Spatial Econometric Models and Methods - Homework 1

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September 7, 2024

1. Consider the spatial linear regression (SLR) model with spatial lag and error (SLE) dependence: $Y_n = \lambda W_{1n} Y_n + X_n \beta + u_n$, $u_n = \rho W_{2n} u_n + \epsilon_n$. (Sec. 2.4, Lecture 2)

   (a) Give the detailed derivations of the quasi Gaussian log-likelihood $\ell_n(\theta)$ in (2.26).

   Solution: Let $A_n(\lambda) = I_n - \lambda W_{1n}$ and $B_n(\rho) = I_n - \rho W_{2n}$. The parameters of interest are summarized by $\theta = (\beta', \sigma^2, \lambda, \rho)'$ whose truth is $\theta_0 = (\beta_0', \sigma_0^2, \lambda_0, \rho_0)'$. And if without specifying a matrix depending on θ but actually it does, we mean the matrix is valued at the true value θ_0, for example, $A_n = A_n(\lambda_0)$, $B_n = B_n(\rho_0)$. So the SARAR model can be written as

$$Y_n = A_n^{-1} X_n \beta + A_n^{-1} B_n^{-1} \epsilon_n, \quad \epsilon_n \sim (0, \sigma_0^2 I_n) \tag{1}$$

   Note $\mathbb{E}(Y_n) = A_n^{-1} X_n \beta$ and $\mathrm{Var}(Y_n) = \sigma^2 A_n^{-1} B_n^{-1} B_n^{-1'} A_n^{-1'}$. Based on the mean and variance of Y_n, the quasi Gaussian log-likelihood for Y_n is

$$\begin{aligned}
\ell_n(\theta) &= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log[\det(\sigma^2 A_n^{-1} B_n^{-1} B_n^{-1'} A_n^{-1'})] \\
&\quad - \frac{1}{2}(Y_n - A_n^{-1} X_n \beta)' \sigma^{-2} A_n' B_n' B_n A_n (Y_n - A_n^{-1} X_n \beta) \\
&= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 + \log|B_n A_n| \\
&\quad - \frac{1}{2\sigma^2}(B_n A_n Y_n - B_n X_n \beta)'(B_n A_n Y_n - B_n X_n \beta) \\
&= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 + \log|A_n(\lambda)| + \log|B_n(\rho)| \\
&\quad - \frac{1}{2\sigma^2}[\mathbb{Y}_n(\delta) - \mathbb{X}_n(\rho)\beta]'[\mathbb{Y}_n(\delta) - \mathbb{X}_n(\rho)\beta]
\end{aligned} \tag{2}$$

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where $\delta = (\lambda, \rho)'$, $\mathbb{Y}_n(\delta) = B_n(\rho)A_n(\lambda)Y_n$, and $\mathbb{X}_n(\rho) = B_n(\rho)X_n$.

(b) Give the detailed derivations of the quasi score function $S_n(\theta)$ in (2.27).

Solution: for the quasi score function, by taking the partial derivative of (2) to different parameters, we have

$$\frac{\partial \ell_n(\theta)}{\partial \beta} = \frac{1}{2\sigma^2} \times 2\mathbb{X}_n'(\rho)[\mathbb{Y}_n(\delta) - \mathbb{X}_n(\rho)\beta] = \frac{1}{\sigma^2}\mathbb{X}_n'(\rho)\epsilon_n(\beta, \delta)$$

$$\frac{\partial \ell_n(\theta)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\epsilon_n'(\beta, \delta)\epsilon_n(\beta, \delta)$$

$$\begin{aligned}\frac{\partial \ell_n(\theta)}{\partial \lambda} &\overset{(i)}{=} \mathrm{tr}[A_n^{-1}(\lambda) \times (-W_{1n})] - \frac{1}{2\sigma^2} \times 2[B_n(\rho)(-W_{1n})Y_n]'\epsilon_n(\beta, \delta) \\ &\overset{(ii)}{=} \frac{1}{\sigma^2}Y_n'W_{1n}'B_n'(\rho)\epsilon_n(\beta, \delta) - \mathrm{tr}[F_n(\lambda)] \end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial \ell_n(\theta)}{\partial \rho} &\overset{(iii)}{=} \mathrm{tr}[B_n^{-1}(\rho) \times (-W_{2n})] - \frac{1}{2\sigma^2} \times 2[-W_{2n}A_n(\lambda)Y_n + W_{2n}X_n\beta]'\epsilon_n(\beta, \delta) \\ &\overset{(iv)}{=} \frac{1}{\sigma^2}\epsilon_n'(\beta, \delta)[G_n(\rho)B_n(\rho)A_n(\lambda)Y_n - G_n(\rho)B_n(\rho)X_n\beta] - \mathrm{tr}[G_n(\rho)] \\ &= \frac{1}{\sigma^2}\epsilon_n'(\beta, \delta)G_n(\rho)\epsilon_n(\beta, \delta) - \mathrm{tr}[G_n(\rho)] \end{aligned} \tag{4}$$

where $\epsilon_n(\beta, \lambda) = \mathbb{Y}_n(\delta) - \mathbb{X}_n(\rho)\beta$, $F_n(\lambda) = W_{1n}A_n^{-1}(\lambda)$ and $G_n(\rho) = W_{2n}B_n^{-1}(\rho)$. Note (i) and (iii) are due to the matrix differential formulae on Page 5 of the Lecture 2 slides, (ii) and (iv) are due to $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Finally,

$$S_n(\theta) = (\frac{\partial \ell_n(\theta)}{\partial \beta}, \frac{\partial \ell_n(\theta)}{\partial \sigma^2}, \frac{\partial \ell_n(\theta)}{\partial \lambda}, \frac{\partial \ell_n(\theta)}{\partial \rho})' \tag{5}$$

(c) Give the detailed derivations of the concentrated quasi score $S_n^c(\delta)$ in (2.31).

Solution: For computational and analytical convenience, we use the concentrated quasi-likelihood method by first concentrating out β and σ^2 for $\theta = (\beta', \sigma^2, \delta')'$. Letting $\frac{\partial \ell_n(\theta)}{\partial \beta} = 0$ and $\frac{\partial \ell_n(\theta)}{\partial \sigma^2} = 0$ gives the QMLEs of β and σ^2

$$\tilde{\beta}_n(\delta) = [\mathbb{X}_n'(\rho)\mathbb{X}_n(\rho)]^{-1}\mathbb{X}_n'(\rho)\mathbb{Y}_n(\delta) \tag{6}$$

$$\begin{aligned}\tilde{\sigma}_n^2(\delta) &= \frac{1}{n}\epsilon_n'(\tilde{\beta}_n(\delta), \delta)\epsilon_n(\tilde{\beta}_n(\delta), \delta) \\ &= \frac{1}{n}\{\mathbb{Y}_n(\delta) - \mathbb{X}_n(\rho)[\mathbb{X}_n'(\rho)\mathbb{X}_n(\rho)]^{-1}\mathbb{X}_n'(\rho)\mathbb{Y}_n(\delta)\}'\epsilon_n(\tilde{\beta}_n(\delta), \delta) \\ &= \frac{1}{n}[\mathbb{M}_n(\rho)\mathbb{Y}_n(\delta)]'[\mathbb{M}_n(\rho\mathbb{Y}_n(\delta)] \\ &= \frac{1}{n}\mathbb{Y}_n'(\delta)\mathbb{M}_n(\rho)\mathbb{Y}_n(\delta) \end{aligned} \tag{7}$$

where $\mathbb{M}_n(\rho) = I_n - \mathbb{X}_n(\rho)[\mathbb{X}_n'(\rho)\mathbb{X}_n(\rho)]^{-1}\mathbb{X}_n'(\rho)$ which is symmetric and idempotent. Substituting (6) and (7) back into the quasi log-likelihood function (2), we get the concentrated quasi log-likelihood for δ,

$$\ell_n^c(\delta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log[\tilde{\sigma}_n^2(\delta)] + \log|A_n(\lambda)| + \log|B_n(\rho)| - \frac{n}{2} \tag{8}$$

Maximizing (8) with respect to δ gives QMLE of δ. On the other hand, one can use the score method. Substituting (6) and (7) to (3) and (4) gives the concentrated quasi score function,

$$\begin{aligned}
\frac{\partial \ell_n^c(\delta)}{\partial \lambda} &= \frac{nY_n'W_{1n}'B_n'(\rho)\mathbb{M}_n(\rho)\mathbb{Y}_n(\delta)}{\mathbb{Y}_n'(\delta)\mathbb{M}_n(\rho)\mathbb{Y}_n(\delta)} - \mathrm{tr}[F_n(\lambda)] \\
&= \frac{n\mathbb{Y}_n'(\delta)\mathbb{M}_n(\rho)B_n(\rho)W_{1n}Y_n}{\mathbb{Y}_n'(\delta)\mathbb{M}_n(\rho)\mathbb{Y}_n(\delta)} - \mathrm{tr}[F_n(\lambda)] \\
&= \frac{n\mathbb{Y}_n'(\delta)\mathbb{M}_n(\rho)B_n(\rho)W_{1n}A_n^{-1}(\lambda)B_n^{-1}(\rho)B_n(\rho)A_n(\lambda)Y_n}{\mathbb{Y}_n'(\delta)\mathbb{M}_n(\rho)\mathbb{Y}_n(\delta)} - \mathrm{tr}[F_n(\lambda)] \\
&= \frac{n\mathbb{Y}_n'(\delta)\mathbb{M}_n(\rho)\bar{F}_n(\delta)\mathbb{Y}_n(\delta)}{\mathbb{Y}_n'(\delta)\mathbb{M}_n(\rho)\mathbb{Y}_n(\delta)} - \mathrm{tr}[F_n(\lambda)] \qquad (9)\\
\frac{\partial \ell_n^c(\delta)}{\partial \rho} &= \frac{n\mathbb{Y}_n'\mathbb{M}_nG_n(\rho)\mathbb{M}_n\mathbb{Y}_n}{\mathbb{Y}_n'(\delta)\mathbb{M}_n(\rho)\mathbb{Y}_n(\delta)} - \mathrm{tr}[G_n(\rho)] \qquad (10)\\
S_n^c(\delta) &= (\frac{\partial \ell_n^c(\delta)}{\partial \lambda}, \frac{\partial \ell_n^c(\delta)}{\partial \rho})' \qquad (11)
\end{aligned}$$

where $\bar{F}_n(\delta) = B_n(\rho)F_n(\lambda)B_n^{-1}(\rho)$.

Note maximizing ℓ_n^c is equivalent to solving $S_n^c(\delta) = 0$.

(d) Give the detailed derivations of the Hessian matrix $\mathcal{J}_n(\theta_0)$ in (2.33).

Solution: We notice that $\mathcal{J}_n(\theta_0) = -\mathbb{E}[\frac{\partial}{\partial\theta'}S_n(\theta_0)]$, henceforth by (5), the diagonal elements of $\mathcal{J}_n$ are

$$\begin{aligned}
\mathcal{J}_{n,11} &= -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial\beta'\partial\beta}]_{\theta=\theta_0} = \frac{1}{\sigma^2}\mathbb{X}_n'\mathbb{X}_n \\
\mathcal{J}_{n,22} &= -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial\sigma^{2'}\partial\sigma^2}]_{\theta=\theta_0} = -\frac{n}{2\sigma_0^4} + \frac{n}{\sigma_0^6}\frac{\epsilon_n'(\beta_0,\delta_0)\epsilon_n(\beta_0,\delta_0)}{n} = \frac{n}{2\sigma_0^4}
\end{aligned}$$

$$\mathcal{J}_{n,33} = -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial\lambda'\partial\lambda}]_{\theta=\theta_0} = \frac{1}{\sigma_0^2}\mathbb{E}[Y_n' W_{1n}' B_n'(\rho_0) B_n(\rho_0) W_{1n} Y_n] + \mathrm{tr}[\frac{\partial}{\partial\lambda} F_n(\lambda)]_{\lambda=\lambda_0}$$

$$\text{(Note } W_{1n}Y_n = F_n(\lambda)X_n\beta_0 + F_n(\lambda)B_n^{-1}\epsilon_n.\text{)}$$

$$= \frac{1}{\sigma_0^2}\beta_0' X_n' F_n'(\lambda_0) B_n'(\rho_0) B_n(\rho_0) F_n(\lambda_0) X_n \beta_0$$

$$+ \frac{1}{\sigma_0^2}\mathbb{E}\{\mathrm{tr}[\epsilon_n' B_n^{-1'} F_n' B_n' B_n F_n B_n^{-1}\epsilon_n]\} + \mathrm{tr}[F_n F_n]$$

$$= \mu_n'\mu_n + \frac{1}{\sigma_0^2}\mathbb{E}\{\mathrm{tr}[B_n F_n B_n^{-1}\epsilon_n\epsilon_n' B_n^{-1'} F_n' B_n']\} + \mathrm{tr}[F_n F_n]$$

$$= \mu_n'\mu_n + \mathrm{tr}[B_n F_n B_n^{-1} B_n^{-1'} F_n' B_n'] + \mathrm{tr}[F_n B_n^{-1} B_n F_n]$$

$$= \mu_n'\mu_n + \mathrm{tr}[\bar{F}_n \bar{F}_n'] + \mathrm{tr}[B_n F_n F_n B_n^{-1}]$$

$$= \mu_n'\mu_n + \mathrm{tr}[\bar{F}_n' \bar{F}_n] + \mathrm{tr}[B_n F_n B_n^{-1} B_n F_n B_n^{-1}]$$

$$= \mu_n'\mu_n + \mathrm{tr}[\bar{F}_n' \bar{F}_n] + \mathrm{tr}[\bar{F}_n \bar{F}_n]$$

$$= \mu_n'\mu_n + \mathrm{tr}(\bar{F}_n^s \bar{F}_n)$$

$$\mathcal{J}_{n,44} = -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial\rho'\partial\rho}]_{\theta=\theta_0}, \text{ note } \frac{\partial\epsilon_n(\rho)}{\partial\rho} = -W_{2n}B_n^{-1}(\rho)\epsilon_n$$

$$= -\frac{1}{\sigma_0^2}\mathbb{E}[\frac{\partial\epsilon_n(\rho)}{\partial\rho}' G_n(\rho)\epsilon_n(\rho) + \epsilon_n'(\rho)G_n(\rho)\frac{\partial\epsilon_n(\rho)}{\partial\rho} + \epsilon_n'(\rho)\frac{\partial[G_n(\rho)]}{\partial\rho}\epsilon_n(\rho)]_{\rho=\rho_0}$$

$$+ \mathrm{tr}[G_n G_n]$$

$$= \frac{1}{\sigma_0^2}\mathbb{E}[\epsilon_n' B_n^{-1'}(\rho) W_{2n}' G_n(\rho)\epsilon_n(\rho) + \epsilon_n'(\rho) G_n(\rho) W_{2n} B_n^{-1}(\rho)\epsilon_n]_{\rho=\rho_0}$$

$$- \frac{1}{\sigma_0^2}\mathbb{E}[\epsilon_n'(\rho)\frac{\partial G_n(\rho)}{\partial\rho}\epsilon_n(\rho)]_{\rho=\rho_0} + \mathrm{tr}[G_n G_n]$$

$$= \frac{1}{\sigma_0^2}\mathbb{E}[\epsilon_n' G_n'(\rho) G_n(\rho)\epsilon_n(\rho) + \epsilon_n'(\rho) G_n(\rho) G_n(\rho)\epsilon_n]_{\rho=\rho_0}$$

$$- \frac{1}{\sigma_0^2}\mathbb{E}[\epsilon_n'(\rho)(W_{2n}B_n^{-1}W_{2n}B^{-1})\epsilon_n(\rho)]_{\rho=\rho_0} + \mathrm{tr}[G_n G_n]$$

$$= \mathrm{tr}(G_n' G_n) + \mathrm{tr}(G_n G_n) - \mathrm{tr}(G_n G_n) + \mathrm{tr}(G_n G_n)$$

$$= \mathrm{tr}(G_n^s G_n)$$

where $\mu_n = \sigma_0^{-1} B_n F_n X_n \beta_0$ and $\bar{F}_n = B_n F_n B_n^{-1}$, $\bar{F}_n^s = \bar{F}_n + \bar{F}_n'$, $G_n^s = G_n + G_n'$. For the off-diagonal elements, we only calculate the up triangular area for $\mathcal{J}_n(\theta_0)$

is symmetric.

$$
\begin{aligned}
\mathcal{J}_{n,12} &= -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial \sigma^{2\prime}\partial \beta}]_{\theta=\theta_0} = -\mathbb{E}[-\frac{1}{\sigma_0^4}\mathbb{X}_n'(\rho_0)\epsilon_n(\beta_0,\delta_0)] = 0 \\
\mathcal{J}_{n,13} &= -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial \lambda'\partial \beta}]_{\theta=\theta_0} = -\mathbb{E}[\frac{1}{\sigma_0^2}\mathbb{X}_n' B_n(-W_{1n})Y_n] \\
&= \frac{1}{\sigma_0^2}\mathbb{X}_n' B_n W_{1n} A_n^{-1} X_n \beta_0 = \frac{1}{\sigma_0}\mathbb{X}_n' \mu_n \\
\mathcal{J}_{n,14} &= -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial \rho'\partial \beta}]_{\theta=\theta_0} = -\frac{1}{\sigma_0^2}\mathbb{E}[-X_n' W_{2n}' \epsilon_n + \mathbb{X}_n'(-W_{2n})B_n^{-1}\epsilon_n] = 0 \\
\mathcal{J}_{n,23} &= -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial \lambda'\partial \sigma^2}]_{\theta=\theta_0} = -\frac{1}{2\sigma_0^4}\mathbb{E}[2Y_n'(-W_{1n}')B_n'\epsilon_n] \\
&= \frac{1}{\sigma_0^4}\mathbb{E}[\epsilon_n' B_n^{-1\prime} A_n^{-1\prime} W_{1n}' B_n' \epsilon_n] = \frac{1}{\sigma_0^2}\mathrm{tr}(F_n) \\
\mathcal{J}_{n,24} &= -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial \rho'\partial \sigma^2}]_{\theta=\theta_0} = -\frac{1}{2\sigma_0^4}\mathbb{E}[2\epsilon_n'(-W_{2n}')B_n^{-1\prime}\epsilon_n] = \frac{1}{\sigma_0^2}\mathrm{tr}(G_n) \\
\mathcal{J}_{n,34} &= -\mathbb{E}[\frac{\partial^2 \ell_n(\theta)}{\partial \rho'\partial \lambda}]_{\theta=\theta_0} = \frac{1}{\sigma_0^2}\mathbb{E}[Y_n' W_{1n}' W_{2n}' \epsilon_n + Y_n' W_{1n}' B_n' W_{2n} B_n^{-1}\epsilon_n] \\
&= \frac{1}{\sigma_0^2}\mathbb{E}[\epsilon_n' B_n^{-1\prime} A_n^{-1\prime} W_{1n}' W_{2n}' \epsilon_n + \epsilon_n' B_n^{-1\prime} A_n^{-1\prime} W_{1n}' B_n' W_{2n} B_n^{-1}\epsilon_n] \\
&= \mathrm{tr}(W_{2n} F_n B_n^{-1}) + \mathrm{tr}(G_n' B_n F_n B_n^{-1}) \\
&= \mathrm{tr}(W_{2n} B_n^{-1} B_n F_n B_n^{-1}) + \mathrm{tr}(G_n' B_n F_n B_n^{-1}) \\
&= \mathrm{tr}(G_n \bar{F}_n) + \mathrm{tr}(G_n' \bar{F}_n) = \mathrm{tr}(G_n^s \bar{F}_n)
\end{aligned}
$$

Therefore,

$$
\mathcal{J}_n = \begin{pmatrix}
\frac{1}{\sigma^2}\mathbb{X}_n'\mathbb{X}_n & 0 & \frac{1}{\sigma_0}\mathbb{X}_n'\mu_n & 0 \\
\sim & \frac{n}{2\sigma_0^4} & \frac{1}{\sigma_0^2}\mathrm{tr}(F_n) & \frac{1}{\sigma_0^2}\mathrm{tr}(G_n) \\
\sim & \sim & \mu_n'\mu_n + \mathrm{tr}(\bar{F}_n^s \bar{F}_n) & \mathrm{tr}(G_n^s \bar{F}_n) \\
\sim & \sim & \sim & \mathrm{tr}(G_n^s G_n)
\end{pmatrix}
$$

(e) Give the detailed derivations of the information matrix $\mathcal{I}_n(\theta)$ in (2.34).

Solution: Here the well-known information matrix equality (IME) does not necessarily hold, that is $\mathcal{J}_n \neq \mathcal{I}_n$, because what we have done so far is based on "quasi" log-likelihood function but not the true log-likelihood function. If the (joint) distribution of ϵ_n is Gaussian shaped, then clearly there will be no difference between $\mathcal{I}_n$ and $\mathcal{J}_n$.

To clarify the notations, we define the following,

$$\epsilon_n = (\varepsilon_1, \cdots, \varepsilon_n)', \quad \varepsilon_i \overset{iid}{\sim} (0, \sigma_0^2), \text{ for } i = 1, \cdots, n$$
$$\gamma_0 = \mathbb{E}(\frac{\varepsilon_i}{\sigma_0})^3, \quad \kappa_0 = \mathbb{E}(\frac{\varepsilon_i}{\sigma_0})^4 - 3$$

We first calculate the diagonal entries of $\mathcal{I}_n - \mathcal{J}_n$.

$$\mathcal{I}_{n,11} - \mathcal{J}_{n,11} = \frac{1}{\sigma_0^4}\mathbb{E}[\mathbb{X}_n'\epsilon_n\epsilon_n'\mathbb{X}_n] - \frac{1}{\sigma_0^2}\mathbb{X}_n'\mathbb{X}_n = 0$$
$$\mathcal{I}_{n,22} - \mathcal{J}_{n,22} = \frac{n^2}{4\sigma_0^4} - \frac{n^2}{2\sigma_0^4} + \frac{1}{4\sigma_0^8}[n(\kappa_0+3)\sigma_0^4 + (n^2-n)\sigma_0^4] - \frac{n}{2\sigma_0^4} = \frac{n\kappa_0}{4\sigma_0^4}$$
$$\begin{aligned}\mathcal{I}_{n,33} - \mathcal{J}_{n,33} &= \mathbb{E}[\frac{1}{\sigma_0^4}(Y_n'W_{1n}'B_n'\epsilon_n)^2 - \frac{2}{\sigma_0^2}Y_n'W_{1n}'B_n'\epsilon_n\mathrm{tr}(F_n)] + [\mathrm{tr}(F_n)]^2 \\ &\quad - \mu_n'\mu - \mathrm{tr}(\bar{F}_n^s\bar{F}_n) \\ &= \frac{1}{\sigma_0^4}\mathbb{E}(\epsilon_n'\bar{F}_n\epsilon_n + \epsilon_n'\bar{F}_nB_nX_n\beta_0)^2 - [\mathrm{tr}(F_n)]^2 - \mu_n'\mu - \mathrm{tr}(\bar{F}_n^s\bar{F}_n) \\ &\overset{(v)}{=} \frac{1}{\sigma_0^4}\{\sigma_0^4\kappa_0\bar{f}_n'\bar{f}_n + \sigma_0^4\mathrm{tr}(\bar{F}_n\bar{F}_n^s) + \sigma_0^4[\mathrm{tr}(F_n)]^2 + 2\sigma_0^4\gamma_0\mu_n'\bar{f}_n + \sigma_0^4\mu_n'\mu_n\} \\ &\quad - [\mathrm{tr}(F_n)]^2 - \mu_n'\mu - \mathrm{tr}(\bar{F}_n^s\bar{F}_n) \\ &= \kappa_0\bar{f}_n'\bar{f}_n + 2\gamma_0\bar{f}_n'\mu_n\end{aligned}$$
$$\begin{aligned}\mathcal{I}_{n,44} - \mathcal{J}_{n,44} &= \mathbb{E}[\frac{1}{\sigma_0^4}(\epsilon_n'G_n\epsilon_n)^2 - \frac{2}{\sigma_0^2}\mathrm{tr}(G_n)\epsilon_n'G_n\epsilon_n] + [\mathrm{tr}(G_n)]^2 - \mathrm{tr}(G_n^sG_n) \\ &\overset{(vi)}{=} \kappa_0g_n'g_n + \mathrm{tr}(G_nG_n^s) + [\mathrm{tr}(G_n)]^2 - 2[\mathrm{tr}(G_n)]^2 + [\mathrm{tr}(G_n)]^2 - \mathrm{tr}(G_n^sG_n) \\ &= \kappa_0g_n'g_n\end{aligned}$$

where $\bar{f}_n = \mathrm{diagv}(\bar{F}_n)$, $g_n = \mathrm{diagv}(G_n)$ and (v) and (vi) are due to the Lemma 2.1.

For the off-diagonal entries of $\mathcal{I}_n - \mathcal{J}_n$, by intensively using Lemma 2.1.

$$\mathcal{I}_{n,12} - \mathcal{J}_{n,12} = \frac{1}{2\sigma_0^6}\mathbb{E}(\mathbb{X}_n'\epsilon_n\epsilon_n'\epsilon_n) = \frac{\gamma_0}{2\sigma_0^3}\mathbb{X}_n'\iota_n$$
$$\begin{aligned}\mathcal{I}_{n,13} - \mathcal{J}_{n,13} &= \frac{1}{\sigma_0^4}\mathbb{E}(Y_n'W_{1n}'B_n'\epsilon_n\mathbb{X}_n'\epsilon_n) - \frac{\mathbb{X}_n'\mu_n}{\sigma_0} \\ &= \frac{1}{\sigma_0^4}\mathbb{E}[(\epsilon_n'\bar{F}_n\epsilon_n + \sigma_0\epsilon_n'\mu_n)\mathbb{X}_n'\epsilon_n] - \frac{\mathbb{X}_n'\mu_n}{\sigma_0} \\ &= \frac{1}{\sigma_0^4}\mathbb{E}(\mathbb{X}_n'\epsilon_n\epsilon_n'\bar{F}_n\epsilon_n) = \frac{1}{\sigma_0^4}\mathbb{E}(\mathbb{X}_{n,\cdot 1}'\bar{F}_{11}^n\varepsilon_1^3 + \cdots + \mathbb{X}_{n,\cdot n}'\bar{F}_{nn}^n\varepsilon_n^3) \\ &= \frac{\gamma_0}{\sigma_0}(\mathbb{X}_{n,\cdot 1}'\bar{F}_{11}^n + \cdots + \mathbb{X}_{n,\cdot n}'\bar{F}_{nn}^n) = \frac{\gamma_0}{\sigma_0}\mathbb{X}_n'\bar{f}_n\end{aligned}$$

$$\mathcal{I}_{n,14} - \mathcal{J}_{n,14} = \frac{1}{\sigma_0^4}\mathbb{E}(\epsilon_n' G_n \epsilon_n \mathbb{X}_n' \epsilon_n) = \frac{\gamma_0}{\sigma_0}\mathbb{X}_n' g_n$$

$$\begin{aligned}
\mathcal{I}_{n,23} - \mathcal{J}_{n,23} &= \mathbb{E}(-\frac{n}{2\sigma_0^4}\epsilon_n' B_n F_n B_n^{-1}\epsilon_n + \frac{1}{2\sigma_0^6} Y_n' W_{1n}' B_n' \epsilon_n \epsilon_n' \epsilon_n) - \frac{1}{\sigma_0^2}\mathrm{tr}(F_n) \\
&= \frac{1}{2\sigma_0^6}\mathbb{E}(\epsilon_n' \bar{F}_n \epsilon_n \epsilon_n' \epsilon_n + \sigma_0 \mu_n' \epsilon_n \epsilon_n' \epsilon_n) - \frac{n}{2\sigma_0^2}\mathrm{tr}(F_n) - \frac{1}{\sigma_0^2}\mathrm{tr}(F_n) \\
&= \frac{1}{2\sigma_0^6}[\sigma_0^4 \kappa_0 \bar{f}_n' \iota_n + 2\sigma_0^4 \mathrm{tr}(F_n) + n\sigma_0^4 \mathrm{tr}(F_n) + \sigma_0^4 \gamma_0 \mu_n' \iota_n] \\
&\quad - \frac{n}{2\sigma_0^2}\mathrm{tr}(F_n) - \frac{1}{\sigma_0^2}\mathrm{tr}(F_n) \\
&= \frac{\kappa_0}{2\sigma_0^2}\mathrm{tr}(F_n) + \frac{\gamma_0}{2\sigma_0^2}\mu_n' \iota_n \\
\mathcal{I}_{n,24} - \mathcal{J}_{n,24} &= \frac{1}{2\sigma_0^6}\mathbb{E}[\epsilon_n' G_n \epsilon_n \epsilon_n' \epsilon_n] - \frac{n}{2\sigma_0^4}\mathbb{E}[\epsilon_n' G_n \epsilon_n] - \frac{1}{\sigma_0^2}\mathrm{tr}(G_n) \\
&= \frac{1}{2\sigma_0^6}[\sigma_0^4 \kappa_0 g_n' \iota_n + 2\sigma_0^4 \mathrm{tr}(G_n) + n\sigma_0^4 \mathrm{tr}(G_n)] - \frac{n}{2\sigma_0^2}\mathrm{tr}(G_n) - \frac{1}{\sigma_0^2}\mathrm{tr}(G_n) \\
&= \frac{\kappa_0}{2\sigma_0^2}\mathrm{tr}(G_n) \\
\mathcal{I}_{n,34} - \mathcal{J}_{n,34} &= \mathbb{E}[\frac{1}{\sigma_0^4} Y_n' W_{1n}' B_n' \epsilon_n \epsilon_n' G_n \epsilon_n - \frac{1}{\sigma_0^2}\epsilon_n' G_n \epsilon_n \mathrm{tr}(F_n)] - \mathrm{tr}(G_n^s \bar{F}_n) \\
&= \frac{1}{\sigma_0^4}\mathbb{E}[\epsilon_n' \bar{F}_n \epsilon_n \epsilon_n' G_n \epsilon_n + \sigma_0 \mu_n' \epsilon_n \epsilon_n' G_n \epsilon_n] - \mathrm{tr}(G_n)\mathrm{tr}(F_n) - \mathrm{tr}(G_n^s \bar{F}_n) \\
&= \frac{1}{\sigma_0^4}[\sigma_0^4 \kappa_0 \bar{f}_n' g_n + \sigma_0^4 \mathrm{tr}(\bar{F}_n G_n^s) + \sigma_0^4 \mathrm{tr}(F_n)\mathrm{tr}(G_n) + \sigma_0^4 \gamma_0 \mu_n' g_n] \\
&\quad - \mathrm{tr}(G_n)\mathrm{tr}(F_n) - \mathrm{tr}(G_n^s \bar{F}_n) \\
&= \kappa_0 g_n' \bar{f}_n + \gamma_0 g_n' \mu_n
\end{aligned}$$

where ι_n is an n-dimensional column vector of ones. Therefore

$$\mathcal{I}_n = \mathcal{J}_n + \begin{pmatrix} 0_{p\times p} & \frac{\gamma_0}{2\sigma_0^3}\mathbb{X}_n' \iota_n & \frac{\gamma_0}{\sigma_0}\mathbb{X}_n' \bar{f}_n & \frac{\gamma_0}{\sigma_0}\mathbb{X}_n' g_n \\ \sim & \frac{n\kappa_0}{4\sigma_0^4} & \frac{\kappa_0}{2\sigma_0^2}\mathrm{tr}(F_n) + \frac{\gamma_0}{2\sigma_0^2}\mu_n' \iota_n & \frac{\kappa_0}{2\sigma_0^2}\mathrm{tr}(G_n) \\ \sim & \sim & \kappa_0 \bar{f}_n' \bar{f}_n + 2\gamma_0 \bar{f}_n' \mu_n & \kappa_0 g_n' \bar{f}_n + \gamma_0 g_n' \mu_n \\ \sim & \sim & \sim & \kappa_0 g_n' g_n \end{pmatrix}$$

2. Consider the GMM estimation of SLR model with SLE (Sec. 2.4.2, Lecture 2).

   (a) Verify the expression for Σ_n given in (2.25).

Solution: Note

$$\epsilon_n(\vartheta) = B_n(\rho)A_n(\lambda)Y_n - B_n(\rho)X_n\beta, \quad \vartheta = (\beta', \lambda, \rho)'.$$

$$\boldsymbol{g}_n(\vartheta) = [\epsilon_n'(\vartheta)\mathbb{Q}_n, \epsilon_n'(\vartheta)P_{1n}\epsilon_n(\vartheta), \cdots, \epsilon_n'(\vartheta)P_{mn}\epsilon_n(\vartheta)]'$$

then

$$\begin{aligned}
-\mathbb{E}[\frac{\partial}{\partial\beta'}\boldsymbol{g}_n(\vartheta_0)] &= [\mathbb{X}_n'\mathbb{Q}_n, \underbrace{\mathbb{E}(\mathbb{X}_n'P_{1n}\epsilon_n + \epsilon_n'P_{1n}\mathbb{X}_n)}_{=0}, 0, \cdots, 0]' \\
-\mathbb{E}[\frac{\partial}{\partial\lambda}\boldsymbol{g}_n(\vartheta_0)] &= [(B_nW_{1n}A_n^{-1}X_n\beta_0)'\mathbb{Q}_n, \mathbb{E}((\bar{F}_n\epsilon_n)'P_{1n}\epsilon_n + \epsilon'P_{1n}\bar{F}_n\epsilon_n), \cdots]' \\
&= [(\bar{F}_nB_nX_n\beta_0)'\mathbb{Q}_n, \mathbb{E}(\epsilon_n'P_{1n}^s\bar{F}_n\epsilon_n), \cdots, \mathbb{E}(\epsilon_n'P_{mn}^s\bar{F}_n\epsilon_n)]' \\
&= [(\bar{F}_nB_nX_n\beta_0)'\mathbb{Q}_n, \sigma_0^2\text{tr}(P_{1n}^s\bar{F}_n), \cdots, \sigma_0^2\text{tr}(P_{mn}^s\bar{F}_n)]' \\
-\mathbb{E}[\frac{\partial}{\partial\rho}\boldsymbol{g}_n(\vartheta_0)] &= [\mathbb{E}(\epsilon_n'B_n^{-1'}W_{2n}'\mathbb{Q}_n), \mathbb{E}(\epsilon_n'B_n^{-1'}W_{2n}'P_{1n}\epsilon_n + \epsilon_n'P_{1n}W_{2n}B_n^{-1}\epsilon_n), \cdots]' \\
&= [\mathbb{E}(\epsilon_n'B_n^{-1'}W_{2n}'\mathbb{Q}_n), \mathbb{E}(\epsilon_n'P_{1n}'W_{2n}B_n^{-1}\epsilon_n + \epsilon_n'P_{1n}W_{2n}B_n^{-1}\epsilon_n), \cdots]' \\
&= [0, \sigma_0^2\text{tr}(P_{1n}^sG_n), \cdots, \sigma_0^2\text{tr}(P_{mn}^sG_n)]'
\end{aligned}$$

Therefore,

$$\Sigma_n = \begin{pmatrix} \mathbb{Q}_n'\mathbb{X}_n & \mathbb{Q}_n'\bar{F}_nB_nX_n\beta_0 & 0 \\ 0 & \sigma_0^2\text{tr}(P_{1n}^s\bar{F}_n) & \sigma_0^2\text{tr}(P_{1n}^sG_n) \\ \vdots & \vdots & \vdots \\ 0 & \sigma_0^2\text{tr}(P_{mn}^s\bar{F}_n) & \sigma_0^2\text{tr}(P_{mn}^sG_n) \end{pmatrix}$$

(b) Verify that Γ_n has an identical expression as that given in (2.24) for the SLR model with only spatial lag dependence.

Solution: Note $\mathbb{E}(\boldsymbol{g}_n(\vartheta)) = 0$. Therefore, by Lemma.2.1.,

$$\begin{aligned}
\Gamma_n &= \mathbb{E}(\boldsymbol{g}_n(\vartheta)\boldsymbol{g}_n'(\vartheta)) \\
&= \begin{pmatrix} \mathbb{E}(\mathbb{Q}_n'\epsilon_n\epsilon_n'\mathbb{Q}_n) & \sigma_0^3\gamma_0\mathbb{Q}_n'p_{1n} & \cdots & \sigma_0^3\gamma_0\mathbb{Q}_n'p_{mn} \\ \sigma_0^3\gamma_0p_{1n}'\mathbb{Q}_n & \sigma_0^4\kappa_0p_{1n}'p_{1n} + \sigma_0^4\text{tr}(P_{1n}P_{1n}^s) & \cdots & \sigma_0^4\kappa_0p_{1n}'p_{mn} + \sigma_0^4\text{tr}(P_{1n}P_{mn}^s) \\ & \sigma_0^4\kappa_0p_{2n}'p_{1n} + \sigma_0^4\text{tr}(P_{2n}P_{1n}^s) & \cdots & \sigma_0^4\kappa_0p_{2n}'p_{mn} + \sigma_0^4\text{tr}(P_{2n}P_{mn}^s) \\ & \vdots & \ddots & \vdots \\ \sigma_0^3\gamma_0p_{mn}'\mathbb{Q}_n & \sigma_0^4\kappa_0p_{mn}'p_{1n} + \sigma_0^4\text{tr}(P_{mn}P_{1n}^s) & \cdots & \sigma_0^4\kappa_0p_{mn}'p_{mn} + \sigma_0^4\text{tr}(P_{mn}P_{mn}^s) \end{pmatrix}
\end{aligned}$$

where $p_{ij} = \text{diagv}(P_{ij})$ and $P_{ij}^s = P_{ij} + P_{ij}'$ for $i, j = 1, \cdots, n$. Now we may write the above in a compact form, that is

$$\Gamma_n = \begin{pmatrix} \sigma_0^2 \mathbb{Q}_n' \mathbb{Q}_n & \sigma_0^3 \gamma_0 \mathbb{Q}_n' \omega_{nm} \\ \sigma_0^3 \gamma_0 \omega_{nm}' \mathbb{Q}_n & \sigma_0^4 (\kappa_0 \omega_{nm}' \omega_{nm} + \Lambda_{mn}) \end{pmatrix}$$

where $\omega_{nm} = (p_{1n}, p_{2n}, \cdots, p_{mn})$ is an $n \times m$ matrix, and $\Lambda_{mn} = (\text{tr}[P_{in} P_{jn}^s])_{i,j \in \{1, \cdots, m\}}$ is an $m \times m$ matrix which also depends on the number of units, n.

3. Consider the **Boston House Price** data considered in Sec. 2.4.4.

   (a) Extend the Matlab code provided for the QML estimation of SLR model with SLE, and fit an SLR model with SLE and spatial Durbin (SD) effects using covariates "crime" and "access".

   Solution: To extend the Matlab code, we modify the Matlab codes in Lab, "Boson_Qmle_SLE.m", by adding the Durbin term $W_n X_n^*$ into regressors where $W_{1n} = W_{2n} = W_n$ and $X_n^* = (\text{crime}, \text{access})$. The regression result is as follows.

```
Command Window
QMLE and inference for Boston Housing Data, SLE model with SD effect
Response: Median value of owner-occupied homes in $1000

                   QMLE     se_QMLE     t_QMLE
constant        28.3412     6.4010     4.4276
crime           -0.1473     0.0305    -4.8354
zoning           0.0393     0.0141     2.7832
industry        -0.0099     0.0731    -0.1355
charlesr        -0.5500     0.8793    -0.6255
noxsq          -18.8540     5.3074    -3.5524
rooms2           4.3384     0.3670    11.8210
houseage        -0.0270     0.0140    -1.9300
distance        -1.6544     0.3181    -5.2016
access           0.3216     0.0812     3.9605
taxrate         -0.0131     0.0036    -3.6501
ptratio         -0.6092     0.1533    -3.9731
blackpop         0.0105     0.0031     3.3664
lowclass        -0.4062     0.0533    -7.6225
durbin_crime    -0.1077     0.0812    -1.3273
durbin_access    0.0586     0.0912     0.6420
sigh^2          14.8575     1.7888     8.3060
SLD              0.0205     0.0755     0.2711
SED              0.6947     0.0576    12.0516
Elapsed time is 1.892461 seconds.
fx >>
```

Figure 1: QMLE results for SLE model with SD effects.

Based on the results, we find the spatial Durbin terms of "crime" and "access" (shows up as "durbin_crime" and "durbin_access") are insignificant. Spatial error dependence is strong while spatial lag dependence is insignificant.

(b) Extend the Matlab code in part (a) to implement the OGMM procedure outlined in Slide 41, Lecture 2 on the data.

Solution: Here I use just only two of instruments, say related to $P_{1n} = W_n$ and $P_{2n} = W_n^2 - \frac{1}{n}\text{tr}(W_n^2)I_n$. Again, the spatial Durbin terms are included. The results of feasible OGMM are as follows.

```
Command Window

Exiting: Maximum number of function evaluations has been exceeded
         - increase MaxFunEvals option.
         Current function value: 1059.917151

OGMM and inference for Boston Housing Data, SLE model with SD effect
Response: Median value of owner-occupied homes in $1000

               OGMM     se_QMLE    t_OGMM
constant     -0.1865   19.4713   -0.0096
crime        -0.0126    0.0595   -0.2116
zoning       -0.0011    0.0250   -0.0428
industry      0.0970    0.1375    0.7060
charlesr     -0.0523    1.5436   -0.0339
noxsq        -0.1795   10.9191   -0.0164
rooms2        0.0731    0.6143    0.1190
houseage     -0.3617    0.0256  -14.1444
distance     -0.2871    0.9215   -0.3115
access       -0.1038    0.1481   -0.7005
taxrate      -0.0226    0.0062   -3.6129
ptratio      -0.1486    0.2821   -0.5266
blackpop      0.0243    0.0056    4.3093
lowclass      0.0401    0.0931    0.4307
durbin_crime  0.1654    0.1758    0.9411
durbin_access-0.3224    0.2617   -1.2318
SLD          -0.2651    0.0977   -2.7128
SED           0.9817    0.0141   69.4387
Elapsed time is 18.247878 seconds.
fx >>
```

Figure 2: Feasible OGMM results for SLE model with SD effects.

(c) Compare the QML and OGMM procedures in terms of point estimates and the standard error estimates of an SLE model on Boston House Price data.

Solution: From the above (a) and (b), the two procedures disagree for a few estimates and their standard errors. One source of the inconsistency could be the iteration times of GMM procedure is not enough as we can find in Figure 2. Based on QMLE, the significant regressors are "crime", "zoning", "noxsq", "rooms2", "distance", "access", "taxrate", "ptratio", "blackpop", "lowclass". And only spatial error dependence is significant. Meanwhile, based on feasible OGMM, the significant explanatory variables are "houseage", "taxrate", "blackpop", and spatial lag dependence contains some energy in understanding housing price. Note that "houseage" is a new significant covariate in negative direction which is reasonable, and "taxrate" and "blackpop", the two elements maintains the direction and significance for explaining house price, which means the two variables are robust variables with explaining power.

4. Repeat Question 3, but using Python program language.

   Solution: This part is incomplete. See "Spatial_HW1_Q4.ipynb" in the submission.