A supplement to "Asymptotic Distributions of Quasi-Maximum Likelihood Estimators for Spatial Autoregressive Models" (for reference only; not for publication)

Appendix A: Some Useful Lemmas

A.1 Uniform Boundedness of Matrices in Row and Column Sums

Lemma A.1 *Suppose that the spatial weights matrix W_n is a non-negative matrix with its (i,j)th element being $w_{n,ij} = \frac{d_{ij}}{\sum_{l=1}^n d_{il}}$ and $d_{ij} \geq 0$ for all i, j.*

(1) If the row sums $\sum_{j=1}^n d_{ij}$ are bounded away from zero at the rate h_n uniformly in i, and the column sums $\sum_{i=1}^n d_{ij}$ are $O(h_n)$ uniformly in j, then $\{W_n\}$ are uniformly bounded in column sums.

(2) If $d_{ij} = d_{ji}$ for all i and j and the row sums $\sum_{j=1}^n d_{ij}$ are $O(h_n)$ and bounded away from zero at the rate h_n uniformly in i, then $\{W_n\}$ are uniformly bounded in column sums.

Proof: (1) Let c_1 and c_2 be positive constants such that $c_1 h_n \leq \sum_{j=1}^n d_{ij}$ for all i and $\sum_{i=1}^n d_{ij} \leq c_2 h_n$ for all j, for large n. It follows that $\sum_{i=1}^n w_{n,ij} = \sum_{i=1}^n \frac{d_{ij}}{\sum_{l=1}^n d_{il}} \leq \frac{1}{c_1 h_n} \sum_{i=1}^n d_{ij} \leq \frac{c_2}{c_1}$ for all i.

(2) This is a special case of (1) because $\sum_{l=1}^n d_{il} = O(h_n)$ and $\sum_{i=1}^n d_{ij} = \sum_{i=1}^n d_{ji}$ imply $\sum_{i=1}^n d_{ij} = O(h_n)$. Q.E.D.

Lemma A.2 *Suppose that* $\limsup_n \| \lambda_0 W_n \| < 1$, *where $\| \cdot \|$ is a matrix norm, then $\{\| S_n^{-1} \|\}$ is uniformly bounded in both row and column sums.*

Proof: For any matrix norm $\| \cdot \|$, $\| \lambda_0 W_n \| < 1$ implies that $S_n^{-1} = \sum_{k=0}^{\infty} (\lambda_0 W_n)^k$ (Horn and Johnson 1985, p.301). Let $c = \sup_n \| \lambda_0 W_n \|$. Then, $\| S_n^{-1} \| \leq \sum_{k=0}^{\infty} \| \lambda_0 W_n \|^k = \sum_{k=0}^{\infty} c^k = \frac{1}{1-c} < \infty$ for all n. Q.E.D.

Lemma A.3 *Suppose that $\{\| W_n \|\}$ and $\{\| S_n^{-1} \|\}$, where $\| \cdot \|$ is a matrix norm, are bounded. Then $\{\| S_n(\lambda)^{-1} \|\}$, where $S_n(\lambda) = I_n - \lambda W_n$, is uniformly bounded in a neighborhood of λ_0.*

Proof: Let c be a constant such that $\| W_n \| \leq c$ and $\| S_n^{-1} \| \leq c$ for all n. We note that $S_n^{-1}(\lambda) = (S_n - (\lambda - \lambda_0) W_n)^{-1} = S_n^{-1}(I_n - (\lambda - \lambda_0) G_n)^{-1}$, where $G_n = W_n S_n^{-1}$. By the submultiplicative property of a matrix norm, $\| G_n \| \leq \| W_n \| \cdot \| S_n^{-1} \| \leq c^2$ for all n.

Let $B_1(\lambda_0) = \{\lambda : |\lambda - \lambda_0| < 1/c^2\}$. It follows that, for any $\lambda \in B_1(\lambda_0)$, $\| (\lambda - \lambda_0) G_n \| \leq |\lambda - \lambda_0| \cdot \| G_n \| < 1$. As $\| (\lambda - \lambda_0) G_n \| < 1$, $I_n - (\lambda - \lambda_0) G_n$ is invertible and $(I_n - (\lambda - \lambda_0) G_n)^{-1} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k G_n^k$. Therefore, $\| (I_n - (\lambda - \lambda_0) G_n)^{-1} \| \leq \sum_{k=0}^{\infty} |\lambda - \lambda_0|^k \| G_n \|^k \leq \sum_{k=0}^{\infty} |\lambda - \lambda_0|^k c^{2k} = \frac{1}{1 - |\lambda - \lambda_0| c^2} < \infty$ for any $\lambda \in B_1(\lambda_0)$. The result follows by taking a close neighborhood $B(\lambda_0)$ contained in $B_1(\lambda_0)$. In $B(\lambda_0)$,

$\sup_{\lambda\in B(\lambda_0)} |\lambda-\lambda_0|c^2 < 1$, and, hence,

$$\sup_{\lambda\in B(\lambda_0)} \| S_n^{-1}(\lambda) \| \leq \| S_n^{-1} \| \cdot \sup_{\lambda\in B(\lambda_0)} \| (I_n-(\lambda-\lambda_0)G_n)^{-1} \| \leq \sup_{\lambda\in B(\lambda_0)} \frac{c}{1-|\lambda-\lambda_0|c^2} < \infty.$$

Q.E.D.

Lemma A.4 *Suppose that* $\| W_n \| \leq 1$ *for all* n*, where* $\| \cdot \|$ *is a matrix norm, then* $\{\| S_n(\lambda)^{-1} \|\}$*, where* $S_n(\lambda) = I_n - \lambda W_n$*, are uniformly bounded in any closed subset of* $(-1, 1)$*.*

Proof: For any $\lambda \in (-1, 1)$, $\| \lambda W_n \| \leq |\lambda| \cdot \| W_n \| < 1$ and, hence, $S_n^{-1}(\lambda) = \sum_{k=0}^{\infty} \lambda^k W_n^k$. It follows that, for any $|\lambda| < 1$, $\| S_n^{-1}(\lambda) \| \leq \sum_{k=0}^{\infty} |\lambda|^k \cdot \| W_n \|^k \leq \sum_{k=0}^{\infty} |\lambda|^k = \frac{1}{1-|\lambda|}$. Hence, for any closed subset B of $(-1, 1)$, $\sup_{\lambda\in B} \| S_n^{-1}(\lambda) \| \leq \sup_{\lambda\in B} \frac{1}{1-|\lambda|} < \infty$. Q.E.D.

Lemma A.5 *Suppose that elements of the* $n \times k$ *matrices* X_n *are uniformly bounded; and the limiting matrix of* $\frac{1}{n}X_n'X_n$ *exists and is nonsingular, then the projectors* M_n *and* $(I_n - M_n)$*, where* $M_n = I_n - X_n(X_n'X_n)^{-1}X_n'$*, are uniformly bounded in both row and column sums.*

Proof: Let $B_n = (\frac{1}{n}X_n'X_n)^{-1}$. From the assumption of the lemma, B_n converges to a finite limit. Therefore, there exists a constant c_b such that $|b_{n,ij}| \leq c_b$ for all n, where $b_{n,ij}$ is the (i, j)th element of B_n. By the uniform boundedness of X_n, there exists a constant c_x such that $|x_{n,ij}| \leq c_x$ for all n. Let $A_n = \frac{1}{n}X_n(\frac{X_n'X_n}{n})^{-1}X_n'$, which can be rewritten as $A_n = \frac{1}{n}\sum_{s=1}^{k}\sum_{r=1}^{k} b_{n,rs}x_{n,r}x_{n,s}'$, where $x_{n,r}$ is the rth column of X_n. It follows that $\sum_{j=1}^{n} |a_{n,ij}| \leq \sum_{j=1}^{n} \frac{1}{n}\sum_{s=1}^{k}\sum_{r=1}^{k} |b_{n,rs}x_{n,ir}x_{n,js}| \leq k^2c_bc_x^2$, for all $i = 1, \cdots, n$. Similarly, $\sum_{i=1}^{n} |a_{n,ij}| \leq \sum_{i=1}^{n} \frac{1}{n}\sum_{s=1}^{k}\sum_{r=1}^{k} |b_{n,rs}x_{n,ir}x_{n,js}| \leq k^2c_bc_x^2$ for all $j = 1, \cdots, n$. That is, $\{X_n(X_n'X_n)^{-1}X_n'\}$ are uniformly bounded in both row and column sums. Consequently, $\{M_n\}$ are also uniformly bounded in both row and column sums. Q.E.D.

A.2 Orders of Some Relevant Quantities

Lemma A.6 *Suppose that the elements of the sequences of vectors* $P_n = (p_{n1}, \cdots, p_{nn})'$ *and* $Q_n = (q_{n1}, \cdots, q_{nn})'$ *are uniformly bounded for all* n*.*

1) If $\{A_n\}$ *are uniformly bounded in either row or column sums, then* $|Q_n'A_nP_n| = O(n)$*.*

2) If the row sums of $\{A_n\}$ *and* $\{Z_n\}$ *are uniformly bounded,* $|z_{i,n}A_nP_n| = O(1)$ *uniformly in* i*, where* $z_{i,n}$ *is the* i*th row of* Z_n*.*

Proof: Let constants c_1 and c_2 such that $|p_{ni}| \leq c_1$ and $|q_{ni}| \leq c_2$. For 1), there exists a constant such that $\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{n,ij}| \leq c_3$. Hence, $|Q_n'A_nP_n| = |\sum_{i=1}^{n}\sum_{j=1}^{n} a_{n,ij}q_{ni}p_{nj}| \leq c_1c_2\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{n,ij}| \leq nc_1c_2c_3$. For 2), let c_4 be a constant such that $\sum_{j=1}^{n} |a_{n,ij}| \leq c_4$ for all n and i. It follows that $|e_{ni}'A_nP_n| = |\sum_{j=1}^{n} a_{n,ij}p_{nj}| \leq c_1\sum_{j=1}^{n} |a_{n,ij}| \leq c_1c_4$ where e_{ni} is the ith unit column vector. Because $\{Z_n\}$ is uniformly bounded in row sums, $\sum_{j=1}^n |z_{n,ij}| \le c_z$ for some constant c_z. It follows that $|z_{i,n} A_n P_n| \le \sum_{j=1}^n |z_{n,ij}| \cdot |e'_{nj} A_n P_n| \le (\sum_{j=1}^n |z_{n,ij}|) c_1 c_4 \le c_z c_1 c_4$. Q.E.D.

Lemma A.7 *Suppose $\{A_n\}$ are uniformly bounded either in row sums or in column sums. Then,*

1) elements $a_{n,ij}$ of A_n are uniformly bounded in i and j,

2) $tr(A_n^m) = O(n)$ for $m \ge 1$, and

3) $tr(A_n A'_n) = O(n)$.

Proof: If A_n is uniformly bounded in row sums, let c_1 be the constant such that $\max_{1 \le i \le n} \sum_{j=1}^n |a_{n,ij}| \le c_1$ for all n. On the other hand, if A_n is uniformly bounded in column sums, let c_2 be the constant such that $\max_{1 \le j \le n} \sum_{i=1}^n |a_{n,ij}| \le c_2$ for all n. Therefore, $|a_{n,ij}| \le \sum_{l=1}^n |a_{n,il}| \le c_1$ if A_n is uniformly bounded in row sums; otherwise, $|a_{n,ij}| \le \sum_{k=1}^n |a_{n,kj}| \le c_2$ if A_n is uniformly bounded in column sums. The result 1) implies immediately that $tr(A_n) = O(n)$. If A_n is uniformly bounded in row (column) sums, then A_n^m for $m \ge 2$ is uniformly bounded in row (column) sums. Therefore, 1) implies $tr(A_n^m) = O(n)$. Finally, as $tr(A_n A'_n) = \sum_{i=1}^n \sum_{j=1}^n a_{n,ij}^2$, $|tr(A_n A'_n)| \le \sum_{i=1}^n (\sum_{j=1}^n |a_{n,ij}|)^2 \le nc_1^2$ if A_n is uniformly bounded in row sums; otherwise $|tr(A_n A'_n)| \le \sum_{j=1}^n (\sum_{i=1}^n |a_{n,ij}|)^2 \le nc_2^2$. Q.E.D.

Lemma A.8 *Suppose that the elements $a_{n,ij}$ of the sequence of $n \times n$ matrices $\{A_n\}$, where $A_n = [a_{n,ij}]$, are $O(\frac{1}{h_n})$ uniformly in all i and j; and $\{B_n\}$ is a sequence of conformable $n \times n$ matrices.*

(1) If $\{B_n\}$ are uniformly bounded in column sums, the elements of $A_n B_n$ have the uniform order $O(\frac{1}{h_n})$.

(2) If $\{B_n\}$ are uniformly bounded in row sums, the elements of $B_n A_n$ have the uniform order $O(\frac{1}{h_n})$. For both cases (1) and (2), $tr(A_n B_n) = tr(B_n A_n) = O(\frac{n}{h_n})$.

Proof: Consider (1). Let $a_{n,ij} = \frac{c_{n,ij}}{h_n}$. Because $a_{n,ij} = O(\frac{1}{h_n})$ uniformly in i and j, there exists a constant c_1 so that $|c_{n,ij}| \le c_1$ for all i, j and n. Because $\{B_n\}$ is uniformly bounded in column sums, there exists a constant c_2 so that $\sum_{k=1}^n |b_{n,kj}| \le c_2$ for all n and j. Let $a_{i,n}$ be the ith row of A_n and $b_{n,l}$ be the lth column of B_n. It follows that $|a_{i,n} b_{n,l}| \le \frac{1}{h_n} \sum_{j=1}^n |c_{n,ij} b_{n,jl}| \le \frac{c_1}{h_n} \sum_{j=1}^n |b_{n,jl}| \le \frac{c_1 c_2}{h_n}$, for all i and l. Furthermore, $|tr(A_n B_n)| = |\sum_{i=1}^n a_{i,n} b_{n,i}| \le \sum_{i=1}^n |a_{i,n} b_{n,i}| \le c_1 c_2 \frac{n}{h_n}$. These prove the results in (1). The results in (2) follow from (1) because $(B_n A_n)' = A'_n B'_n$ and the uniform boundedness in row sums of $\{B_n\}$ is equivalent to the uniform boundedness in column sums of $\{B'_n\}$. Q.E.D.

Lemma A.9 *Suppose that A_n are uniformly bounded in both row and column sums. Elements of the $n \times k$ matrices X_n are uniformly bounded; and $\lim_{n \to \infty} \frac{X'_n X_n}{n}$ exists and is nonsingular. Let $M_n = I_n - X_n (X'_n X_n)^{-1} X'_n$. Then* *(i)* $tr(M_nA_n) = tr(A_n) + O(1)$,

(ii) $tr(A_n'M_nA_n) = tr(A_n'A_n) + O(1)$,

(iii) $tr[(M_nA_n)^2] = tr(A_n^2) + O(1)$, *and*

(iv) $tr[(A_n'M_nA_n)^2] = tr[(M_nA_nA_n')^2] = tr[(A_nA_n')^2] + O(1)$.

Furthermore, if $A_{n,ij} = O(\frac{1}{h_n})$ *for all* i *and* j*, then*

(a) $tr^2(M_nA_n) = tr^2(A_n) + O(\frac{n}{h_n})$ *and*

(b) $\sum_{i=1}^n ((M_nA_n)_{ii})^2 = \sum_{i=1}^n (A_{n,ii})^2 + O(\frac{1}{h_n})$.

Proof: The assumptions imply that elements of the $k \times k$ matrix $(\frac{1}{n}X_n'X_n)^{-1}$ are uniformly bounded for large enough n. Lemma A.6 implies that elements of the $k \times k$ matrices $\frac{1}{n}X_n'A_nX_n$, $\frac{1}{n}X_n'A_nA_n'X_n$ and $\frac{1}{n}X_n'A_n^2X_n$ are also uniformly bounded. It follows that

$$tr(M_nA_n) = tr(A_n) - tr[(X_n'X_n)^{-1}X_n'A_nX_n] = tr(A_n) + O(1),$$

$$tr(A_n'M_nA_n) = tr(A_n'A_n) - tr[(X_n'X_n)^{-1}X_n'A_nA_n'X_n] = tr(A_n'A_n) + O(1),$$

and $tr((M_nA_n)^2) = tr(A_n^2) - 2tr[(X_n'X_n)^{-1}X_n'A_n^2X_n] + tr([(X_n'X_n)^{-1}X_n'A_nX_n]^2) = tr(A_n^2) + O(1)$. By (iii), $tr[(A_n'M_nA_n)^2] = tr[(M_nA_nA_n')^2] = tr[(A_nA_n')^2] + O(1)$.

When $A_{n,ij} = O(\frac{1}{h_n})$, from (i), $tr^2(M_nA_n) = (tr(A_n) + O(1))^2 = tr^2(A_n) + 2tr(A_n) \cdot O(1) + O(1) = tr^2(A_n) + O(\frac{n}{h_n})$. Because A_n is uniformly bounded in column sums and elements of X_n are uniformly bounded, $X_n'A_ne_{ni} = O(1)$ for all i. Hence, $\sum_{i=1}^n (M_nA_n)_{ii}^2 = \sum_{i=1}^n (A_{n,ii} - x_{i,n}(X_n'X_n)^{-1}X_n'A_ne_{ni})^2 = \sum_{i=1}^n (A_{n,ii} + O(\frac{1}{n}))^2 = \sum_{i=1}^n [(A_{n,ii})^2 + 2A_{n,ii} \cdot O(\frac{1}{n}) + O(\frac{1}{n^2})] = \sum_{i=1}^n (A_{n,ii})^2 + O(\frac{1}{h_n})$. Q.E.D.

Lemma A.10 *Suppose that* A_n *is an* $n \times n$ *matrix with its column sums being uniformly bounded and elements of the* $n \times k$ *matrix* C_n *are uniformly bounded. Elements* $v_i's$ *of* $V_n = (v_1, \cdots, v_n)'$ *are i.i.d.*$(0, \sigma^2)$*. Then,* $\frac{1}{\sqrt{n}}C_n'A_nV_n = O_P(1)$*, Furthermore, if the limit of* $\frac{1}{n}C_n'A_nA_n'C_n$ *exists and is positive definite, then* $\frac{1}{\sqrt{n}}C_n'A_nV_n \xrightarrow{D} N(0, \sigma_0^2 \lim_{n\to\infty} \frac{1}{n}C_n'A_nA_n'C_n)$.

Proof: This is Lemma A.2 in Lee (2002). These results can be established by Chebyshev's inequality and Liapounov double array central limit theorem. Q.E.D.

A.3 First and Second Moments of Quadratic Forms and Limiting Distribution

For the lemmas in this subsection, v's in $V_n = (v_1, \cdots, v_n)'$ are i.i.d. with zero mean, variance σ^2 and finite fourth moment μ_4.

Lemma A.11 *Let* $A_n = [a_{ij}]$ *be an* n*-dimensional square matrix. Then*

1) $E(V_n'A_nV_n) = \sigma^2 tr(A_n)$*,*

2) $E(V_n'A_nV_n)^2 = (\mu_4 - 3\sigma^4)\sum_{i=1}^n a_{ii}^2 + \sigma^4[tr^2(A_n) + tr(A_nA_n') + tr(A_n^2)]$*, and*

3) $\text{var}(V_n'A_nV_n) = (\mu_4 - 3\sigma^4)\sum_{i=1}^n a_{ii}^2 + \sigma^4[tr(A_nA_n') + tr(A_n^2)]$*.*

In particular, if v*'s are normally distributed, then* $E(V_n'A_nV_n)^2 = \sigma^4[tr^2(A_n) + tr(A_nA_n') + tr(A_n^2)]$ *and* $\text{var}(V_n'A_nV_n) = \sigma^4[tr(A_nA_n') + tr(A_n^2)]$*.*

Proof: The result in 1) is trivial. For the second moment,

$$E(V_n'A_nV_n)^2 = E(\sum_{i=1}^n\sum_{j=1}^n a_{ij}v_iv_j)^2 = E(\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n a_{ij}a_{kl}v_iv_jv_kv_l).$$

Because v's are i.i.d. with zero mean, $E(v_iv_jv_kv_l)$ will not vanish only when $i=j=k=l$, $(i=j)\neq(k=l)$, $(i=k)\neq(j=l)$, and $(i=l)\neq(j=k)$. Therefore,

$$\begin{aligned} E(V_n'A_nV_n)^2 &= \sum_{i=1}^n a_{ii}^2E(v_i^4) + \sum_{i=1}^n\sum_{j\neq i}^n a_{ii}a_{jj}E(v_i^2v_j^2) + \sum_{i=1}^n\sum_{j\neq i}^n a_{ij}^2E(v_i^2v_j^2) + \sum_{i=1}^n\sum_{j\neq i}^n a_{ij}a_{ji}E(v_i^2v_j^2) \\ &= (\mu_4 - 3\sigma^4)\sum_{i=1}^n a_{ii}^2 + \sigma^4[\sum_{i=1}^n\sum_{j=1}^n a_{ii}a_{jj} + \sum_{i=1}^n\sum_{j=1}^n a_{ij}^2 + \sum_{i=1}^n\sum_{j=1}^n a_{ij}a_{ji}] \\ &= (\mu_4 - 3\sigma^4)\sum_{i=1}^n a_{ii}^2 + \sigma^4[tr^2(A_n) + tr(A_nA_n') + tr(A_n^2)]. \end{aligned}$$

The result 3) follows from $\text{var}(V_n'A_nV_n) = E(V_n'A_nV_n)^2 - E^2(V_n'A_nV_n)$ and those of 1) and 2). When v's are normally distributed, $\mu_4 = 3\sigma^2$. Q.E.D.

Lemma A.12 *Suppose that* $\{A_n\}$ *are uniformly bounded in either row and column sums, and the elements* $a_{n,ij}$ *of* A_n *are* $O(\frac{1}{h_n})$ *uniformly in all* i *and* j*. Then,* $E(V_n'A_nV_n) = O(\frac{n}{h_n})$*,* $\text{var}(V_n'A_nV_n) = O(\frac{n}{h_n})$ *and* $V_n'A_nV_n = O_P(\frac{n}{h_n})$*. Furthermore, if* $\lim_{n\to\infty}\frac{h_n}{n} = 0$*,* $\frac{h_n}{n}V_n'A_nV_n - \frac{h_n}{n}E(V_n'A_nV_n) = o_P(1)$*.*

Proof: $E(V_n'A_nV_n) = \sigma^2 tr(A_n) = O(\frac{n}{h_n})$. From Lemma A.11, the variance of $V_n'A_nV_n$ is $\text{var}(V_n'A_nV_n) = (\mu_4 - 3\sigma^4)\sum_{i=1}^n a_{n,ii}^2 + \sigma^4[tr(A_nA_n') + tr(A_n^2)]$. Lemma A.8 implies that $tr(A_n^2)$ and $tr(A_nA_n')$ are $O(\frac{n}{h_n})$. As $\sum_{i=1}^n a_{n,ii}^2 \leq tr(A_nA_n')$, it follows that $\sum_{i=1}^n a_{n,ii}^2 = O(\frac{n}{h_n})$. Hence, $\text{var}(V_n'A_nV_n) = O(\frac{n}{h_n})$. As $E((V_n'A_nV_n)^2) = \text{var}(V_n'A_nV_n) + E^2(V_n'A_nV_n) = O((\frac{n}{h_n})^2)$, the generalized Chebyshev inequality implies that $P(\frac{h_n}{n}|V_n'A_nV_n| \geq M) \leq \frac{1}{M^2}(\frac{h_n}{n})^2E((V_n'A_nV_n)^2) = \frac{1}{M^2}O(1)$ and, hence, $\frac{h_n}{n}V_n'A_nV_n = O_P(1)$.

Finally, because $\text{var}(\frac{h_n}{n}V_n'A_nV_n) = O(\frac{h_n}{n}) = o(1)$ when $\lim_{n\to\infty}\frac{h_n}{n} = 0$, the Chebyshev inequality implies that $\frac{h_n}{n}V_n'A_nV_n - \frac{h_n}{n}E(V_n'A_nV_n) = o_P(1)$. Q.E.D.

Lemma A.13 *Suppose that* $\{A_n\}$ *is a sequence of symmetric matrices with row and column sums uniformly bounded and* $\{b_n\}$ *is a sequence of constant vectors with its elements uniformly bounded. The moment* $E(|v|^{4+2\delta})$ *for some* $\delta > 0$ *of* v *exists. Let* $\sigma_{Q_n}^2$ *be the variance of* Q_n *where* $Q_n = b_n'V_n + V_n'A_nV_n -$

$\sigma^2 tr(A_n)$. *Assume that the variance* $\sigma^2_{Q_n}$ *is* $O(\frac{n}{h_n})$ *with* $\{\frac{h_n}{n}\sigma^2_{Q_n}\}$ *bounded away from zero, the elements of* A_n *are of uniform order* $O(\frac{1}{h_n})$ *and the elements of* b_n *of uniform order* $O(\frac{1}{\sqrt{h_n}})$. *If* $\lim_{n\to\infty} \frac{h_n^{1+\frac{2}{\delta}}}{n} = 0$, *then* $\frac{Q_n}{\sigma_{Q_n}} \xrightarrow{D} N(0,1)$.

Proof: The asymptotic distribution of the quadratic random form Q_n can be established via the martingale central limit theorem. Our proof of this Lemma follows closely the original arguments in Kelejian and Prucha (2001). In their paper, $\sigma^2_{Q_n}$ is assumed to be bounded away from zero with the n-rate. Our subsequent arguments modify theirs to take into account the different rate of $\sigma^2_{Q_n}$.

The Q_n can be expanded into $Q_n = \sum_{i=1}^n b_{ni}v_i + \sum_{i=1}^n a_{n,ii}v_i^2 + 2\sum_{i=1}^n \sum_{j=1}^{i-1} a_{n,ij}v_iv_j - \sigma^2 tr(A_n) = \sum_{i=1}^n Z_{ni}$, where $Z_{ni} = b_{ni}v_i + a_{n,ii}(v_i^2 - \sigma^2) + 2v_i \sum_{j=1}^{i-1} a_{n,ij}v_j$. Define σ-fields $\mathcal{J}_i =< v_1, \ldots, v_i >$ generated by $v_1, \ldots, v_i$. Because vs are i.i.d. with zero mean and finite variance, $E(Z_{ni}|\mathcal{J}_{i-1}) = b_{ni}E(v_i) + a_{n,ii}(E(v_i^2) - \sigma^2) + 2E(v_i)\sum_{j=1}^{i-1} a_{n,ij}v_j = 0$. The $\{(Z_{ni}, \mathcal{J}_i)|1 \le i \le n, 1 \le n\}$ forms a martingale difference double array. We note that $\sigma^2_{Q_n} = \sum_{i=1}^n E(Z_{ni}^2)$ as Z_{ni} are martingale differences. Also $\frac{h_n}{n}\sigma^2_{Q_n} = O(1)$. Define the normalized variables $Z^*_{ni} = Z_{ni}/\sigma_{Q_n}$. The $\{(Z^*_{ni}, \mathcal{J}_i)|1 \le i \le n\}$ is a martingale difference double array and $\frac{Q_n}{\sigma_{Q_n}} = \sum_{i=1}^n Z^*_{ni}$. In order for the martingale central limit theorem to be applicable, we would show that there exists a $\delta^* >$ such that $\sum_{i=1}^n E|Z^*_{ni}|^{2+\delta^*}$ tends to zero as n goes to infinity. Second, it will be shown that $\sum_{i=1}^n E(Z^{*2}_{ni}|\mathcal{J}_{i-1}) \xrightarrow{p} 1$.

For any positive constants p and q such that $\frac{1}{p} + \frac{1}{q} = 1$, $|Z_{ni}| \le |a_{n,ii}| \cdot |v_i^2 - \sigma^2| + |v_i|(|b_{ni}| + 2\sum_{j=1}^{i-1} |a_{n,ij}| \cdot |v_j|) = |a_{n,ii}|^{\frac{1}{p}}|a_{n,ii}|^{\frac{1}{q}}|v_i^2 - \sigma^2| + |v_i|(|b_{ni}|^{\frac{1}{p}}|b_{ni}|^{\frac{1}{q}} + 2\sum_{j=1}^{i-1}|a_{n,ij}|^{\frac{1}{p}}|a_{n,ij}|^{\frac{1}{q}}|v_j|)$. The Holder inequality for inner products applied to the last term implies that

$$|Z_{ni}|^q \le \left\{ \left[(|b_{ni}|^{\frac{1}{p}})^p + \sum_{j=1}^{i} (|a_{n,ij}|^{\frac{1}{p}})^p \right]^{\frac{1}{p}} \left[(|b_{ni}|^{\frac{1}{q}}|v_i|)^q + (|a_{n,ii}|^{\frac{1}{q}}|v_i^2 - \sigma^2|)^q + \sum_{j=1}^{i-1} (|a_{n,ij}|^{\frac{1}{q}} 2|v_i| \cdot |v_j|)^q \right]^{\frac{1}{q}} \right\}^q$$

$$= \left[|b_{ni}| + \sum_{j=1}^{i} |a_{n,ij}| \right]^{\frac{q}{p}} \left[|b_{ni}| \cdot |v_i|^q + |a_{n,ii}| \cdot |v_i^2 - \sigma^2|^q + \sum_{j=1}^{i-1} |a_{n,ij}| 2^q |v_i|^q |v_j|^q \right].$$

As $\{A_n\}$ are uniformly bounded in row sums and elements of b_n are uniformly bounded, there exists a constant c_1 such that $\sum_{j=1}^n |a_{n,ij}| \le c_1/2$ and $|b_{ni}| \le c_1/2$ for all i and n. Hence $|Z_{ni}|^q \le 2^q c_1^{\frac{q}{p}} (|b_{ni}| \cdot |v_i|^q + |a_{n,ii}| \cdot |v_i^2 - \sigma^2|^q + |v_i|^q \sum_{j=1}^{i-1} |a_{n,ij}||v_j|^q)$. Take $q = 2 + \delta$. Let $c_q > 1$ be a finite constant such that $E(|v|) \le c_q$, $E(|v|^q) \le c_q$ and $E(|v^2 - \sigma^2|^q) \le c_q$. Such a constant exists under the moment conditions of v. It follows that $\sum_{i=1}^n E|Z_{ni}|^q \le 2^q c_1^{\frac{q}{p}} c_q^2 \sum_{i=1}^n (|b_{ni}| + \sum_{j=1}^i |a_{n,ij}|) = O(n)$. As $\sum_{i=1}^n E|Z^*_{ni}|^{2+\delta} = \frac{1}{\sigma^{2+\delta}_{Q_n}} \sum_{i=1}^n E|Z_{ni}|^{2+\delta}$ and $\sigma^{2+\delta}_{Q_n} = (\frac{h_n}{n}\sigma^2_{Q_n})^{1+\frac{\delta}{2}} (\frac{n}{h_n})^{1+\frac{\delta}{2}} \ge c \cdot (\frac{n}{h_n})^{1+\frac{\delta}{2}}$ for some constant $c > 0$ when n is large, $\sum_{i=1}^n E|Z^*_{ni}|^{2+\delta} =$

$O(\frac{h_n^{1+\frac{\delta}{2}}}{n^{\frac{\delta}{2}}}) = O(\frac{h_n^{1+\frac{2}{\delta}}}{n})^{\frac{\delta}{2}}$, which goes to zero as n tends to infinity.

It remains to show that $\sum_{i=1}^{n} E(Z_{ni}^{*2}|\mathcal{J}_{i-1}) \xrightarrow{p} 0$. As

$$E(Z_{ni}^2|\mathcal{J}_{i-1}) = (\mu_4 - \sigma^4)a_{n,ii}^2 + \sigma^2(b_{ni} + 2\sum_{j=1}^{i-1} a_{n,ij}v_j)^2 + 2\mu_3 a_{n,ii}(b_{ni} + 2\sum_{j=1}^{i-1} a_{n,ij}v_j),$$

and $E(Z_{ni}^2) = (\mu_4 - \sigma^4)a_{n,ii}^2 + 4\sigma^4 \sum_{j=1}^{i-1} a_{n,ij}^2 + \sigma^2 b_{ni}^2 + 2\mu_3 a_{n,ii}b_{ni}$, because $E(b_{ni} + 2\sum_{j=1}^{i-1} a_{n,ij}v_j)^2 = b_{ni}^2 + 4\sigma^2 \sum_{j=1}^{i-1} a_{n,ij}^2$. Hence,

$$E(Z_{ni}^2|\mathcal{J}_{i-1}) - E(Z_{ni}^2) = 4\sigma^2(\sum_{j=1}^{i-1}\sum_{k\neq j}^{i-1} a_{n,ij}a_{n,ik}v_jv_k + \sum_{j=1}^{i-1} a_{n,ij}^2(v_j^2 - \sigma^2)) + 4(\sigma^2 b_{ni} + \mu_3 a_{n,ii})(\sum_{j=1}^{i-1} a_{n,ij}v_j)$$

and $\sum_{i=1}^{n} E(Z_{ni}^{*2}|\mathcal{J}_{i-1}) - 1 = \frac{1}{\sigma_{Q_n}^2}\sum_{i=1}^{n}[E(Z_{ni}^2|\mathcal{J}_{i-1}) - E(Z_{ni}^2)] = \frac{4\sigma^2}{\frac{h_n}{n}\sigma_{Q_n}^2}(H_{1n} + H_{2n}) + \frac{4}{\frac{h_n}{n}\sigma_{Q_n}^2}H_{3n}$, where

$$H_{1n} = \frac{h_n}{n}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{k\neq j}^{i-1} a_{n,ij}a_{n,ik}v_jv_k, \qquad H_{2n} = \frac{h_n}{n}\sum_{i=1}^{n}\sum_{j=1}^{i-1} a_{n,ij}^2(v_j^2 - \sigma^2),$$

and $H_{3n} = \frac{h_n}{n}\sum_{i=1}^{n}(\sigma^2 b_{ni} + \mu_3 a_{n,ii})\sum_{j=1}^{i-1} a_{n,ij}v_j$. We would like to show that H_{jn} for $j = 1, 2, 3$, converge in probability to zero. It is obvious that $E(H_{3n}) = 0$. By exchanging summations, $\sum_{i=1}^{n}(\sigma^2 b_{ni} + \mu_3 a_{n,ii})\sum_{j=1}^{i-1} a_{n,ij}v_j = \sum_{j=1}^{n-1}(\sum_{i=j+1}^{n}(\sigma^2 b_{ni} + \mu_3 a_{n,ii})a_{n,ij})v_j$. Thus,

$$E(H_{3n}^2) = \frac{\sigma^4 h_n^2}{n^2}\sum_{j=1}^{n-1}(\sum_{i=j+1}^{n}(b_{ni} + \frac{\mu_3}{\sigma^2}a_{n,ii})a_{n,ij})^2 \leq \frac{\sigma^4 h_n^2}{n^2}(\max_{1\leq i\leq n}|b_{ni} + \frac{\mu_3}{\sigma^2}a_{n,ii}|)^2\sum_{j=1}^{n-1}(\sum_{i=j+1}^{n}|a_{n,ij}|)^2 = O(\frac{h_n}{n})$$

because $\max_{i,j}|a_{n,ij}| = O(\frac{1}{h_n})$, $\max_i |b_{ni}| = O(\frac{1}{\sqrt{h_n}})$ and $\sum_{j=1}^{n-1}(\sum_{i=j+1}^{n}|a_{n,ij}|)^2 = O(n)$. $E(H_{2n}) = 0$ and H_{2n} can be rewritten into $H_{2n} = \frac{h_n}{n}\sum_{j=1}^{n-1}(\sum_{i=j+1}^{n} a_{n,ij}^2)(v_j^2 - \sigma^2)$. Thus

$$E(H_{2n}^2) = (\frac{h_n}{n})^2(\mu_4 - \sigma^4)\sum_{j=1}^{n-1}(\sum_{i=j+1}^{n} a_{n,ij}^2)^2 \leq (\frac{h_n}{n})^2(\mu_4 - \sigma^4)\max_{1\leq i,j\leq n}|a_{n,ij}|^2 \cdot \sum_{j=1}^{n-1}(\sum_{i=j+1}^{n}|a_{n,ij}|)^2 = O(\frac{1}{n}).$$

We conclude that $H_{3n} = o_P(1)$ and $H_{2n} = o_P(1)$. $E(H_{1n}) = 0$ but its variance is relatively more complex than that of H_{2n} and H_{3n}. By rearranging terms, $H_{1n} = \frac{h_n}{n}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{k\neq j}^{i-1} a_{n,ij}a_{n,ik}v_jv_k = \frac{h_n}{n}\sum_{j=1}^{n-1}\sum_{k\neq j}^{n-1}\bar{S}_{n,jk}v_jv_k$, where $\bar{S}_{n,jk} = \sum_{i=\max\{j,k\}+1}^{n} a_{n,ij}a_{n,ik}$. The variance of H_{1n} is

$$E(H_{1n}^2) = (\frac{h_n}{n})^2\sum_{j=1}^{n-1}\sum_{k\neq j}^{n-1}\sum_{r=1}^{n-1}\sum_{s\neq r}^{n-1}\bar{S}_{n,jk}\bar{S}_{n,rs}E(v_jv_kv_rv_s).$$

As $k \neq j$ and $s \neq r$, $E(v_jv_kv_rv_s) \neq 0$ only for the cases that $(j = r) \neq (k = s)$ and $(j = s) \neq (k = r)$. The variance of H_{1n} can be simplified and

$$\begin{aligned}
E(H_{1n}^2) &= 2\sigma^4(\frac{h_n}{n})^2\sum_{j=1}^{n-1}\sum_{k\neq j}^{n-1}\bar{S}_{n,jk}^2 \leq 2\sigma^4(\frac{h_n}{n})^2\sum_{j=1}^{n-1}\sum_{k\neq j}^{n-1}(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}|a_{n,i_1j}a_{n,i_1k}a_{n,i_2j}a_{n,i_2k}|) \\
&\leq 2\sigma^4(\frac{h_n}{n})^2\sum_{i_1=1}^{n}(\sum_{j=1}^{n}\sum_{k\neq j}^{n}|a_{n,i_1j}a_{n,i_1k}|)\cdot\max_{1\leq j\leq n}\sum_{i_2=1}^{n}|a_{n,i_2j}|\cdot\max_{i_2,k}|a_{n,i_2k}| \\
&\leq 2\sigma^4(\frac{h_n}{n})^2\max_{1\leq j\leq n}\sum_{i_2=1}^{n}|a_{n,i_2j}|\cdot\max_{i_2,k}|a_{n,i_2k}|\cdot\sum_{i_1=1}^{n}(\sum_{j=1}^{n}|a_{n,i_1j}|\cdot\sum_{k=1}^{n}|a_{n,i_1k}|) = O(\frac{h_n}{n}),
\end{aligned}$$

because A_n is uniformly bounded in row and column sums and $a_{n,ij} = O(\frac{1}{h_n})$ uniformly in i and j. Thus, $H_{1n} = o_P(1)$ as $\lim_{n\to\infty} \frac{h_n}{n} = 0$ implied by the condition $\lim_{n\to\infty} \frac{h_n^{(1+\frac{2}{\delta})}}{n} = 0$.

As H_{jn}, $j = 1, 2, 3$, are $o_P(1)$ and $\lim_{n\to\infty} \frac{h_n}{n}\sigma^2_{Q_n} > 0$, $\sum_{i=1}^n E(Z^{*2}_{ni}|\mathcal{J}_{i-1})$ converges in probability to 1. The central limit theorem for the martingale difference double array is thus applicable (see, Hall and Heyde, 1980; Potscher and Prucha, 1997) to establish the result. Q.E.D.

Lemma A.14 *Suppose that A_n is a constant $n \times n$ matrix uniformly bounded in both row and column sums. Let c_n be a column vector of constants. If $\frac{h_n}{n}c'_n c_n = o(1)$, then $\sqrt{\frac{h_n}{n}}c'_n A_n V_n = o_P(1)$. On the other hand, if $\frac{h_n}{n}c'_n c_n = O(1)$, then $\sqrt{\frac{h_n}{n}}c'_n A_n V_n = O_P(1)$.*

Proof: The first result follows from Chebyshev's inequality if $var(\sqrt{\frac{h_n}{n}}c'_n A_n V_n) = \sigma_0^2 \frac{h_n}{n} c'_n A_n A'_n c_n$ goes to zero. Let Λ_n be the diagonal matrix of eigenvalues of $A_n A'_n$ and Γ_n be the orthonormal matrix of eigenvectors. As eigenvalues in absolute values are bounded by any norm of the matrix, eigenvalues in Λ_n in absolute value are uniformly bounded because $\parallel A_n \parallel_\infty$ (or $\parallel A_n \parallel_1$) are uniformly bounded. Hence, $\frac{h_n}{n}c'_n A_n A'_n c_n \leq \frac{h_n}{n}c'_n \Gamma_n \Gamma'_n c_n \cdot |\lambda_{n,max}| = \frac{h_n}{n}c'_n c_n |\lambda_{n,max}| = o(1)$, where $\lambda_{n,max}$ is the eigenvalue of $A_n A'_n$ with the largest absolute value.

When $\frac{h_n}{n}c'_n c_n = O(1)$, $\frac{h_n}{n}c'_n A_n A'_n c_n \leq \frac{h_n}{n}c'_n c_n|\lambda_{n,max}| = O(1)$. In this case, $\text{var}(\sqrt{\frac{h_n}{n}}c'_n A_n V_n) = \sigma^2 \frac{h_n}{n}c'_n A_n A'_n c_n = O(1)$. Therefore, $\sqrt{\frac{h_n}{n}}c'_n A_n V_n = O_P(1)$. Q.E.D.

Appendix B: Detailed Proofs:

Proof of Consistency (Theorem 3.1 and Theorem 4.1)

We shall prove that $\frac{1}{n}\ln L_n(\lambda) - \frac{1}{n}Q_n(\lambda)$ converges in probability to zero uniformly on Λ, and the identification uniqueness condition holds, i.e., for any $\epsilon > 0$, $\limsup_{n\to\infty}[\max_{\lambda\in\bar{N}_\epsilon(\lambda_0)} \frac{1}{n}(Q_n(\lambda) - \frac{1}{n}Q_n(\lambda_0)] < 0$ where $\bar{N}_\epsilon(\lambda_0)$ is the complement of an open neighborhood of λ_0 in Λ with radius ϵ.

For the proof of these properties, it is useful to establish some properties for $\ln|S_n(\lambda)|$ and $\sigma_n^2(\lambda)$, where $\sigma_n^2(\lambda) = \frac{\sigma_o^2}{n}tr(S_n^{'-1}S'_n(\lambda)S_n(\lambda)S_n^{-1}) = \sigma_0^2[1 + 2(\lambda_0 - \lambda)\frac{1}{n}tr(G_n) + (\lambda_0 - \lambda)^2\frac{1}{n}tr(G_n G'_n)]$.

There is also an auxiliary model which has useful implications. Denote $Q_{p,n}(\lambda) = -\frac{n}{2}(\ln(2\pi) + 1) - \frac{n}{2}\ln\sigma_n^2(\lambda) + \ln|S_n(\lambda)|$. The log likelihood function of a SAR process $Y_n = \lambda W_n Y_n + V_n$, where $V_n \sim N(0, \sigma_0^2 I_n)$, is $\ln L_{p,n}(\lambda, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 + \ln|S_n(\lambda)| - \frac{1}{2\sigma^2}Y'_n S'_n(\lambda)S_n(\lambda)Y_n$. It is apparent that $Q_{p,n}(\lambda) = \max_{\sigma^2} E_p(\ln L_{p,n}(\lambda, \sigma^2))$, where E_p is the expectation under this SAR process. By the Jensen inequality, $Q_{p,n}(\lambda) \leq E_p(\ln L_{p,n}(\lambda_0, \sigma_0^2)) = Q_{p,n}(\lambda_0)$ for all λ. This implies that $\frac{1}{n}(Q_{p,n}(\lambda) - Q_{p,n}(\lambda_0)) \leq 0$ for all λ.

Let λ_1 and λ_2 be in Λ. By the mean value theorem, $\frac{1}{n}(\ln|S_n(\lambda_2)|-\ln|S_n(\lambda_1)|) = \frac{1}{n}tr(W_nS_n^{-1}(\bar{\lambda}_n))(\lambda_2-\lambda_1)$ where $\bar{\lambda}_n$ lies between λ_1 and λ_2. By the uniform boundedness of Assumption 7, Lemma A.8 implies that $\frac{1}{n}tr(W_nS^{-1}(\bar{\lambda}_n)) = O(\frac{1}{h_n})$. Thus, $\frac{1}{n}\ln|S_n(\lambda)|$ is uniformly equicontinuous in λ in Λ. As Λ is a bounded set, $\frac{1}{n}(\ln|S_n(\lambda_2)|-\ln|S_n(\lambda_1)|) = O(1)$ uniformly in λ_1 and λ_2 in Λ.

The $\sigma_n^2(\lambda)$ is uniformly bounded away from zero on Λ. This can be established by a counter argument. Suppose that $\sigma_n^2(\lambda)$ were not uniformly bounded away from zero on Λ. Then, there would exist a sequence $\{\lambda_n\}$ in Λ such that $\lim_{n\to\infty}\sigma_n^2(\lambda_n)=0$. We have shown that $\frac{1}{n}(Q_{p,n}(\lambda)-Q_{p,n}(\lambda_0)) \leq 0$ for all λ, and $\frac{1}{n}(\ln|S_n(\lambda_0)|-\ln|S_n(\lambda)|) = O(1)$ uniformly on Λ. This implies that $-\frac{1}{2}\ln\sigma_n^2(\lambda) \leq -\frac{1}{2}\ln\sigma_0^2+\frac{1}{n}(\ln|S_n(\lambda_0)|-\ln|S_n(\lambda)|) = O(1)$. That is, $-\ln\sigma_n^2(\lambda_n)$ is bounded from above, a contradiction. Therefore, $\sigma_n^2(\lambda)$ must be bounded always from zero uniformly on Λ.

(uniform convergence) Show that $\sup_{\lambda\in\Lambda}|\frac{1}{n}\ln L_n(\lambda)-\frac{1}{n}Q_n(\lambda)| = o_P(1)$.

Note that $\frac{1}{n}\ln L_n(\lambda)-\frac{1}{n}Q_n(\lambda) = -\frac{1}{2}(\ln\hat{\sigma}_n^2(\lambda)-\ln\sigma_n^{*2}(\lambda))$. Because $M_nS_n(\lambda)Y_n = (\lambda_0-\lambda)M_nG_nX_n\beta_0+M_nS_n(\lambda)S_n^{-1}V_n$,

$$\hat{\sigma}_n^2(\lambda) = \frac{1}{n}Y_n'S_n'(\lambda)M_nS_n(\lambda)Y_n = (\lambda_0-\lambda)^2\frac{1}{n}(G_nX_n\beta_0)'M_n(G_nX_n\beta_0)+2(\lambda_0-\lambda)H_{1n}(\lambda)+H_{2n}(\lambda), \quad (B.1)$$

where $H_{1n}(\lambda) = \frac{1}{n}(G_nX_n\beta_0)'M_nS_n(\lambda)S_n^{-1}V_n$ and $H_{2n}(\lambda) = \frac{1}{n}V_n'S_n^{'-1}S_n'(\lambda)M_nS_n(\lambda)S_n^{-1}V_n$. As $H_{1n}(\lambda) = \frac{1}{n}(G_nX_n\beta_0)'M_nV_n + (\lambda_0-\lambda)\frac{1}{n}(G_nX_n\beta_0)'M_nG_nV_n$, Lemma A.10 and the linearity of $H_{1n}(\lambda)$ in λ imply $H_{1n}(\lambda) = o_P(1)$ uniformly in $\lambda\in\Lambda$. Note that

$$H_{2n}(\lambda)-\sigma_n^2(\lambda) = \frac{1}{n}V_n'S_n^{'-1}S_n'(\lambda)S_n(\lambda)S_n^{-1}V_n - \frac{\sigma_0^2}{n}tr(S_n^{'-1}S_n'(\lambda)S_n(\lambda)S_n^{-1}) - T_n(\lambda),$$

where $T_n(\lambda) = \frac{1}{n}V_n'S_n^{'-1}S_n'(\lambda)X_n(X_n'X_n)^{-1}X_n'S_n(\lambda)S_n^{-1}V_n$. By Lemma A.10,

$$\frac{1}{\sqrt{n}}X_n'S_n(\lambda)S_n^{-1}V_n = \frac{1}{\sqrt{n}}X_n'S_n^{-1}V_n - \lambda\frac{1}{\sqrt{n}}X_n'G_nV_n = O_p(1).$$

Thus, $T_n(\lambda) = \frac{1}{n}(\frac{1}{\sqrt{n}}X_n'S_n(\lambda)S_n^{-1}V_n)'(\frac{X_n'X_n}{n})^{-1}(\frac{1}{\sqrt{n}}X_n'S_n(\lambda)S_n^{-1}V_n)' = o_P(1)$. By Lemma A.12,

$$\frac{1}{n}[V_n'S_n^{'-1}S_n'(\lambda)S_n(\lambda)S_n^{-1}V_n - \sigma_0^2tr(S_n^{'-1}S_n'(\lambda)S_n(\lambda)S_n^{-1})] = o_P(1)$$

uniformly in $\lambda\in\Lambda$. These convergences are uniform on Λ because λ appears simply as linear or quadratic factors in those terms. That is, $H_{2n}(\lambda)-\sigma_n^2(\lambda) = o_P(1)$ uniformly on Λ. Therefore, $\hat{\sigma}_n^2(\lambda)-\sigma_n^{*2}(\lambda) = o_P(1)$ uniformly on Λ. By the Taylor expansion, $|\ln\hat{\sigma}_n^2(\lambda)-\ln\sigma_n^{*2}(\lambda)| = |\hat{\sigma}_n^2(\lambda)-\sigma_n^{*2}(\lambda)|/\tilde{\sigma}_n^2(\lambda)$, where $\tilde{\sigma}_n^2(\lambda)$ lies between $\hat{\sigma}_n^2(\lambda)$ and $\sigma_n^{*2}(\lambda)$. Note that $\sigma_n^{*2}(\lambda) \geq \sigma_n^2(\lambda)$ because $\sigma_n^{*2}(\lambda) = (\lambda_0-\lambda)^2\frac{1}{n}(G_nX_n\beta_0)'M_n(G_nX_n\beta_0)+$

$\sigma_n^2(\lambda)$. As $\sigma_n^2(\lambda)$ is uniformly bounded away from zero on Λ, $\sigma_n^{*2}(\lambda)$ will be so too. It follows that, because $\hat{\sigma}_n^2(\lambda) - \sigma_n^{*2}(\lambda) = o_P(1)$ uniformly on Λ, $\hat{\sigma}_n^2(\lambda)$ will be bounded away from zero uniformly on Λ in probability. Hence, $|\ln \hat{\sigma}_n^2(\lambda) - \ln \sigma_n^{*2}(\lambda)| = o_P(1)$ uniformly on Λ. Consequently, $\sup_{\lambda \in \Lambda} |\frac{1}{n} \ln L_n(\lambda) - \frac{1}{n} Q_n(\lambda)| = o_P(1)$.

(uniform equicontinuity) We will show that $\frac{1}{n} \ln Q_n(\lambda) = -\frac{1}{2}(\ln(2\pi) + 1) - \frac{1}{2} \ln \sigma_n^{*2}(\lambda) + \frac{1}{n} \ln |S_n(\lambda)|$ is uniformly equicontinuous on Λ. The $\sigma_n^{*2}(\lambda)$ is uniformly continuous on Λ. This is so, because $\sigma_n^{*2}(\lambda)$ is a quadratic form of λ and its coefficients, $\frac{1}{n}(G_n X_n \beta_0)' M_n (G_n X_n \beta_0)$, $\frac{1}{n} tr(G_n)$ and $\frac{1}{n} tr(G_n' G_n)$ are bounded by Lemmas A.6 and A.8. The uniform continuity of $\ln \sigma_n^{*2}(\lambda)$ on Λ follows because $\frac{1}{\sigma_n^{*2}(\lambda)}$ is uniformly bounded on Λ. Hence $\frac{1}{n} \ln Q_n(\lambda)$ is uniformly equicontinuous on Λ.

(identification uniqueness) At λ_0, $\sigma_n^{*2}(\lambda_0) = \sigma_0^2$. Therefore,

$$
\begin{aligned}
\frac{1}{n} Q_n(\lambda) - \frac{1}{n} Q_n(\lambda_0) &= -\frac{1}{2}(\ln \sigma_n^2(\lambda) - \ln \sigma_0^2) + \frac{1}{n}(\ln |S_n(\lambda)| - \ln |S_n(\lambda_0)|) - \frac{1}{2}[\ln \sigma_n^*(\lambda) - \ln \sigma_n^2(\lambda)] \\
&= \frac{1}{n}(Q_{p,n}(\lambda) - Q_{p,n}(\lambda_0)) - \frac{1}{2}[\ln \sigma_n^{*2}(\lambda) - \ln \sigma_n^2(\lambda)].
\end{aligned}
$$

Suppose that the identification uniqueness condition would not hold. Then, there would exist an $\epsilon > 0$ and a sequence λ_n in $\bar{N}_\epsilon(\lambda_0)$ such that $\lim_{n\to\infty}[\frac{1}{n} Q_n(\lambda_n) - \frac{1}{n} Q_n(\lambda_0)] = 0$. Because $\bar{N}_\epsilon(\lambda_0)$ is a compact set, there would exist a convergent subsequence $\{\lambda_{n_m}\}$ of $\{\lambda_n\}$. Let λ_+ be the limit point of λ_{n_m} in Λ. As $\frac{1}{n} Q_n(\lambda)$ is uniformly equicontinuous in λ, $\lim_{n_m\to\infty}[\frac{1}{n_m} Q_{n_m}(\lambda_+) - \frac{1}{n_m} Q_{n_m}(\lambda_0)] = 0$. Because $(Q_{p,n}(\lambda) - Q_{p,n}(\lambda_0)) \leq 0$ and $-[\ln \sigma_n^{*2}(\lambda) - \ln \sigma_n^2(\lambda)] \leq 0$, this is possible only if $\lim_{n_m\to\infty}(\sigma_{n_m}^{*2}(\lambda_+) - \sigma_{n_m}^2(\lambda_+)) = 0$ and $\lim_{n_m\to\infty}(\frac{1}{n_m} Q_{p,n_m}(\lambda_+) - \frac{1}{n_m} Q_{p,n_m}(\lambda_0)) = 0$. The $\lim_{n_m\to\infty}(\sigma_{n_m}^{*2}(\lambda_+) - \sigma_{n_m}^2(\lambda_+)) = 0$ is a contradiction when $\lim_{n\to\infty} \frac{1}{n}(G_n X_n \beta_0)' M_n (G_n X_n \beta_0) \neq 0$. In the event that $\lim_{n\to\infty} \frac{1}{n}(G_n X_n \beta_0)' M_n (G_n X_n \beta_0) = 0$, the contradiction follows from the relation $\lim_{n\to\infty}(\frac{1}{n} Q_{p,n}(\lambda_+) - \frac{1}{n} Q_{p,n}(\lambda_0)) = 0$ under Assumption 9. This is so, because, in this event, Assumption 9 is equivalent to that $\lim_{n\to\infty}[\frac{1}{n}(\ln |S_n(\lambda)| - \ln |S_n|) - \frac{1}{2}(\ln \sigma_n^2(\lambda) - \ln \sigma_0^2)] = \lim_{n\to\infty} \frac{1}{n}[Q_{p,n}(\lambda) - Q_{p,n}(\lambda_0)] \neq 0$ for $\lambda \neq \lambda_0$. Therefore, the identification uniqueness condition must hold.

The consistency of $\hat{\lambda}_n$ and, hence, $\hat{\theta}_n$ follow from this identification uniqueness and uniform convergence (White 1994, Theorem 3.4). Q.E.D.

Proof of Theorem 3.2

(Show that Σ_θ is nonsingular): Let $\alpha = (\alpha_1', \alpha_2, \alpha_3)'$ be a column vector of constants such that $\Sigma_\theta \alpha = 0$. It is sufficient to show that $\alpha = 0$. From the first row block of the linear equation system $\Sigma_\theta \alpha = 0$, one has $\lim_{n\to\infty} \frac{X_n' X_n}{n} \alpha_1 + \lim_{n\to\infty} \frac{X_n' G_n X_n \beta_0}{n} \alpha_2 = 0$ and, therefore, $\alpha_1 = -\lim_{n\to\infty}(X_n' X_n)^{-1} X_n' G_n X_n \beta_0 \cdot \alpha_2$. From the last equation of the linear system, one has $\alpha_3 = -2\sigma_0^2 \lim_{n\to\infty} \frac{tr(G_n)}{n} \cdot \alpha_2$. By eliminating α_1 and

α_3, the remaining equation becomes

$$\left\{ \lim_{n\to\infty} \frac{1}{n\sigma_0^2}(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) + \lim_{n\to\infty} \frac{1}{n}\left[tr(G_n'G_n) + tr(G_n^2) - \frac{2}{n}tr^2(G_n)\right]\right\}\alpha_2 = 0. \qquad (B.2)$$

Because $tr(G_nG_n') + tr(G_n^2) - \frac{2}{n}tr^2(G_n) = \frac{1}{2}tr[(C_n' + C_n)(C_n' + C_n)'] \geq 0$ and Assumption 8 implies that $\lim_{n\to\infty} \frac{1}{n}(G_nX_n\beta_0)'M_n(G_nX_n\beta_0)$ is positive definite, it follows that $\alpha_2 = 0$ and, so, $\alpha = 0$.

(the limiting distribution of $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(\theta_0)}{\partial\theta}$): The matrix G_n is uniformly bounded in row sums. As the elements of X_n are bounded, the elements of $G_nX_n\beta_0$ for all n are uniformly bounded by Lemma A.6. With the existence of high order moments of v in Assumption 1, the central limit theorem for quadratic forms of double arrays of Kelejian and Prucha (2001) can be applied and the limiting distribution of the score vector follows.

(Show that $\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde\theta_n)}{\partial\theta\partial\theta'} - \frac{1}{n}\frac{\partial^2 \ln L_n(\theta_0)}{\partial\theta\partial\theta'} \xrightarrow{p} 0$): The second-order derivatives are

$$\frac{\partial^2 \ln L_n(\theta)}{\partial\beta\partial\beta'} = -\frac{1}{\sigma^2}X_n'X_n, \quad \frac{\partial^2 \ln L_n(\theta)}{\partial\beta\partial\lambda} = -\frac{1}{\sigma^2}X_n'W_nY_n, \quad \frac{\partial^2 \ln L_n(\theta)}{\partial\beta\partial\sigma^2} = -\frac{1}{\sigma^4}X_n'V_n(\delta),$$

$$\frac{\partial^2 \ln L_n(\theta)}{\partial\lambda^2} = -tr([W_nS_n^{-1}(\lambda)]^2) - \frac{1}{\sigma^2}Y_n'W_n'W_nY_n, \quad \frac{\partial^2 \ln L_n(\theta)}{\partial\sigma^2\partial\lambda} = -\frac{1}{\sigma^4}Y_n'W_n'V_n(\delta),$$

and $\frac{\partial^2 \ln L_n(\theta)}{\partial\sigma^2\partial\sigma^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}V_n'(\delta)V_n(\delta)$. As $\frac{X_n'X_n}{n} = O(1)$, $\frac{X_n'W_nY_n}{n} = O_P(1)$ and $\tilde\sigma_n^2 \xrightarrow{p} \sigma_0^2$, it follows that

$$\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde\theta_n)}{\partial\beta\partial\beta'} - \frac{1}{n}\frac{\partial^2 \ln L_n(\theta_0)}{\partial\beta\partial\beta'} = (\frac{1}{\sigma_0^2} - \frac{1}{\tilde\sigma_n^2})\frac{X_n'X_n}{n} = o_p(1),$$

and

$$\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde\theta_n)}{\partial\beta\partial\lambda} - \frac{1}{n}\frac{\partial^2 \ln L_n(\theta_0)}{\partial\beta\partial\lambda} = (\frac{1}{\sigma_0^2} - \frac{1}{\tilde\sigma_n^2})\frac{X_n'W_nY_n}{n} = o_p(1).$$

As $V_n(\tilde\delta_n) = Y_n - X_n\tilde\beta_n - \tilde\lambda_nW_nY_n = X_n(\beta_0 - \tilde\beta_n) + (\lambda_0 - \tilde\lambda_n)W_nY_n + V_n$,

$$\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde\theta_n)}{\partial\beta\partial\sigma^2} - \frac{1}{n}\frac{\partial^2 \ln L_n(\theta_0)}{\partial\beta\partial\sigma^2} = (\frac{1}{\sigma_0^4} - \frac{1}{\tilde\sigma_n^4})\frac{X_n'V_n}{n} + \frac{X_n'X_n}{n\tilde\sigma_n^4}(\tilde\beta_n - \beta_0) + \frac{X_n'W_nY_n}{\tilde n\sigma_n^4}(\tilde\lambda_n - \lambda_0) = o_p(1).$$

Let $G_n(\lambda) = W_nS_n^{-1}(\lambda)$. By the mean value theorem, $tr(G_n^2(\tilde\lambda_n)) = tr(G_n^2) + 2tr(G_n^3(\bar\lambda_n)) \cdot (\tilde\lambda_n - \lambda_0)$, therefore, $\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde\theta_n)}{\partial\lambda^2} - \frac{1}{n}\frac{\partial^2 \ln L_n(\theta_0)}{\partial\lambda^2} = -2\frac{tr(G_n^3(\bar\lambda_n))}{n}(\tilde\lambda_n - \lambda_0) + (\frac{1}{\sigma_0^2} - \frac{1}{\tilde\sigma_n^2})\frac{Y_n'W_n'W_nY_n}{n} = o_p(1)$, because $tr(G_n^3(\bar\lambda_n)) = O(\frac{n}{h_n})$ and $Y_n'W_n'W_nY_n = O_P(\frac{n}{h_n})$. Note that $G_n(\bar\lambda_n)$ is uniformly bounded in row and column sums uniformly in a neighborhood of λ_0 by Lemma A.3 under Assumption 5. Therefore, $tr(G_n^3(\bar\lambda_n)) = O(\frac{n}{h_n})$. On the other hand, because

$$\frac{1}{n}Y_n'W_n'V_n(\tilde\delta_n) = \frac{Y_n'W_n'X_n}{n}(\beta_0 - \tilde\beta_n) + (\lambda_0 - \tilde\lambda_n)\frac{Y_n'W_n'W_nY_n}{n} + \frac{Y_n'W_n'V_n}{n} = \frac{Y_n'W_n'V_n}{n} + o_P(1)$$

and

$$\frac{1}{n}V_n'(\tilde{\delta}_n)V_n(\tilde{\delta}_n) = (\tilde{\beta}_n - \beta_0)'\frac{X_n'X_n}{n}(\tilde{\beta}_n - \beta_0) + (\tilde{\lambda}_n - \lambda_0)^2\frac{Y_n'W_n'W_nY_n}{n} + \frac{V_n'V_n}{n}$$
$$+ 2(\tilde{\lambda}_n - \lambda_0)(\tilde{\beta}_n - \beta_0)'\frac{X_n'W_nY_n}{n} + 2(\beta_0 - \tilde{\beta}_n)'\frac{X_n'V_n}{n} + 2(\lambda_0 - \tilde{\lambda}_n)\frac{Y_n'W_n'V_n}{n}$$
$$= \frac{V_n'V_n}{n} + o_P(1),$$

it follows that

$$\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde{\theta}_n)}{\partial\sigma^2\partial\lambda} - \frac{1}{n}\frac{\partial^2 \ln L_n(\theta_0)}{\partial\sigma^2\partial\lambda} = -\frac{Y_n'W_n'V_n(\tilde{\delta}_n)}{\tilde{\sigma}_n^4 n} + \frac{Y_n'W_n'V_n}{\sigma_0^4 n}$$
$$= \frac{Y_n'W_n'X_n}{\tilde{\sigma}_n^4 n}(\tilde{\beta}_n - \beta_0) + \frac{Y_n'W_n'W_nY_n}{\tilde{\sigma}_n^4 n}(\tilde{\lambda}_n - \lambda_0) + (\frac{1}{\sigma_0^4} - \frac{1}{\tilde{\sigma}_n^4})\frac{Y_n'W_n'V_n}{n} = o_p(1),$$

and

$$\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde{\theta}_n)}{\partial\sigma^2\partial\sigma^2} - \frac{1}{n}\frac{\partial^2 \ln L_n(\theta_0)}{\partial\sigma^2\partial\sigma^2} = \frac{1}{2\tilde{\sigma}_n^4} - \frac{V_n'(\tilde{\delta}_n)V_n(\tilde{\delta}_n)}{n\tilde{\sigma}_n^6} - \frac{1}{2\sigma_0^4} + \frac{V_n'V_n}{n\sigma_0^6}$$
$$= \frac{1}{2}(\frac{1}{\tilde{\sigma}_n^4} - \frac{1}{\sigma_0^4}) + (\frac{1}{\sigma_0^6} - \frac{1}{\tilde{\sigma}_n^6})\frac{V_n'V_n}{n} + o_P(1) = o_p(1).$$

(Show $\frac{1}{n}\frac{\partial^2 \ln L_n(\theta_0)}{\partial\theta\partial\theta'} - E(\frac{1}{n}\frac{\partial^2 \ln L_n(\theta_0)}{\partial\theta\partial\theta'}) \xrightarrow{p} 0$): By Lemma A.10, $\frac{1}{n}X_n'G_nV_n = o_P(1)$ and $\frac{1}{n}X_n'G_n'G_nV_n = o_P(1)$. It follows that $\frac{1}{n}X_n'W_nY_n = \frac{1}{n}X_n'G_nX_n\beta_0 + o_P(1)$, $\frac{1}{n}Y_n'W_n'V_n = \frac{1}{n}V_n'G_n'V_n + o_P(1)$, and

$$\frac{1}{n}Y_n'W_n'W_nY_n = \frac{1}{n}(X_n\beta_0)'G_n'G_nX_n\beta_0 + \frac{1}{n}V_n'G_n'G_nV_n + o_P(1).$$

Lemmas A.11 and A.8 imply $E(V_n'G_n'V_n) = \sigma_0^2 tr(G_n)$ and

$$\text{var}(\frac{1}{n}V_n'G_n'V_n) = \frac{(\mu_4 - 3\sigma_0^4)}{n^2}\sum_{i=1}^n G_{n,ii}^2 + \frac{\sigma_0^4}{n^2}[tr(G_nG_n') + tr(G_n^2)] = O(\frac{1}{nh_n}).$$

Similarly, $E(V_n'G_n'G_nV_n) = \sigma_0^2 tr(G_n'G_n)$ and

$$\text{var}(\frac{1}{n}V_n'G_n'G_nV_n) = \frac{(\mu_4 - 3\sigma_0^4)}{n^2}\sum_{i=1}^n (G_n'G_n)_{ii}^2 + 2\frac{\sigma_0^4}{n^2}tr((G_n'G_n)^2) = O(\frac{1}{nh_n}).$$

By the law of large numbers, $\frac{1}{n}V_n'V_n \xrightarrow{p} \sigma_0^2$. With these properties, the convergence result follows.

Finally, from the expansion $\sqrt{n}(\hat{\theta}_n - \theta_0) = -\left(\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde{\theta}_n)}{\partial\theta\partial\theta'}\right)^{-1}\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(\theta_0)}{\partial\theta}$, the asymptotic distribution of $\hat{\theta}_n$ follows. Q.E.D.

Proof of Theorem 4.2 The nonsingularity of Σ_θ will now be guaranteed by Assumption 9 instead of Assumption 8. With (B.2) in the proof of Theorem 3.2, under Assumption 8$'$, one arrives at

$$\lim_{n\to\infty}\frac{1}{n}\left[tr(G_n'G_n) + tr(G_n^2) - 2\frac{tr^2(G_n)}{n}\right]\alpha_2 = 0.$$

Because $\frac{1}{n}[tr(G_nG_n')+tr(G_n^2)-\frac{2}{n}tr^2(G_n)] = \frac{1}{2n}tr[(C_n'+C_n)(C_n'+C_n)'] > 0$ for large n implied by Assumption 9, it follows that $\alpha_2 = 0$. Hence Σ_θ is nonsingular. The remaining arguments are similar as in the proof of Theorem 3.2. Q.E.D.

Proof of Theorem 5.1

(Show that $\frac{h_n}{n}[\ln L_n(\lambda) - \ln L_n(\lambda_0) - (Q_n(\lambda) - Q_n(\lambda_0)] \xrightarrow{p} 0$ uniformly on Λ): From (2.7) and (3.3), by the mean value theorem,

$$\begin{aligned}
&\frac{h_n}{n}[\ln L_n(\lambda) - \ln L_n(\lambda_0) - (Q_n(\lambda) - Q_n(\lambda_0))] \\
&= -\frac{h_n}{2}[\ln \hat{\sigma}_n^2(\lambda) - \ln \hat{\sigma}_n^2(\lambda_0) - (\ln \sigma_n^{*2}(\lambda) - \ln \sigma_n^{*2}(\lambda_0))] \\
&= -\frac{h_n}{2}[(\ln \hat{\sigma}_n^2(\lambda) - \ln \sigma_n^{*2}(\lambda)) - (\ln \hat{\sigma}_n^2(\lambda_0) - \ln \sigma_n^{*2}(\lambda_0))] = -\frac{h_n}{2}\frac{\partial[\ln \hat{\sigma}_n^2(\bar{\lambda}_n) - \ln \sigma_n^{*2}(\bar{\lambda}_n)]}{\partial \lambda}(\lambda - \lambda_0).
\end{aligned}$$

With the expressions of $\hat{\sigma}_n^2(\lambda)$ in (2.6) and $\sigma_n^{*2}(\lambda)$ in (3.2), it follows that $\frac{\partial \hat{\sigma}_n^2(\lambda)}{\partial \lambda} = -\frac{2}{n}Y_n'W_n'M_nS_n(\lambda)Y_n$, and

$$\frac{\partial \sigma_n^{*2}(\lambda)}{\partial \lambda} = \frac{1}{n}\{2(\lambda - \lambda_0)(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) - 2\sigma_0^2 tr[G_n'S_n(\lambda)S_n^{-1}]\}.$$

These imply that

$$\begin{aligned}
&\frac{h_n}{n}[(\ln L_n(\lambda) - \ln L_n(\lambda_0)) - (Q_n(\lambda) - Q_n(\lambda_0))] \\
&= \frac{1}{\hat{\sigma}_n^2(\bar{\lambda}_n)}\frac{h_n}{n}\{Y_n'W_n'M_nS_n(\bar{\lambda}_n)Y_n - \frac{\hat{\sigma}_n^2(\bar{\lambda}_n)}{\sigma_n^{*2}(\bar{\lambda}_n)}[(\lambda_0 - \bar{\lambda}_n)(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) \\
&\quad + \sigma_0^2 tr(G_n'S_n(\bar{\lambda}_n)S_n^{-1})]\}(\lambda - \lambda_0) \\
&= \frac{1}{\hat{\sigma}_n^2(\bar{\lambda}_n)}\frac{h_n}{n}\{Y_n'W_n'M_nS_n(\bar{\lambda}_n)Y_n - [(\lambda_0 - \bar{\lambda}_n)(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) + \sigma_0^2 tr(G_n'S_n(\bar{\lambda}_n)S_n^{-1})] \\
&\quad - \frac{\hat{\sigma}_n^2(\bar{\lambda}_n) - \sigma_n^{*2}(\bar{\lambda}_n)}{\sigma_n^{*2}(\bar{\lambda}_n)}[(\lambda_0 - \bar{\lambda}_n)(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) + \sigma_0^2 tr(G_n'S_n(\bar{\lambda}_n)S_n^{-1})]\}(\lambda - \lambda_0).
\end{aligned}$$

By using $S_n(\lambda)S_n^{-1} = I_n + (\lambda_0 - \lambda)G_n$, the model implies that

$$\begin{aligned}
&Y_n'W_n'M_nS_n(\lambda)Y_n \\
&= (G_nX_n\beta_0)'M_nS_n(\lambda)S_n^{-1}X_n\beta_0 + (G_nX_n\beta_0)'M_nS_n(\lambda)S_n^{-1}V_n + V_n'G_n'M_nS_n(\lambda)S_n^{-1}X_n\beta_0 \\
&\quad + V_n'G_n'M_nS_n(\lambda)S_n^{-1}V_n \\
&= (G_nX_n\beta_0)'M_n(G_nX_n\beta_0)(\lambda_0 - \lambda) + (G_nX_n\beta_0)'M_nV_n + 2(G_nX_n\beta_0)'M_nG_nV_n(\lambda_0 - \lambda) \\
&\quad + V_n'G_n'M_nV_n + V_n'G_n'M_nG_nV_n(\lambda_0 - \lambda).
\end{aligned}$$

Lemma A.9 implies that $tr(M_nG_n) = tr(G_n) + O(1)$ and $tr(G_n'M_nG_n) = tr(G_n'G_n) + O(1)$. The law of large numbers in Lemma A.12 shows that $\frac{h_n}{n}(V_n'M_nG_nV_n - \sigma_0^2 tr(G_n)) = o_P(1)$ and $\frac{h_n}{n}(V_n'G_n'M_nG_nV_n -$

$\sigma_0^2 tr(G_n' G_n)) = o_P(1)$. Under Assumption 10, $\frac{h_n}{n}(G_n X_n \beta_0)' M_n V_n = o_P(1)$ and $\frac{h_n}{n}(G_n X_n \beta_0)' M_n G_n V_n = o_P(1)$ by Lemma A.14. Therefore,

$$
\begin{aligned}
&\frac{h_n}{n}\{Y_n' W_n' M_n S_n(\lambda) Y_n - (G_n X_n \beta_0)' M_n (G_n X_n \beta_0)(\lambda_0 - \lambda) - \sigma_0^2 tr(G_n' S_n(\lambda) S_n^{-1})\} \\
&= \frac{h_n}{n}\{(G_n X_n \beta_0)' M_n V_n + 2(\lambda_0 - \lambda)(G_n X_n \beta_0)' M_n G_n V_n \\
&\quad + V_n' G_n' M_n V_n + (\lambda_0 - \lambda) V_n' G_n' M_n G_n V_n - \sigma_0^2 tr(G_n') - \sigma_0^2 (\lambda_0 - \lambda) tr(G_n' G_n)\} \\
&= o_P(1).
\end{aligned}
$$

From (B.1) and (3.2),

$$
\begin{aligned}
\hat{\sigma}_n^2(\lambda) - \sigma_n^{*2}(\lambda) &= 2(\lambda_0 - \lambda)\frac{1}{n}(G_n X_n \beta_0)' M_n S_n(\lambda) S_n^{-1} V_n \\
&+ \frac{1}{n}\{V_n' S_n'^{-1} S_n'(\lambda) M_n S_n(\lambda) S_n^{-1} V_n - \sigma_0^2 tr[S_n'^{-1} S_n'(\lambda) M_n S_n(\lambda) S_n^{-1}]\} \\
&+ \frac{1}{n}\sigma_0^2\{tr[S_n'^{-1} S_n'(\lambda) M_n S_n(\lambda) S_n^{-1}] - tr[S_n'^{-1} S_n'(\lambda) S_n(\lambda) S_n^{-1}]\} \\
&= o_P(1),
\end{aligned}
$$

uniformly in λ by Chebyshev's LLN, Lemma A.12 and Lemma A.9. Note that $\frac{h_n}{n}(G_n X_n \beta_0)' M_n (G_n X_n \beta_0) = O(1)$ and $\frac{h_n}{n} tr(G_n' S_n(\lambda) S_n^{-1}) = O(1)$. When $h_n \to \infty$, $\sigma_n^2(\lambda) = \sigma_0^2[1 + 2(\lambda_0 - \lambda)\frac{tr(G_n)}{n} + (\lambda - \lambda_0)^2 \frac{tr(G_n G_n')}{n}] \to \sigma_0^2$ uniformly on Λ. As $\sigma_n^{*2}(\lambda) \geq \sigma_n^2(\lambda)$ and $\sigma_0^2 > 0$, $\frac{1}{\sigma_n^{*2}(\lambda_n)}$ is $O(1)$ and $\frac{1}{\hat{\sigma}_n^2(\lambda_n)}$ is $O_P(1)$. In conclusion, $\frac{h_n}{n}[(\ln L_n(\lambda) - \ln L_n(\lambda_0)) - (Q_n(\lambda) - Q_n(\lambda_0))] = o_P(1)$ uniformly in $\lambda \in \Lambda$.

(Show the uniform equicontinuity of $\frac{h_n}{n}(Q_n(\lambda) - Q_n(\lambda_0))$): Recall that

$$
\frac{h_n}{n}(Q_n(\lambda) - Q_n(\lambda_0)) = -\frac{h_n}{2}(\ln \sigma_n^{*2}(\lambda) - \ln \sigma_0^2) + \frac{h_n}{n}(\ln|S_n(\lambda)| - \ln|S_n(\lambda_0)|).
$$

As $tr(S_n'^{-1} S_n'(\lambda) S_n(\lambda) S_n^{-1}) - n = (\lambda_0 - \lambda) tr(G_n' + G_n) + (\lambda - \lambda_0)^2 tr(G_n' G_n)$,

$$
\begin{aligned}
&h_n(\sigma_n^{*2}(\lambda) - \sigma_0^2) \\
&= (\lambda - \lambda_0)^2 \frac{h_n}{n}(G_n X_n \beta_0)' M_n (G_n X_n \beta_0) + \sigma_0^2 \frac{h_n}{n}\{tr[S_n'^{-1} S_n'(\lambda) S_n(\lambda) S_n^{-1}] - n\} \\
&= (\lambda - \lambda_0)^2 \frac{h_n}{n}(G_n X_n \beta_0)' M_n (G_n X_n \beta_0) + \sigma_0^2(\lambda_0 - \lambda)\frac{h_n}{n} tr(G_n' + G_n) + \sigma_0^2(\lambda_0 - \lambda)^2 \frac{h_n}{n} tr(G_n' G_n)
\end{aligned}
$$

is uniformly equicontinous in $\lambda \in \Lambda$. By the mean value theorem, $h_n(\ln \sigma_n^{*2}(\lambda) - \ln \sigma_0^2) = \frac{h_n}{\bar{\sigma}_n^2(\lambda)}(\sigma_n^{*2}(\lambda) - \sigma_0^2)$ where $\bar{\sigma}_n^2(\lambda)$ lies between $\sigma_n^{*2}(\lambda)$ and σ_0^2. As $\sigma_n^{*2}(\lambda)$ is uniformly bounded away from zero on Λ and $\sigma_0^2 > 0$, $\bar{\sigma}_n^2(\lambda)$ is uniformly bounded from above. Hence, $h_n(\ln \sigma_n^{*2}(\lambda) - \ln \sigma_0^2)$ is uniformly equicontinuous on Λ. The $\frac{h_n}{n}(\ln|S_n(\lambda)| - \ln|S_n(\lambda_0)|) = \frac{h_n}{n} tr(W_n S_n^{-1}(\bar{\lambda}_n))(\lambda - \lambda_0)$ is uniformly equicontinuous on Λ because $\frac{h_n}{n} tr(W_n S_n^{-1}(\bar{\lambda}_n)) = O_P(1)$. In conclusion, $\frac{h_n}{n}(Q_n(\lambda) - Q_n(\lambda_0))$ is uniformly equicontinuous on Λ.

(uniqueness identification): For identification, let $D_n(\lambda) = -\frac{h_n}{2}(\ln \sigma_n^2(\lambda) - \ln \sigma_0^2) + \frac{h_n}{n}(\ln |S_n(\lambda)| - \ln |S_n(\lambda_0)|)$. Then, $\frac{h_n}{n}(Q_n(\lambda) - Q_n(\lambda_0)) = D_n(\lambda) - \frac{h_n}{2}(\ln \sigma_n^{*2}(\lambda) - \ln \sigma_n^2(\lambda))$. By the Taylor expansion,

$$\ln \sigma_n^{*2}(\lambda) - \ln \sigma_n^2(\lambda) = \frac{(\sigma_n^{*2}(\lambda) - \sigma_n^2(\lambda))}{\bar{\sigma}_n^{*2}(\lambda)} = \frac{(\lambda - \lambda_0)^2}{\bar{\sigma}_n^{*2}(\lambda)} \frac{h_n}{n}(G_n X_n \beta_0)' M_n (G_n X_n \beta_0),$$

where $\bar{\sigma}_n^{*2}$ lies between $\sigma_n^{*2}(\lambda)$ and $\sigma_n^2(\lambda)$. Because $\sigma_n^{*2}(\lambda) \geq \sigma_n^2(\lambda)$ for all $\lambda \in \Lambda$,

$$h_n(\ln \sigma_n^{*2}(\lambda) - \ln \sigma_n^2(\lambda)) \geq \frac{1}{\sigma_n^{*2}(\lambda)} \frac{h_n}{n}(\lambda - \lambda_0)^2 (G_n X_n \beta_0)' M_n (G_n X_n \beta_0).$$

For the situation in Assumption 8′, $\sigma_n^{*2}(\lambda) - \sigma_n^2(\lambda) = o_P(1)$ uniformly on Λ. Thus, $\lim_{n\to\infty} \sigma_n^{*2}(\lambda) = \sigma_0^2$. Therefore, under Assumption 10(a),

$$\begin{aligned} -\lim_{n\to\infty} h_n(\ln \sigma_n^{*2}(\lambda) - \ln \sigma_n^2(\lambda)) &\leq -\lim_{n\to\infty} \frac{1}{\sigma_n^{*2}(\lambda)}(\lambda - \lambda_0)^2 \frac{h_n}{n}(G_n X_n \beta_0)' M_n (G_n X_n \beta_0) \\ &= -\frac{(\lambda - \lambda_0)^2}{\sigma_0^2} \lim_{n\to\infty} \frac{h_n}{n}(G_n X_n \beta_0)' M_n (G_n X_n \beta_0) < 0, \end{aligned}$$

for any $\lambda \neq \lambda_0$. Furthermore, under the situation in Assumption 10(b), $D_n(\lambda) < 0$ whenever $\lambda \neq \lambda_0$. It follows that $\lim_{n\to\infty} \frac{h_n}{n}(Q_n(\lambda) - Q_n(\lambda_0)) < 0$ whenever $\lambda \neq \lambda_0$. As $\frac{h_n}{n}(Q_n(\lambda) - Q_n(\lambda_0))$ is uniformly equicontinuous, the identification uniqueness condition holds and θ_0 is identifiably unique.

The consistency of $\hat{\lambda}_n$ follows from the uniform convergence and the identification uniqueness condition.

For the pure SAR process, $\beta = 0$ is imposed in the estimation. The consistency of the QMLE of λ follows by similar arguments above. For the pure process, $\hat{\sigma}_n^2(\lambda) = \frac{1}{n} Y_n' S_n'(\lambda) S_n(\lambda) Y_n$ and the concentrated log likelihood function is $\ln L_n(\lambda) = -\frac{n}{2}(\ln(2\pi) + 1) - \frac{n}{2} \ln \hat{\sigma}_n^2(\lambda) + \ln |S_n(\lambda)|$. For the pure process, $Q_n(\lambda)$ happens to have the same expression as that of the case where $G_n X_n \beta_0$ is multicollinear with X_n. The simpler analysis corresponds to setting $X_n = 0$ and $M_n = I_n$ in the preceding arguments. Q.E.D.

Proof of Theorem 5.2

(Show $\frac{h_n}{n}(\frac{\partial^2 \ln L_n(\tilde{\lambda}_n)}{\partial \lambda^2} - \frac{\partial^2 \ln L_n(\lambda_0)}{\partial \lambda^2}) = o_P(1)$): The first and second order derivatives of the concentrated log likelihood are

$$\frac{\partial \ln L_n(\lambda)}{\partial \lambda} = \frac{1}{\hat{\sigma}_n^2(\lambda)} Y_n' W_n' M_n S_n(\lambda) Y_n - tr(W_n S_n^{-1}(\lambda)),$$

and

$$\frac{\partial^2 \ln L_n(\lambda)}{\partial \lambda^2} = \frac{2}{n \hat{\sigma}_n^4(\lambda)} (Y_n' W_n' M_n S_n(\lambda) Y_n)^2 - \frac{1}{\hat{\sigma}_n^2(\lambda)} Y_n' W_n' M_n W_n Y_n - tr([W_n S_n^{-1}(\lambda)]^2),$$

where $\hat{\sigma}_n^2(\lambda) = \frac{1}{n} Y_n' S_n'(\lambda) M_n S_n(\lambda) Y_n$. For the pure SAR process, $\beta_0 = 0$ and the corresponding derivatives are similar with M_n replaced by the identity I_n. So it is sufficient to consider the regressive model.

Because $M_nX_n = 0$ and $S_n(\lambda) = S_n + (\lambda_0 - \lambda)W_n$, one has

$$Y_n'W_n'M_nW_nY_n = (G_nX_n\beta_0)'M_n(G_nX_n\beta_0) + 2(G_nX_n\beta_0)'M_nG_nV_n + V_n'G_n'M_nG_nV_n,$$

and

$$\begin{aligned}Y_n'W_n'M_nS_n(\lambda)Y_n &= Y_n'W_n'M_nS_nY_n + (\lambda_0-\lambda)Y_n'W_n'M_nW_nY_n = Y_n'W_n'M_nV_n + (\lambda_0-\lambda)Y_n'W_n'M_nW_nY_n \\ &= (G_nX_n\beta_0)'M_nV_n + V_n'G_n'M_nV_n + (\lambda_0-\lambda)[(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) \\ &\quad + 2(G_nX_n\beta_0)'M_nG_nV_n + V_n'G_n'M_nG_nV_n].\end{aligned}$$

As shown in the proof of Theorem 5.1, $\frac{h_n}{n}(G_nX_n\beta_0)'M_nG_nV_n = o_P(1)$. Hence,

$$\frac{h_n}{n}Y_n'W_n'M_nW_nY_n = \frac{h_n}{n}(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) + \frac{h_n}{n}V_n'G_n'M_nG_nV_n + o_P(1)$$

and

$$\frac{h_n}{n}Y_n'W_n'M_nS_n(\lambda)Y_n = \frac{h_n}{n}V_n'G_n'M_nV_n + (\lambda_0-\lambda)[\frac{h_n}{n}(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) + \frac{h_n}{n}V_n'G_n'M_nG_nV_n] + o_P(1).$$

Lemma A.12 implies that $V_n'G_n'M_nV_n = O_p(\frac{n}{h_n})$ and $V_n'G_n'M_nG_nV_n = O_p(\frac{n}{h_n})$. Thus $\frac{h_n}{n}Y_n'W_n'M_nS_n(\lambda)Y_n = O_P(1)$ uniformly on Λ. From the proof of Theorem 5.1, $\hat{\sigma}_n^2(\lambda) = \sigma_n^2(\lambda) + o_P(1) = \sigma_0^2 + o_P(1)$ uniformly on Λ, when $\lim_{n\to\infty} h_n = \infty$. Thus, $\frac{1}{\hat{\sigma}_n^2(\lambda)}\frac{h_n}{n}V_n'G_n'M_nG_nV_n = \frac{1}{\sigma_0^2}\frac{h_n}{n}V_n'G_n'M_nG_nV_n + o_P(1)$ uniformly on Λ. Therefore, one has

$$\frac{h_n}{n}\frac{\partial^2 \ln L_n(\lambda)}{\partial\lambda^2} = -\frac{1}{\sigma_0^2}[\frac{h_n}{n}(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) + \frac{h_n}{n}V_n'G_n'M_nG_nV_n] - \frac{h_n}{n}tr([W_nS_n^{-1}(\lambda)]^2) + o_P(1),$$

uniformly on Λ. By Lemma A.8, under Assumption 7, $\frac{h_n}{n}tr(G_n^3(\lambda)) = O(1)$ uniformly on Λ. Therefore, by the Taylor expansion, $\frac{h_n}{n}(\frac{\partial^2 \ln L_n(\tilde{\lambda}_n)}{\partial\lambda^2} - \frac{\partial^2 \ln L_n(\lambda_0)}{\partial\lambda^2}) = -\frac{h_n}{n}\{tr([W_nS_n^{-1}(\tilde{\lambda}_n)]^2) - tr(G_n^2)\} + o_P(1) = -2\frac{h_n}{n}tr(G_n^3(\bar{\lambda}_n))(\tilde{\lambda}_n - \lambda_0) + o_P(1) = o_P(1)$, for any $\tilde{\lambda}_n$ which converges in probability to λ_0.

(Show $\frac{h_n}{n}(\frac{\partial^2 \ln L_n(\lambda_0)}{\partial\lambda^2} - E(P_n(\lambda_0))) \xrightarrow{p} 0$):

Define $P_n(\lambda_0) = -\frac{1}{\sigma_0^2}[(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) + V_n'G_n'M_nG_nV_n] - tr(G_n^2)$. Then $\frac{h_n}{n}\frac{\partial^2 \ln L_n(\lambda_0)}{\partial\lambda^2} = \frac{h_n}{n}P_n(\lambda_0) + o_P(1)$. From Lemma A.9, $tr(G_n'M_nG_n) = tr(G_n'G_n) + O(1)$, $tr[(G_n'M_nG_n)^2] = tr[(G_nG_n')^2] + O(1)$ and $\sum_{i=1}^n((G_n'M_nG_n)_{ii})^2 = \sum_{i=1}^n((G_n'G_n)_{ii})^2 + O(\frac{1}{h_n})$. The $tr(G_n'M_nG_n)$ and $tr((G_n'M_nG_n)^2)$ are $O(\frac{n}{h_n})$ and $\sum_{i=1}^n((G_n'G_n)_{ii})^2 = O(\frac{n}{h_n^2})$ from Lemma A.8. Therefore,

$$E(P_n(\lambda_0)) = -\frac{1}{\sigma_0^2}(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) - [tr(G_nG_n') + tr(G_n^2)] + O(1).$$

As $\frac{h_n}{n}[P_n(\lambda_0)-E(P_n(\lambda_0))] = -\frac{1}{\sigma_0^2}\Delta_n + o(1)$, where $\Delta_n = \frac{h_n}{n}[V_n'G_n'M_nG_nV_n - \sigma_0^2 tr(G_n'M_nG_n)]$, $\frac{h_n}{n}[P_n(\lambda_0)-E(P_n(\lambda_0))] = o_P(1)$ if $\Delta_n = o_P(1)$. By Lemma A.11 and the orders of relevant terms,

$$E(\Delta_n^2) = (\frac{h_n}{n})^2 \text{var}(V_n'G_n'M_nG_nV_n) = (\frac{h_n}{n})^2[(\mu_4 - 3\sigma_0^4)\sum_{i=1}^n (G_{n,i}'M_nG_{n,i})^2 + 2\sigma_0^4 tr((G_n'M_nG_n)^2)] = O(\frac{h_n}{n}),$$

which goes to zero. Therefore, $\frac{h_n}{n}\left(\frac{\partial^2 \ln L_n(\lambda_0)}{\partial\lambda^2} - E(P_n(\lambda_0))\right) = o_P(1)$.

(Show $\sqrt{\frac{h_n}{n}}\frac{\partial \ln L_n(\lambda_0)}{\partial\lambda} \xrightarrow{p} N(0,\sigma_\lambda^2)$ when $\lim_{n\to\infty} h_n = \infty$):

Let $q_n = V_n'C_n'M_nV_n$. Thus,

$$\sqrt{\frac{h_n}{n}}\frac{\partial \ln L_n(\lambda_0)}{\partial\lambda} = \frac{1}{\hat\sigma_n^2(\lambda_0)}\sqrt{\frac{h_n}{n}}[(G_nX_n\beta_0)'M_nV_n + q_n].$$

The mean of q_n is $E(q_n) = \sigma_0^2 tr(M_nC_n) = -\sigma_0^2 \cdot tr[(X_n'X_n)^{-1}X_n'C_nX_n] = O(1)$ because $\frac{X_n'X_n}{n} = O(1)$ and $\frac{X_n'C_nX_n}{n} = O(1)$ from Lemma A.6. The variance of q_n from Lemma A.11 is

$$\begin{aligned}\sigma_{q_n}^2 &= (\mu_4 - 3\sigma_0^4)\sum_{i=1}^n ((C_n'M_n)_{ii})^2 + \sigma_0^4[tr(C_n'M_nC_n) + tr((C_n'M_n)^2)] \\ &= (\mu_4 - 3\sigma_0^4)\sum_{i=1}^n (C_{n,ii})^2 + \sigma_0^4[tr(C_n'C_n) + tr(C_n^2)] + O(1),\end{aligned}$$

where the last expression follows because

$$\sum_{i=1}^n ((C_n'M_n)_{ii})^2 = \sum_{i=1}^n (C_{n,ii})^2 + O(\frac{1}{h_n}),\quad tr(C_n'M_nC_n) = tr(C_n'C_n) + O(1) \quad \text{and } tr[(C_n'M_n)^2] = tr(C_n^2) + O(1)$$

from Lemma A.9. The covariance of a linear term Q_nV_n and a quadratic form $V_n'P_nV_n$ is $E(Q_n'V_n \cdot V_n'P_nV_n) = Q_n'\sum_{i=1}^n\sum_{j=1}^n p_{n,ij}E(V_nv_{ni}v_{nj}) = Q_n'\text{vec}_D(P_n)\mu_3$. Denote $\sigma_{lq_n}^2 = \text{var}((G_nX_n\beta_0)'M_nV_n + q_n)$. Thus,

$$\sigma_{lq_n}^2 = \sigma_0^2(G_nX_n\beta_0)'M_n(G_nX_n\beta_0) + \sigma_{q_n}^2 + 2(G_nX_n\beta_0)'M_n\text{vec}_D(C_n'M_n)\mu_3.$$

As $\sqrt{\frac{h_n}{n}}E(q_n) = O(\sqrt{\frac{h_n}{n}})$, which goes to zero,

$$\begin{aligned}\sqrt{\frac{h_n}{n}}\frac{\partial \ln L_n(\lambda_0)}{\partial\lambda} &= \frac{\sqrt{\frac{h_n}{n}}\sigma_{lq_n}}{\hat\sigma_n^2(\lambda_0)} \cdot \frac{[(G_nX_n\beta_0)'M_nV_n + q_n - E(q_n)]}{\sigma_{lq_n}} + \sqrt{\frac{h_n}{n}}\frac{E(q_n)}{\hat\sigma_n^2(\lambda_0)} \\ &= \frac{\sqrt{\frac{h_n}{n}}\sigma_{lq_n}}{\hat\sigma_n^2(\lambda_0)} \cdot \frac{[(G_nX_n\beta_0)'M_nV_n + q_n - E(q_n)]}{\sigma_{lq_n}} + o_P(1) \xrightarrow{p} N(0, \lim_{n\to\infty}\frac{h_n}{n}\frac{\sigma_{lq_n}^2}{\sigma_0^4}).\end{aligned}$$

As $(C_{n,ii})^2 = O(\frac{1}{h_n^2})$, $\frac{h_n}{n}\sum_{i=1}^n (C_{n,ii})^2 = O(\frac{1}{h_n})$ which goes to zero when $\lim_{n\to\infty} h_n = \infty$. Finally, as $\frac{h_n}{n^2}tr^2(G_n) = O(\frac{1}{h_n}) = o(1)$,

$$\lim_{n\to\infty}\frac{h_n}{n}[tr(C_nC_n') + tr(C_n^2)] = \lim_{n\to\infty}\frac{h_n}{n}[tr(G_nG_n') + tr(G_n^2) - \frac{2}{n}tr^2(G_n)] = \lim_{n\to\infty}\frac{h_n}{n}[tr(G_nG_n') + tr(G_n^2)].$$

The limiting distribution of $\sqrt{n}(\hat{\lambda}_n - \lambda_0)$ follows from the Taylor expansion and the convergence results above. Q.E.D.

Proof of Theorem 5.3 As $S_n(\lambda) = S_n + (\lambda_0 - \hat{\lambda}_n)W_n$,

$$\begin{aligned}\hat{\beta}_n - \beta_0 &= (X_n'X_n)^{-1}X_n'V_n - (\hat{\lambda}_n - \lambda_0)(X_n'X_n)^{-1}X_n'W_nY_n \\ &= (X_n'X_n)^{-1}X_n'V_n - (\hat{\lambda}_n - \lambda_0)(X_n'X_n)^{-1}X_n'G_nX_n\beta_0 - (\hat{\lambda}_n - \lambda_0)(X_n'X_n)^{-1}X_n'G_nV_n.\end{aligned}$$

As $(\hat{\lambda}_n - \lambda_0)(X_n'X_n)^{-1}X_n'G_nV_n = O_P(\frac{\sqrt{h_n}}{n})$ because $\frac{X_n'X_n}{n} = O(1)$ and $\frac{X_n'G_nV_n}{\sqrt{n}} = O_P(1)$ and $\hat{\lambda}_n - \lambda_0 = O_P(\sqrt{\frac{h_n}{n}})$ by Theorem 5.2, $\hat{\beta}_n - \beta_0 = (X_n'X_n)^{-1}X_n'V_n - (\hat{\lambda}_n - \lambda_0)(X_n'X_n)^{-1}X_n'G_nX_n\beta_0 + O_p(\frac{\sqrt{h_n}}{n})$. In general,

$$\begin{aligned}\sqrt{\frac{n}{h_n}}(\hat{\beta}_n - \beta_0) &= \frac{1}{\sqrt{h_n}}(\frac{X_n'X_n}{n})^{-1}\frac{X_n'V_n}{\sqrt{n}} - \sqrt{\frac{n}{h_n}}(\hat{\lambda}_n - \lambda_0)\cdot(X_n'X_n)^{-1}X_n'G_nX_n\beta_0 + O_P(\frac{1}{\sqrt{n}}) \\ &= -\sqrt{\frac{n}{h_n}}(\hat{\lambda}_n - \lambda_0)\cdot(X_n'X_n)^{-1}X_n'G_nX_n\beta_0 + O_P(\frac{1}{\sqrt{h_n}}).\end{aligned}$$

If β_0 is zero, $\sqrt{n}(\hat{\beta}_n - \beta_0) = (\frac{X_n'X_n}{n})^{-1}\frac{X_n'V_n}{\sqrt{n}} + O_P(\sqrt{\frac{h_n}{n}}) \xrightarrow{D} N(0, \sigma_0^2 \lim_{n\to\infty}(\frac{X_n'X_n}{n})^{-1})$.

As

$$\begin{aligned}\hat{\sigma}_n^2 &= \frac{1}{n}Y_n'S_n'(\hat{\lambda}_n)M_nS_n(\hat{\lambda}_n)Y_n \\ &= \frac{1}{n}Y_n'S_n'M_nS_nY_n + 2(\lambda_0 - \hat{\lambda}_n)\frac{1}{n}Y_n'W_n'M_nS_nY_n + (\lambda_0 - \hat{\lambda}_n)^2\frac{1}{n}Y_n'W_n'M_nW_nY_n\end{aligned}$$

and $\frac{1}{n}Y_n'S_n'M_nS_nY_n = \frac{1}{n}V_n'M_nV_n$, it follows that

$$\begin{aligned}\sqrt{n}(\hat{\sigma}_n^2 - \sigma_0^2) &= \frac{1}{\sqrt{n}}(V_n'V_n - n\sigma_0^2) - \frac{1}{\sqrt{n}}V_n'X_n(X_n'X_n)^{-1}X_n'V_n \\ &\quad - 2\sqrt{\frac{n}{h_n}}(\hat{\lambda}_n - \lambda_0)\cdot\frac{\sqrt{h_n}}{n}Y_n'W_n'M_nS_nY_n + \sqrt{\frac{n}{h_n}}(\hat{\lambda}_n - \lambda_0)^2\cdot\frac{\sqrt{h_n}}{n}Y_n'W_n'M_nW_nY_n.\end{aligned}$$

Because $M_nX_n = 0$, and $\sqrt{\frac{h_n}{n}}(G_nX_n\beta_0)'M_nV_n = O_P(1)$ and $\frac{h_n}{n}(G_nX_n\beta_0)'M_nG_nV_n = O_P(1)$ under Assumption 10,

$$\frac{\sqrt{h_n}}{n}Y_n'W_n'M_nS_nY_n = \frac{\sqrt{h_n}}{n}(G_nX_n\beta_0 + G_nV_n)'M_nV_n = \frac{\sqrt{h_n}}{n}V_n'G_n'M_nV_n + O_P(\frac{1}{\sqrt{n}}) = O_P(\frac{1}{\sqrt{h_n}})$$

and

$$\begin{aligned}\frac{\sqrt{h_n}}{n}Y_n'W_n'M_nW_nY_n &= \frac{\sqrt{h_n}}{n}(G_nX_n\beta_0 + G_nV_n)'M_n(G_nX_n\beta_0 + G_nV_n) \\ &= \frac{\sqrt{h_n}}{n}V_n'G_n'M_nG_nV_n + O_P(\frac{1}{\sqrt{h_n}}) = O_P(\frac{1}{\sqrt{h_n}})\end{aligned}$$

by Lemmas A.12 and A.14. As $E(\frac{1}{\sqrt{n}}V_n'X_n(X_n'X_n)^{-1}X_n'V_n) = \frac{\sigma_0^2}{\sqrt{n}}tr(X_n(X_n'X_n)^{-1}X_n') = \frac{\sigma_0^2 k}{\sqrt{n}}$ goes to zero, the Markov inequality implies that $\frac{1}{\sqrt{n}}V_n'X_n(X_n'X_n)^{-1}X_n'V_n = o_P(1)$. Hence, as $\lim_{n\to\infty} h_n = \infty$, $\sqrt{n}(\hat{\sigma}_n^2 - \sigma_0^2) = \frac{1}{\sqrt{n}}(V_n'V_n - n\sigma_0^2) + o_P(1) \xrightarrow{D} N(0, \mu_4 - \sigma^4)$. Q.E.D.

Proof of Theorem 5.4

Let $X_n = (X_{1n}, X_{2n})$, $M_{1n} = I_n - X_{1n}(X'_{1n}X_{1n})^{-1}X'_{1n}$ and $M_{2n} = I_n - X_{2n}(X'_{2n}X_{2n})^{-1}X'_{2n}$. Using a matrix partition for $(X'_nX_n)^{-1}$, $\hat{\beta}_{n1} - \beta_{01} = (X'_{1n}M_{2n}X_{1n})^{-1}X'_{1n}M_{2n}V_n - c_{1n}(\hat{\lambda}_n - \lambda_0) + O_P(\frac{\sqrt{h_n}}{n})$, and $\hat{\beta}_{n2} - \beta_{20} = (X'_{2n}M_{1n}X_{2n})^{-1}X'_{2n}M_{1n}V_n + O_P(\frac{\sqrt{h_n}}{n})$. Therefore,

$$\sqrt{\frac{n}{h_n}}(\hat{\beta}_{n1} - \beta_{01}) = \frac{1}{\sqrt{h_n}}(\frac{1}{n}X'_{1n}M_{2n}X_{1n})^{-1}\frac{1}{\sqrt{n}}X'_{1n}M_{2n}V_n - c_{1n} \cdot \sqrt{\frac{n}{h_n}}(\hat{\lambda}_n - \lambda_0) + O_P(\frac{1}{\sqrt{n}})$$
$$= -c_{1n} \cdot \sqrt{\frac{n}{h_n}}(\hat{\lambda}_n - \lambda_0) + O_P(\frac{1}{\sqrt{h_n}}),$$

and $\sqrt{n}(\hat{\beta}_{n2} - \beta_{20}) = (\frac{1}{n}X'_{2n}M_{1n}X_{2n})^{-1} \cdot \frac{1}{\sqrt{n}}X'_{2n}M_{1n}V_n + O_P(\sqrt{\frac{h_n}{n}})$. The asymptotic distributions of $\hat{\beta}_{n1}$ and $\hat{\beta}_{n2}$ follow. Q.E.D.