Assignment 2 – ECON747 Spatial Econometric Models and Methods

1. For the SLR model with both spatial lag (SL) and spatial error (SE) dependence, or simply the SLE model, treated in Section 3.4, Lecture 3, for hypothesis testing:

$$Y_n = \lambda W_{1n} Y_n + X_n \beta + u_n, \quad u_n = \rho W_{2n} u_n + \epsilon_n, \tag{1}$$

we are interested in three hypotheses concerning the spatial effects in the SLE model:

   (a) $H_0^{\texttt{SLE}}$: $\delta_0 = 0$ in the SLE model,

   (b) $H_0^{\texttt{SL|SE}}$: $\lambda_0 = 0$ in the SLE model, and

   (c) $H_0^{\texttt{SE|SL}}$: $\rho_0 = 0$ in the SLE model, where $\delta_0 = (\lambda_0, \rho_0)'$.

   (i) Following the instructions given in Slide 53, Lecture 3, and the ideas behind (3.34) and (3.35), develop fully the two variants of the standardized LM tests: $\text{SLM}^{\circ}_{\texttt{SLE}}$ and $\text{SLM}^{\texttt{MD}}_{\texttt{SLE}}$, for testing the joint hypothesis $H_0^{\texttt{SLE}}$: $\delta_0 = 0$.

   (ii) Following the instructions given in Slide 54, Lecture 3, develop the two standardized LM tests, $\text{SLM}^{\circ}_{\texttt{SL|SE}}$ and $\text{SLM}^{\circ}_{\texttt{SE|SL}}$, for testing $H_0^{\texttt{SL|SE}}$: $\lambda_0 = 0$ and $H_0^{\texttt{SE|SL}}$: $\rho_0 = 0$, respectively.

   (iii) Develop the MD variants of the two tests in (ii) above, $\text{SLM}^{\texttt{MD}}_{\texttt{SL|SE}}$ and $\text{SLM}^{\texttt{MD}}_{\texttt{SE|SL}}$.

Solution:

   (i) Write Model (1) in reduced form:

$$B_n(\rho) A_n(\lambda) Y_n = B_n(\rho) X_n \beta + \epsilon_n.$$

where $A_n(\lambda) = I_n - \lambda W_{1n}$ and $B_n(\rho) = I_n - \rho W_{2n}$. This leads to Gaussian quasi loglikelihood function of $\theta = (\beta', \sigma^2, \lambda, \rho)'$:

$$\ell_n(\theta) = -\tfrac{n}{2} \log(2\pi\sigma^2) + \log|A_n(\lambda)| + \log|B_n(\rho)| - \tfrac{1}{2\sigma^2} \epsilon_n'(\beta, \delta) \epsilon_n(\beta, \delta), \tag{2}$$

where $\epsilon_n(\beta, \delta) = \mathbb{Y}_n(\delta) - \mathbb{X}_n(\rho)\beta$, $\delta = (\lambda, \rho)'$, $\mathbb{X}_n(\rho) = B_n(\rho) X_n$, and $\mathbb{Y}_n(\delta) = B_n(\rho) A_n(\lambda) Y_n$, and the quasi Gaussian score function:

$$S_n(\theta) = \begin{cases} \frac{1}{\sigma^2} \mathbb{X}_n'(\rho) \epsilon_n(\beta, \delta), \\ \frac{1}{2\sigma^4} \epsilon_n'(\beta, \delta) \epsilon_n(\beta, \delta) - \frac{n}{2\sigma^2}, \\ \frac{1}{\sigma^2} \epsilon_n'(\beta, \delta) B_n W_{1n}(\rho) Y_n - \text{tr}[F_n(\lambda)], \\ \frac{1}{\sigma^2} \epsilon_n'(\beta, \delta) G_n(\rho) \epsilon_n(\beta, \delta) - \text{tr}[G_n(\rho)], \end{cases} \tag{3}$$

where $F_n(\lambda) = W_{1n} A_n^{-1}(\lambda)$, and $G_n(\rho) = W_{2n} B_n^{-1}(\rho)$.

Given δ, solving the first two components of the quasi score equations, $S_n(\theta) = 0$, we obtain the constrained (Q)MLEs of β and σ^2:

$$\tilde{\beta}_n(\delta) = [\mathbb{X}_n'(\rho) \mathbb{X}_n(\rho)]^{-1} \mathbb{X}_n'(\rho) \mathbb{Y}_n(\delta), \tag{4}$$

$$\tilde{\sigma}_n^2(\delta) = \tfrac{1}{n} \mathbb{Y}_n'(\delta) \mathbb{M}_n(\rho) \mathbb{Y}_n(\delta), \tag{5}$$

where $\mathbb{M}_n(\rho) = I_n - \mathbb{X}_n(\rho)[\mathbb{X}_n'(\rho)\mathbb{X}_n(\rho)]^{-1}\mathbb{X}_n'(\rho)$. Substituting $\tilde{\beta}_n(\delta)$ and $\tilde{\sigma}_n^2(\delta)$ back into the δ-component, $S_{n,\delta}(\theta)$, of $S_n(\theta)$, we obtain the concentrated quasi score (CQS) function of δ, $S_n^c(\delta) = S_{n,\delta}(\tilde{\beta}_n(\delta), \tilde{\sigma}_n^2(\delta), \delta) \equiv \tilde{S}_{n,\delta}(\delta)$:

$$\tilde{S}_{n,\delta}(\delta) = \begin{cases} \frac{1}{\tilde{\sigma}_n^2(\delta)}\tilde{\epsilon}_n'(\delta)B_n(\rho)W_{1n}Y_n - \text{tr}(F_n(\lambda)), \\ \frac{1}{\tilde{\sigma}_n^2(\delta)}\tilde{\epsilon}_n'(\delta)G_n(\rho)\tilde{\epsilon}_n(\delta) - \text{tr}(G_n(\rho)), \end{cases} \tag{6}$$

where $\tilde{\epsilon}_n(\delta) = \epsilon_n(\tilde{\beta}_n(\delta), \delta) = \mathbb{M}_n(\rho)\mathbb{Y}_n(\delta)$.

At $\delta = 0$, we have $A_n(0) = B_n(0) = I_n$, $F_n(0) = W_{1n}$, $G_n(0) = W_{2n}$, and

$$\tilde{S}_{\texttt{SLE},\delta}(0) \;\propto\; \begin{cases} \tilde{\epsilon}_n' W_{1n}Y_n = \epsilon_n' M_n W_{1n}\epsilon_n + \epsilon_n' M_n W_{1n}X_n\beta_0, \\ \tilde{\epsilon}_n' W_{2n}\tilde{\epsilon}_n = \epsilon_n M_n W_{2n} M_n \epsilon_n \end{cases} \tag{7}$$

where $\tilde{\epsilon}_n = \tilde{\epsilon}_n(0) = M_n\epsilon_n = M_nY_n$ and $M_n = \mathbb{M}_n(0)$. It is easy to see that $\text{E}[\tilde{S}_{\texttt{SLE},\delta}(0)] = \sigma_0^2(\texttt{tr}(M_nW_{1n}),\ \texttt{tr}(M_nW_{2n}))' \neq 0$, and thus $\tilde{S}_{\texttt{SLE},\delta}(0)$ needs to be centered! An obvious centered version of $\tilde{S}_{\texttt{SLE},\delta}(0)$ that is feasible takes the form:

$$\tilde{S}^*_{\texttt{SLE},\delta} = \begin{cases} \tilde{\epsilon}_n' W_{1n}Y_n - \frac{1}{n-k}\tilde{\epsilon}_n'\tilde{\epsilon}_n \texttt{tr}(M_nW_{1n}) \\ \tilde{\epsilon}_n' W_{2n}\tilde{\epsilon}_n - \frac{1}{n-k}\tilde{\epsilon}_n'\tilde{\epsilon}_n \texttt{tr}(M_nW_{2n}) \end{cases} = \begin{cases} \epsilon_n'\Phi_1\epsilon_n + \epsilon_n'\eta \\ \epsilon_n\Phi_2\epsilon_n, \end{cases} \tag{8}$$

where $\Phi_1 = M_nW_{rn} - c_rM_n$, $\Phi_2 = M_nW_{2n}M_n - c_2M_n$, $c_r = \frac{1}{n-k}\texttt{tr}(M_nW_{rn}), r = 1, 2$, and $\eta = M_nW_{1n}X_n\beta_0$.

SLM Test based on Analytical VC Matrix.

The second expression given in (8) greatly facilitate the derivation and presentation of the VC matrix of $\tilde{S}^*_{\texttt{SLE},\delta}$ at $\delta = 0$. We have by Lemma 2.1,

$$\text{Var}(\tilde{S}^*_{\texttt{SLE},\delta}|_{\delta=0}) = \begin{pmatrix} \omega_{11} + \pi_1, & \omega_{12} + \pi_2 \\ \sim & \omega_{22} \end{pmatrix},$$

where $\omega_{jr} = \sigma_0^4\kappa_0\phi_j'\phi_r + \sigma_0^4\texttt{tr}(\Phi_j\Phi_r^s), j, r = 1, 2$, $\pi_1 = 2\sigma_0^3\gamma_0\phi_1'\eta + \sigma_0^2\eta'\eta$, $\pi_2 = \sigma_0^3\gamma_0\phi_2'\eta$, ϕ_r is the vector of diagonal elements of Φ_r, $\Phi_r^s = \Phi_r + \Phi_r'$, $r = 1, 2$, and γ and κ are the measures of skewness and excess kurtosis of ϵ_{ni}.

The standardized LM (SLM) test for testing $H_0^{\texttt{SLE}}$: $\delta_0 = 0$ takes the form:

$$\text{SLM}^\circ_{\texttt{SLE}} = \begin{pmatrix} \tilde{\epsilon}_n' W^\circ_{1n}Y_n \\ \tilde{\epsilon}_n' W^\circ_{2n}\tilde{\epsilon}_n \end{pmatrix}' \begin{pmatrix} \omega_{11} + \pi_1, & \omega_{12} + \pi_2 \\ \sim & \omega_{22} \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\epsilon}_n' W^\circ_{1n}Y_n \\ \tilde{\epsilon}_n' W^\circ_{2n}\tilde{\epsilon}_n \end{pmatrix} \tag{9}$$

where $W^\circ_{rn} = W_{rn} - c_rI_n, r = 1, 2$, and the tilde-quantities are the *plug-in* estimates by plugging in the null estimates: $\tilde{\beta}_n(0)$ for β_0, $\frac{n}{n-k}\tilde{\sigma}_n^2(0)$ for σ_0^2, and the sample skewness and excess kurtosis of $\tilde{\epsilon}_n$ for γ and κ.

Remark 1: The above results give a conditional test of $H_0^{\texttt{SE}}$: $\rho_0 = 0$ given $\lambda = 0$:

$$\text{SLM}_{\texttt{SE}}^{\circ} = \frac{(n-k)\tilde{\epsilon}_n' W_{2n}^{\circ}\tilde{\epsilon}_n}{[\tilde{\kappa}_n \phi_2'\phi_2 + \texttt{tr}(\Phi_2\Phi_2^s)]^{\frac{1}{2}}\tilde{\epsilon}_n'\tilde{\epsilon}_n},$$

which is the same in form as (3.34) given in Lecture 3, except $n-k$ is used in place of n for better finite sample performance.

Remark 2: Similarly, the conditional test of $H_0^{\texttt{SL}}$: $\lambda_0 = 0$ given $\rho = 0$ is

$$\text{SLM}_{\texttt{SL}}^{\circ} = \frac{\tilde{\epsilon}_n' W_{1n}^{\circ} Y_n}{[\tilde{\sigma}_n^4\tilde{\kappa}_n\phi_1'\phi_1 + \tilde{\sigma}_n^4\texttt{tr}(\Phi_1\Phi_1^s) + 2\tilde{\sigma}_n^3\tilde{\gamma}_n\phi_1'\tilde{\eta} + \tilde{\sigma}_n^2\tilde{\eta}'\tilde{\eta}]^{\frac{1}{2}}},$$

where $\tilde{\sigma}_n^2 = \frac{n}{n-k}\tilde{\sigma}_n^2(0)$. This fills in a gap indicated in (3.50) of Lecture 3.

SLM Test based on OPMD Estimate of VC Matrix.

To derive MD (martingale difference) version of the SLM test, write

$$\tilde{S}_{\texttt{SLE},\delta}^{*} = \begin{cases} \epsilon_n'\Phi_1\epsilon_n + \epsilon_n'\eta \\ \epsilon_n\Phi_2\epsilon_n \end{cases} = \begin{cases} \epsilon_n'\Phi_1\epsilon_n + \epsilon_n'\eta - \sigma_0^2\texttt{tr}(\Phi_1), \\ \epsilon_n\Phi_2\epsilon_n - \sigma_0^2\texttt{tr}(\Phi_2). \end{cases}$$

Following Lemma 3.1 in Lecture 3, we have,

$$\tilde{S}_{\texttt{SLE},\delta}^{*} = \begin{cases} \sum_{i=1}^n [\epsilon_{ni}\xi_{1i} + \phi_{1,ii}(\epsilon_{ni}^2 - \sigma_0^2) + \eta_i\epsilon_{ni}] \equiv \sum_{i=1}^n g_{1,ni} \\ \sum_{i=1}^n [\epsilon_{ni}\xi_{2i} + \phi_{2,ii}(\epsilon_{ni}^2 - \sigma_0^2)] \equiv \sum_{i=1}^n g_{ni} \end{cases} \equiv \sum_{i=1}^n \mathbf{g}_{ni},$$

where $\phi_{r,ii}$ are the diagonal elements of $\Phi_r, r = 1, 2$, η_i the elements of η, and ξ_{ri} the elements of $(\Phi_r^{u\prime} + \Phi_r^l)\epsilon_n$ with Φ_r^u and Φ_r^l being the strictly upper and lower triangular matrices of Φ_r. As $\{\mathbf{g}_{ni}\}$ are uncorrelated with mean 0, $\sum_{i=1}^n \tilde{\mathbf{g}}_{2,ni}\tilde{\mathbf{g}}_{ni}'$ gives an OPMD estimate of $\text{Var}(\tilde{S}_{\texttt{SLE},\delta}^{*}|_{\delta=0})$ that is consistent, where $\tilde{\mathbf{g}}_{2,ni}$ is the (null) plug-in estimate of $\mathbf{g}_{2,ni}$ ($\tilde{\epsilon}_{ni}$ for ϵ_{ni}, $\tilde{\beta}_n(0)$ for β_0, $\frac{n}{n-k}\tilde{\sigma}_n^2(0)$ for σ_0^2).

The OPMD version of the SLM test takes the form:

$$\begin{pmatrix} \tilde{\epsilon}_n' W_{1n}^{\circ} Y_n \\ \tilde{\epsilon}_n' W_{2n}^{\circ}\tilde{\epsilon}_n \end{pmatrix}' \left(\sum_{i=1}^n \tilde{\mathbf{g}}_{2,ni}\tilde{\mathbf{g}}_{ni}'\right)^{-1} \begin{pmatrix} \tilde{\epsilon}_n' W_{1n}^{\circ} Y_n \\ \tilde{\epsilon}_n' W_{2n}^{\circ}\tilde{\epsilon}_n \end{pmatrix}. \tag{10}$$

Remark 3. Both $\texttt{tr}(\Phi_1)$ and $\texttt{tr}(\Phi_1)$ are zero, and thus with or without these terms does not affect $\tilde{S}_{\texttt{SLE},\delta}^{*}$, but it does affect the MD decomposition and thus potentially improves the OPMD estimate of $\text{Var}(\tilde{S}_{\texttt{SLE},\delta}^{*})$.

Remark 4. With (10), the two OPMD versions of the conditional tests of $H_0^{\texttt{SE}}$: $\lambda_0 = 0$ given $\rho = 0$ and $H_0^{\texttt{SE}}$: $\rho_0 = 0$ given $\lambda = 0$ can be easily obtained:

$$\text{SLM}_{\texttt{SL}}^{\texttt{MD}} = \frac{\tilde{\epsilon}_n' W_{1n}^{\circ} Y_n}{\sqrt{\sum_{i=1}^n \tilde{g}_{1,ni}^2}} \quad \text{and} \quad \text{SLM}_{\texttt{SE}}^{\texttt{MD}} = \frac{(n-k)\tilde{\epsilon}_n' W_{2n}^{\circ}\tilde{\epsilon}_n}{\sqrt{\sum_{i=1}^n \tilde{g}_{2,ni}^2}},$$

These fill in two gaps as indicated in Lecture 3.

(ii) To develop the two marginal SLM tests, $\mathrm{SLM}^{\circ}_{\mathtt{SL|SE}}$ for testing $H_0^{\mathtt{SL|SE}}$: $\lambda_0 = 0$, and $\mathrm{SLM}^{\circ}_{\mathtt{SE|SL}}$ for testing $H_0^{\mathtt{SE|SL}}$: $\rho_0 = 0$, one may choose to work with either the joint quasi score function given in (3), or the concentrated quasi score (CQS) function given in (6), following the principle laid out in Slides 9 and 10 of Lecture 3.

We choose to work with the CQS function given in (6) for its apparent advantage: being able take into account the estimation of β and σ^2.

It is sufficient to re-center and re-scale the two stochastic quantities in (6):

$$\tilde{\epsilon}_n'(\delta)B_n(\rho)W_{1n}Y_n \quad \text{and} \quad \tilde{\epsilon}_n'(\delta)G_n(\rho)\tilde{\epsilon}_n(\delta),$$

which have expectations at δ_0: $\sigma_0^2\mathrm{tr}(\mathbb{M}_n(\rho_0)\bar{F}_n(\lambda_0))$ and $\sigma_0^2\mathrm{tr}(\mathbb{M}_n(\rho_0)G_n(\lambda_0))$. This suggests one should first center these two quantities:

$$\begin{cases} \tilde{\epsilon}_n'(\delta_0)B_n(\rho_0)W_{1n}Y_n - \sigma_0^2\mathrm{tr}(\mathbb{M}_n(\rho_0)\bar{F}_n(\lambda_0)), \\ \tilde{\epsilon}_n'(\delta_0)G_n(\rho_0)\tilde{\epsilon}_n(\delta_0) - \sigma_0^2\mathrm{tr}(\mathbb{M}_n(\rho_0)G_n(\rho_0)), \end{cases}$$

where $\bar{F}_n(\lambda_0) = B_n(\rho_0)F_n(\lambda_0)B_n^{-1}(\rho_0)$, and work with their feasible versions by replacing σ_0^2 by its unbiased estimate $\frac{1}{n-k}\tilde{\epsilon}_n'(\delta_0)\tilde{\epsilon}_n(\delta_0) = \frac{1}{n-k}\tilde{\epsilon}_n'(\delta_0)\mathbb{Y}_n(\delta_0)$:

$$\tilde{S}^*_{n,\delta}(\delta_0) = \begin{cases} \tilde{\epsilon}_n'(\delta_0)\bar{F}_n(\lambda_0)\mathbb{Y}_n(\delta_0) - c_1(\delta_0)\tilde{\epsilon}_n'(\delta_0)\mathbb{Y}_n(\delta_0), \\ \tilde{\epsilon}_n'(\delta_0)G_n(\rho_0)\tilde{\epsilon}_n(\delta_0) - c_2(\rho_0)\tilde{\epsilon}_n'(\delta_0)\tilde{\epsilon}_n(\delta_0), \end{cases} \tag{11}$$

where $c_1(\delta_0) = \frac{1}{n-k}\mathrm{tr}(\mathbb{M}_n(\rho_0)\bar{F}_n(\lambda_0))$ and $c_2(\rho_0) = \frac{1}{n-k}\mathrm{tr}(\mathbb{M}_n(\rho_0)G_n(\rho_0))$. This leads to the unbiased estimation function of δ_0, which takes the form at general δ:

$$\tilde{S}^*_{n,\delta}(\delta) = \begin{cases} \tilde{\epsilon}_n'(\delta)[\bar{F}_n(\delta) - c_1(\delta)I_n]\mathbb{Y}_n(\delta), \\ \tilde{\epsilon}_n'(\delta)[G_n(\rho) - c_2(\rho)I_n]\tilde{\epsilon}_n(\delta). \end{cases} \tag{12}$$

With (12), we are ready to derive the two marginal SLM tests by applying the principle laid out in Slides 9 and 10, Lecture 3.

Let $\mathcal{I}_n = \mathrm{Var}[\tilde{S}^*_{n,\delta}(\delta_0)]$ and $\mathcal{J}_n = -\mathrm{E}[\frac{\partial}{\partial\delta}\tilde{S}^*_{n,\delta}(\delta_0)]$. Partition according to (λ, ρ): $\mathcal{I}_n = [\mathcal{I}_{n,\lambda\lambda}, \mathcal{I}_{\lambda\rho}; \mathcal{I}_{n,\rho\lambda}, \mathcal{I}_{n,\rho\rho}]$ and $\mathcal{J}_n = [\mathcal{J}_{n,\lambda\lambda}, \mathcal{J}_{\lambda\rho}; \mathcal{J}_{n,\rho\lambda}, \mathcal{J}_{n,\rho\rho}]$. The detailed analytical expressions of $\mathcal{I}_n$ and $\mathcal{J}_n$ can easily be obtained.

Then, the SLM test for testing $H_0^{\mathtt{SL|SE}}$: $\lambda_0 = 0$ has the general form:

$$\mathrm{SLM}^{\circ}_{\mathtt{SL|SE}} = \frac{\tilde{\epsilon}_n'(\tilde{\delta}_n)[\bar{F}_n(\tilde{\delta}_n) - c_1(\tilde{\delta}_n)I_n]\mathbb{Y}_n(\tilde{\delta}_n)}{(\tilde{\mathcal{I}}_{n,\lambda\lambda} - 2\tilde{\pi}_{1n}\tilde{\mathcal{I}}_{n,\lambda\rho} + \tilde{\pi}_{1n}^2\tilde{\mathcal{I}}_{n,\rho\rho})^{\frac{1}{2}}},$$

where $\tilde{\delta}_n = (0, \tilde{\rho}_n)'$ and $\pi_{1n} = \mathcal{J}_{\lambda\rho}/\mathcal{J}_{n,\rho\rho}$.

Similarly, the SLM test for testing $H_0^{\texttt{SE|SL}}$: $\rho_0 = 0$ has the general form:

$$\text{SLM}^{\circ}_{\texttt{SE|SL}} = \frac{\tilde{\epsilon}'_n(\tilde{\delta}_n)[G_n(0) - c_2(0)I_n]\tilde{\epsilon}_n(\tilde{\delta}_n)}{(\tilde{\mathcal{I}}_{n,\rho\rho} - 2\tilde{\pi}_{2n}\tilde{\mathcal{I}}_{n,\rho\lambda} + \tilde{\pi}^2_{2n}\tilde{\mathcal{I}}_{n,\lambda\lambda})^{\frac{1}{2}}},$$

where $\tilde{\delta}_n = (\tilde{\lambda}_n, 0)'$ and $\pi_{2n} = \mathcal{J}_{\rho\lambda}/\mathcal{J}_{n,\lambda\lambda}$. These fill in two gaps as indicated in (3.69) and (3.70), Lecture 3.

(iii) The OPMD versions of the two marginal SLM tests can be obtained following the general principle laid out in (3.13) and (3.14), Lecture 3. By (11), $\tilde{S}^*_{n,\delta}(\delta_0)$ can be written in the same form as the second expression of (8):

$$\tilde{S}^*_{n,\delta}(\delta_0) = \begin{cases} \epsilon'_n\Phi_1\epsilon_n + \epsilon'_n\eta, \\ \epsilon_n\Phi_2\epsilon_n, \end{cases}$$

but with Φ_r and $c_r, r = 1, 2$, and $\eta = M_n W_{1n} X_n \beta_0$ being redefined.

Again, $\texttt{tr}(\Phi_r) = 0, r = 1, 2$, but their elements are not. Therefore, the MD decomposition is based on

$$\tilde{S}^*_{n,\delta}(\delta_0) = \begin{cases} \epsilon'_n\Phi_1\epsilon_n + \epsilon'_n\eta - \sigma_0^2\texttt{tr}(\Phi_1) \equiv \sum_{i=1}^n g_{1,ni}, \\ \epsilon_n\Phi_2\epsilon_n - \sigma_0^2\texttt{tr}(\Phi_2) \equiv \sum_{i=1}^n g_{2,ni}. \end{cases}$$

The OPMD version of the marginal SLM test for testing $H_0^{\texttt{SL|SE}}$: $\lambda_0 = 0$ is,

$$\text{SLM}^{\circ}_{\texttt{SL|SE}} = \frac{\sum_{i=1}^n(\tilde{g}_{1,ni} - \pi_1\tilde{g}_{2,ni})}{\sqrt{\sum_{i=1}^n(\tilde{g}_{1,ni} - \pi_1\tilde{g}_{2,ni})^2}},$$

where $\tilde{g}_{r,ni}$ is the null estimate of $g_{r,ni}$ at $\tilde{\delta}_n = (0, \tilde{\rho}_n)'$, $\tilde{\beta}_n(\tilde{\delta}_n)$, $\tilde{\sigma}^2_n(\tilde{\delta}_n)$, and $\tilde{\epsilon}_n(\tilde{\delta}_n)$.

The OPMD version of the marginal SLM test for testing $H_0^{\texttt{SE|SL}}$: $\rho_0 = 0$ is,

$$\text{SLM}^{\circ}_{\texttt{SE|SL}} = \frac{\sum_{i=1}^n(\tilde{g}_{2,ni} - \pi_2\tilde{g}_{1,ni})}{\sqrt{\sum_{i=1}^n(\tilde{g}_{2,ni} - \pi_2\tilde{g}_{1,ni})^2}},$$

where $\tilde{g}_{r,ni}$ is the null estimate of $g_{r,ni}$ at $\tilde{\delta}_n = (\tilde{\lambda}_n, 0)'$, $\tilde{\beta}_n(\tilde{\delta}_n)$, $\tilde{\sigma}^2_n(\tilde{\delta}_n)$, and $\tilde{\epsilon}_n(\tilde{\delta}_n)$. These fill in another two gaps as indicated in (3.71) and (3.72), Lecture 3.

2. Consider fitting an SLE model to the **Boston House Price** data, introduced in Lectures 1 and 2, using all the covariates listed. Refer to Section 3.4, Lecture 3. Use both Matlab and Python to carry out the following analyses:

   (i) Based on (3.54) and the description below it, carry out the three LR tests, $\text{LR}_{\texttt{SL|SE}}$, $\text{LR}_{\texttt{SE|SL}}$ and $\text{LR}_{\texttt{SLE}}$, for testing the three hypotheses in Problem 1. Comment.

   (ii) Based on (3.59) and the description below it, carry out the three Wald tests, $\text{T}_{\texttt{SL|SE}}$, $\text{T}_{\texttt{SE|SL}}$, and $\text{T}_{\texttt{SLE}}$, for testing the three hypotheses in Problem 1. Comment.

(iii) Based on (3.66) and the descriptions below it, carry out the three LM tests $\mathtt{LM}^{\mathtt{FI}}_{\mathtt{SLE}}$, $\mathtt{LM}^{\mathtt{FI}}_{\mathtt{SL|SE}}$ and $\mathtt{LM}^{\mathtt{FI}}_{\mathtt{SE|SL}}$, for testing the three hypotheses in Problem 1. Comment.

(iv) Implement the six standardized LM tests developed in Problem 1, $\mathrm{SLM}^{\circ}_{\mathtt{SLE}}$ and $\mathrm{SLM}^{\mathtt{MD}}_{\mathtt{SLE}}$, $\mathrm{SLM}^{\circ}_{\mathtt{SL|SE}}$ and $\mathrm{SLM}^{\mathtt{MD}}_{\mathtt{SL|SE}}$, and $\mathrm{SLM}^{\circ}_{\mathtt{SE|SL}}$ and $\mathrm{SLM}^{\mathtt{MD}}_{\mathtt{SE|SL}}$. Comment.

(v) Based on the tests performed, which SLR model is most suitable for the data?

Solution: