0.0105
	0.25	0.4110	0.3110	0.1420	0.4815	0.3870	0.2155	0.5820	0.4975	0.3165
	0.50	0.9970	0.9920	0.9700	0.9920	0.9875	0.9605	0.9980	0.9960	0.9820
$\mathtt{RS}_{uu}$	-0.50	0.8165	0.7015	0.3865	0.8390	0.7290	0.4475	0.8660	0.7785	0.5550
	-0.25	0.4980	0.3615	0.1340	0.5055	0.3700	0.1500	0.5770	0.4570	0.2170
	0.00	0.1105	0.0520	0.0075	0.1050	0.0535	0.0115	0.1045	0.0465	0.0085
	0.25	0.4020	0.3030	0.1300	0.4790	0.3835	0.2120	0.5800	0.4900	0.3215
	0.50	0.9970	0.9905	0.9690	0.9930	0.9875	0.9655	0.9980	0.9960	0.9825

Heteroskedasticity = Group Size/Mean Group Size; MC: Monte Carlo critical values, $M = 30,000$.
Note: With larger values of $|\lambda|$, the patterns on bootstrap critical values remain.

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Table 4.3c. Bootstrap Critical Values and Rejection Frequencies for $LM^{\mathtt{OPG}}_{\mathtt{SARAR}}$, H_0: $\lambda = \rho = 0$
Group Interaction with $g = n^{0.5}$, $\sigma = 1$, $n = 50$, XVal-B

			Normal Error			Normal Mixture p=.1			Lognormal Error		
Method	λ	ρ	10%	5%	1%	10%	5%	1%	10%	5%	1%
			Bootstrap Critical Values								
$\mathtt{RS}_{rr}$	-0.50	-0.20	5.829	6.922	9.137	5.352	6.356	8.384	5.200	6.192	8.254
	-0.25	-0.10	5.809	6.894	9.080	5.375	6.370	8.375	5.158	6.141	8.152
	0.00	0.00	5.766	6.841	9.030	5.324	6.313	8.324	5.197	6.164	8.155
	0.25	0.10	5.791	6.888	9.088	5.399	6.412	8.450	5.281	6.278	8.319
	0.50	0.20	5.793	6.912	9.159	5.563	6.632	8.782	5.547	6.609	8.761
$\mathtt{RS}_{ur}$	-0.50	-0.20	5.815	6.904	9.121	5.340	6.341	8.363	5.196	6.187	8.260
	-0.25	-0.10	5.805	6.882	9.068	5.370	6.364	8.367	5.151	6.135	8.148
	0.00	0.00	5.764	6.839	9.030	5.322	6.311	8.315	5.193	6.161	8.146
	0.25	0.10	5.795	6.893	9.091	5.403	6.414	8.456	5.284	6.278	8.321
	0.50	0.20	5.801	6.925	9.171	5.571	6.644	8.797	5.561	6.626	8.770
$\mathtt{RS}_{ru}$	-0.50	-0.20	5.793	6.883	9.085	5.368	6.377	8.411	5.240	6.235	8.272
	-0.25	-0.10	5.806	6.900	9.090	5.421	6.428	8.482	5.254	6.251	8.308
	0.00	0.00	5.791	6.883	9.089	5.390	6.399	8.450	5.294	6.297	8.336
	0.25	0.10	5.796	6.887	9.096	5.412	6.433	8.485	5.276	6.280	8.347
	0.50	0.20	5.806	6.911	9.147	5.427	6.452	8.515	5.287	6.291	8.355
$\mathtt{RS}_{uu}$	-0.50	-0.20	5.779	6.866	9.069	5.356	6.358	8.389	5.230	6.223	8.250
	-0.25	-0.10	5.799	6.890	9.077	5.415	6.423	8.474	5.245	6.240	8.295
	0.00	0.00	5.790	6.880	9.087	5.388	6.394	8.447	5.289	6.292	8.326
	0.25	0.10	5.799	6.893	9.098	5.416	6.437	8.493	5.275	6.278	8.342
	0.50	0.20	5.816	6.920	9.152	5.436	6.458	8.529	5.294	6.297	8.357
MC	0.00	0.00	5.753	6.876	9.291	5.312	6.383	8.632	5.284	6.451	8.856
			Rejection Frequencies								
ACR	-0.50	-0.20	.6135	.3945	.0785	.6045	.3725	.0780	.6410	.4295	.1155
	-0.25	-0.10	.4000	.2135	.0235	.3915	.2015	.0250	.3840	.2160	.0325
	0.00	0.00	.2055	.0840	.0070	.1735	.0695	.0100	.1750	.0835	.0080
	0.25	0.10	.2815	.1695	.0385	.3325	.2015	.0425	.4715	.3260	.1175
	0.50	0.20	.8590	.7710	.5010	.8995	.8380	.6015	.9250	.8745	.7150
$\mathtt{RS}_{rr}$	-0.50	-0.20	.4090	.2610	.0785	.4810	.3290	.1195	.5515	.4030	.1685
	-0.25	-0.10	.2320	.1225	.0290	.2735	.1525	.0410	.3000	.1845	.0590
	0.00	0.00	.0965	.0495	.0100	.1080	.0575	.0165	.1100	.0580	.0130
	0.25	0.10	.1905	.1195	.0410	.2595	.1705	.0690	.4075	.3115	.1755
	0.50	0.20	.7810	.7000	.4990	.8515	.7975	.6460	.8920	.8525	.7495
$\mathtt{RS}_{uu}$	-0.50	-0.20	.4155	.2705	.0830	.4800	.3240	.1205	.5430	.3950	.1555
	-0.25	-0.10	.2330	.1275	.0275	.2640	.1480	.0400	.2865	.1750	.0510
	0.00	0.00	.0925	.0470	.0095	.1040	.0535	.0170	.1010	.0530	.0090
	0.25	0.10	.1870	.1175	.0470	.2565	.1705	.0695	.4015	.3080	.1735
	0.50	0.20	.7790	.6950	.4975	.8580	.8000	.6650	.8995	.8565	.7555

Heteroskedasticity = Group Size/Mean Group Size; MC: Monte Carlo critical values, $M = 30,000$.

Note: With larger values of $|\lambda|$ and $|\rho|$, the patterns on bootstrap critical values remain.