Online Supplement to ‘Specification Tests based on MCMC
Output’

Yong Li
Renmin University of China

Jun Yu
Singapore Management University

Tao Zeng
Zhejiang University

February 12, 2018

The purpose of this online supplement is to prove Theorem 3.2 in Li, et al (2017),
that is, to show under $H_0$ that $\widehat{BIMT}$ has the same asymptotic distribution as BIMT
and that $\widehat{BMT}$ has the same asymptotic distribution as BMT. Based on Proposition
3.1, the relationship $\widehat{BIMT} = BIMT + o_p(n^{-1/2})$ is enough to guarantee that $\widehat{BIMT}$
and BIMT have the same asymptotic distribution. Based on Theorem 3.1, $\widehat{J}_1 = J_1 + o_p(1)$
and $\widehat{J}_0 = J_0 + o_p(1)$ are enough to guarantee that $\widehat{BMT}$ and BMT will have the same
asymptotic distribution. Therefore, what we try to find are an order condition for $M$ to
ensure $\widehat{BIMT} = BIMT + o_p(n^{-1/2})$ and order conditions for $M$ and $M_L$ to ensure $\widehat{J}_1 =
J_1 + o_p(1)$ and $\widehat{J}_0 = J_0 + o_p(1)$. Note that $\widehat{BIMT}$ and $\widehat{J}_0$ are based on MCMC output
obtained from the null model while $\widehat{J}_1$ is based on MCMC output obtained from both the
null model and the expanded model because $J_1 = \text{tr} \{ C_E (y, (\widehat{\theta}, \theta_E = 0)) V_E (\widehat{\theta}_E) \}$.

We organize this supplement as follows. In Section 1, we give an order condition for $M$
to ensure $\widehat{BIMT} = BIMT + o_p(n^{-1/2})$. In Section 2, we give order conditions for $M$
and $M_L$ to ensure that $\widehat{J}_1 = J_1 + o_p(1)$. In Section 3, we give an order condition for $M$
to ensure that $\widehat{J}_0 = J_0 + o_p(1)$. Section 4 proves Theorem 3.2. Throughout this supplement,
the sample size $n$ is assumed to go to infinity.

1 Order Condition for $M$ to Ensure $\widehat{BIMT} = BIMT + o_p(n^{-1/2})$

Under $H_0$, $\widehat{J}_n(\widehat{\theta}) = O_p(1)$ and $nV(\widehat{\theta}) = O_p(1)$. If $\widehat{J}_n(\widehat{\theta}) - \widehat{J}_n(\theta) = o_p(n^{-1/2})$ and
$n \left( \bar{V}(\widehat{\theta}) - V(\widehat{\theta}) \right) = o_p(n^{-1/2})$, then we will have

$$\widehat{BIMT} = \text{tr} \left\{ \widehat{J}_n \left( \widehat{\theta} \right) \bar{V} \left( \widehat{\theta} \right) \right\} = \text{tr} \left\{ \widehat{J}_n \left( \theta \right) n\bar{V} \left( \theta \right) \right\}$$

$$= \text{tr} \left\{ \left[ \widehat{J}_n \left( \theta \right) + o_p(n^{-1/2}) \right] \left[ n \bar{V} \left( \theta \right) - V \left( \theta \right) \right] + nV \left( \theta \right) \right\}$$

$$= \text{tr} \left\{ \left[ \widehat{J}_n \left( \theta \right) + o_p(n^{-1/2}) \right] \left[ nV \left( \theta \right) + o_p(n^{-1/2}) \right] \right\}$$

$$= \text{tr} \left\{ n\widehat{J}_n \left( \theta \right) V \left( \theta \right) \right\} + \text{tr} \left\{ \widehat{J}_n \left( \theta \right) o_p(n^{-1/2}) \right\} + \text{tr} \left\{ nV \left( \theta \right) o_p(n^{-1/2}) \right\} + o_p(n^{-1})$$
= \text{BIMT} + o_p(n^{-1/2}) = \text{IOS}_A + o_p(n^{-1/2}) = q \times \text{IR} + o_p(n^{-1/2}).

Together with Proposition 3.1, this will ensure that \( \hat{BIMT} \) has the same asymptotic distribution as BIMT. In Section 1.1 we give an order condition for \( M \) to ensure \( \hat{J}_n(\theta) - \hat{J}_n(\bar{\theta}) = o_p(n^{-1/2}) \). In Section 1.2 we then give an order condition for \( M \) to ensure \( n\left( \hat{V}(\theta) - V(\bar{\theta}) \right) = o_p(n^{-1/2}) \).

1.1 Order condition for \( M \) to ensure \( \hat{J}_n(\tilde{\theta}) - \hat{J}_n(\bar{\theta}) = o_p(n^{-1/2}) \)

Let us first assume \( \theta \) is a scalar. Let \( \sigma^2_{1n} \) be the long run variance of Markov chain, \( \left\{ \theta^{(m)}_n \right\}_{m=1}^{M} \), i.e., \( \sigma^2_{1n} = \text{Var}\left( \theta^{(1)}_n \right| y) + 2 \sum_{k=1}^{\infty} \gamma_n(k|y) \) where \( \gamma_n(k|y) \) is the \( k^{th} \) order autocovariance. Note that \( \text{Var}\left( \theta^{(1)}_n \right| y) \) is the posterior variance \( V(\bar{\theta}) \). We can rewrite \( \sigma^2_{1n} \) as

\[
\sigma^2_{1n} = \text{Var}\left( \theta^{(1)}_n \right| y) + 2 \sum_{k=1}^{\infty} \gamma_n(k|y) = 2 \sum_{k=0}^{\infty} \gamma_n(k|y) - \text{Var}\left( \theta^{(1)}_n \right| y) = \left( 2 \sum_{k=0}^{\infty} \frac{\gamma_{1n}(k|y)}{\text{Var}\left( \theta^{(1)}_n \right| y)} - 1 \right) \text{Var}\left( \theta^{(1)}_n \right| y) = \left( 2 \sum_{k=0}^{\infty} \rho(k) - 1 \right) \text{Var}\left( \theta^{(1)}_n \right| y).
\]

According to Jones (2004), under Assumption 13, as \( M \to \infty \), we have

\[
\sqrt{M} \sigma_{1n}^{-1}(\bar{\theta} - \tilde{\theta}) \overset{d}{\to} N(0,1).
\]

By the Taylor expansion and (1), we have

\[
\hat{J}_n(\theta) = \frac{1}{n} \sum_{t=1}^{n} s_t(\theta)^2 = \frac{1}{n} \sum_{t=1}^{n} \left[ s_t(\theta) + h_t(\tilde{\theta}_4)(\theta - \tilde{\theta}) \right]^2
\]

\[
= \frac{1}{n} \sum_{t=1}^{n} s_t(\theta)^2 + \frac{2}{n} \sum_{t=1}^{n} h_t(\tilde{\theta}_4)(\theta - \bar{\theta}) s_t(\bar{\theta}) + \frac{1}{n} \sum_{t=1}^{n} h_t(\tilde{\theta}_4)(\theta - \bar{\theta})^2
\]

\[
= \hat{J}_n(\theta) + O \left( \frac{1}{\sqrt{M}} \sigma_{1n} \right) 2 \sum_{t=1}^{n} h_t(\tilde{\theta}_4) s_t(\bar{\theta}) + O \left( \frac{1}{M} \sigma^2_{1n} \right) \frac{1}{n} \sum_{t=1}^{n} h_t(\tilde{\theta}_4)^2
\]

\[
= \hat{J}_n(\theta) + O \left( \frac{1}{\sqrt{M}} \sigma_{1n} \right) O_p(1) + O \left( \frac{1}{M} \sigma^2_{1n} \right) O_p(1),
\]

where \( \tilde{\theta}_4 \) lies between \( \theta \) and \( \bar{\theta} \) and \( \frac{2}{n} \sum_{t=1}^{n} h_t(\tilde{\theta}_4) s_t(\theta) = O_p(1) \) and \( \frac{1}{n} \sum_{t=1}^{n} h_t(\tilde{\theta}_4)^2 = O_p(1) \) by Assumptions 10-12.

To show

\[
\hat{J}_n(\tilde{\theta}) = \hat{J}_n(\theta) + o_p(n^{-1/2}),
\]

it is enough to have

\[
\frac{1}{\sqrt{M}} \sigma_{1n} = o_p \left( n^{-1/2} \right),
\]

(3)
which is equivalent to

\[ M = O\left( n^{1+c_1^2\sigma_{1n}^2} \right), \quad \text{for any } c_1^2 > 0. \]  

(4)

The condition (3) is also used in Chen, Gao and Phillips (2017) to obtain the asymptotic normality of \( \tilde{\theta} \) when \( M \to \infty \) and \( n \to \infty \).

In this paper, \( \theta \) is a \( q \)-dimensional vector. That is why we require

\[ M = n^{1+c_1^2\sigma_{1n}^2}, \]  

(5)

with \( \sigma_{1n}^2 = \max_{a \in \{1, \ldots, q\}} \sigma_{n,a}^2 \) and \( \sigma_{n,a}^2 \) being the long run variance of \( \{\theta_a^{(m)}\}_{m=1}^M \) for \( a = 1, \ldots, q \).

1.2 Order condition for \( M \) to ensure \( n \left[ \tilde{V}(\tilde{\theta}) - V(\tilde{\theta}) \right] = o_p(n^{-1/2}) \)

Again, let us first assume \( \theta \) is a scalar. Let \( \sigma_{2n}^2 = \text{Var} \left( \left( \theta_n^{(1)} - \bar{\theta} \right)^2 \mid y \right) + 2 \sum_{k=1}^\infty \gamma_{2n}(k \mid y) \) be the long run variance of \( \left\{ \left( \theta_n^{(m)} - \bar{\theta} \right)^2 \right\}_{m=1}^M \) where \( \gamma_{2n}(k \mid y) \) is the \( k \)th order autocovariance of \( \left( \theta_n^{(m)} - \bar{\theta} \right)^2 \).

Note that

\[
\tilde{V}(\tilde{\theta}) = \frac{1}{M} \sum_{m=1}^M \left( \theta_n^{(m)} - \bar{\theta} \right)^2 = \frac{1}{M} \sum_{m=1}^M \left( \theta_n^{(m)} - \bar{\theta} + \bar{\theta} - \bar{\theta} \right)^2 \\
= \frac{1}{M} \sum_{m=1}^M \left( \theta_n^{(m)} - \bar{\theta} \right)^2 - 2 \frac{1}{M} \sum_{m=1}^M \left( \theta_n^{(m)} - \bar{\theta} \right) \left( \bar{\theta} - \bar{\theta} \right) + \frac{1}{M} \sum_{m=1}^M \left( \bar{\theta} - \bar{\theta} \right)^2 \\
= \frac{1}{M} \sum_{m=1}^M \left( \theta_n^{(m)} - \bar{\theta} \right)^2 - \frac{1}{M} \sum_{m=1}^M \left( \bar{\theta} - \bar{\theta} \right)^2.
\]

Then we have

\[
\sqrt{M} \left( \tilde{V}(\tilde{\theta}) - V(\tilde{\theta}) \right) = \frac{1}{\sqrt{M}} \sum_{m=1}^M \left( \theta_n^{(m)} - \bar{\theta} \right)^2 - \sqrt{M} V(\tilde{\theta}) - \sqrt{M} \left( \bar{\theta} - \bar{\theta} \right)^2 \\
= \sqrt{M} \left( \frac{1}{M} \sum_{m=1}^M \left( \theta_n^{(m)} - \bar{\theta} \right)^2 - V(\tilde{\theta}) \right) - \sqrt{M} \left( \bar{\theta} - \bar{\theta} \right)^2.
\]

Thus,

\[
\sqrt{M} \sigma_{2n}^{-1} \left( \tilde{V}(\tilde{\theta}) - V(\tilde{\theta}) \right) = \sqrt{M} \sigma_{2n}^{-1} \left( \frac{1}{M} \sum_{m=1}^M \left( \theta_n^{(m)} - \bar{\theta} \right)^2 - V(\tilde{\theta}) \right) - \sqrt{M} \sigma_{2n}^{-1} \left( \bar{\theta} - \bar{\theta} \right)^2.
\]

By (1), \( \sqrt{M} \sigma_{2n}^{-1} \left( \bar{\theta} - \bar{\theta} \right)^2 \xrightarrow{p} 0 \) as \( M \to \infty \). Note that

\[
\sqrt{M} \sigma_{2n}^{-1} \left( \frac{1}{M} \sum_{m=1}^M \left( \theta_n^{(m)} - \bar{\theta} \right)^2 - V(\tilde{\theta}) \right) \xrightarrow{d} N(0, 1),
\]

(6)
by the central limit theorem for Markov chains (Jones, 2004) under Assumption 13. Hence, we have
\[
\sqrt{M} \sigma_{2n}^{-1} \left( \tilde{V} \left( \bar{\theta} \right) - V \left( \bar{\theta} \right) \right) \xrightarrow{d} N \left( 0, 1 \right),
\]
by the Slusky Theorem. It can be shown that
\[
n \left( \tilde{V} \left( \bar{\theta} \right) - V \left( \bar{\theta} \right) \right) = \frac{n}{\sqrt{M}} \sigma_{2n}^{-1} \left( \sqrt{M} \sigma_{2n}^{-1} \left( \tilde{V} \left( \bar{\theta} \right) - V \left( \bar{\theta} \right) \right) \right) = o_p \left( n^{-1/2} \right),
\]
if
\[
n \sqrt{M} \sigma_{2n}^{-1} = o \left( n^{-1/2} \right),
\]
which is equivalent to
\[
M = O \left( n^{3+c^*_2 \sigma^2_{2n}} \right), \text{ for any } c^*_2 > 0.
\]
Since \( \theta \) is \( q \)-dimensional, we require
\[
M = n^{3+c^*_2 \sigma^2_{2n}},
\]
with \( \sigma^2_{2n} = \max_{b \in \{1, \ldots, r\}} \sigma^2_{2n,b} \) and \( \sigma^2_{2n,b} \) being the long run variance of \( \left\{ \phi_b^{(m)} \right\} \left\{ M \right\}_{m=1} \) for \( b = 1, \ldots, r \), where \( \phi = vech \left[ (\theta - \bar{\theta}) (\theta - \bar{\theta})' \right] \).

2 Order Conditions for \( M \) and \( M_L \) to Ensure \( \tilde{J}_1 - J_1 = o_p(1) \)

Following Li, et al (2015), \( \frac{1}{n} C \left( y, (\tilde{\theta}, \theta_E = 0) \right) = O_p(1) \) and \( n V(\bar{\theta}_L) = O_p(1) \). If
\[
\frac{1}{n} \left[ C \left( y, (\tilde{\theta}, \theta_E = 0) \right) - C \left( y, (\bar{\theta}, \theta_E = 0) \right) \right] = o_p(1),
\]
and
\[
n \left( \tilde{V}(\bar{\theta}_L) - V(\bar{\theta}_L) \right) = o_p(1),
\]
we will have
\[
\tilde{J}_1 = \text{tr} \left\{ C_E \left( y, (\tilde{\theta}, \theta_E = 0) \right) \tilde{V}_E \left( \bar{\theta}_L \right) \right\} = \text{tr} \left\{ \frac{1}{n} C_E \left( y, (\tilde{\theta}, \theta_E = 0) \right) n \tilde{V}_E \left( \bar{\theta}_L \right) \right\} = \text{tr} \left\{ \left[ \frac{1}{n} C_E \left( y, (\tilde{\theta}, \theta_E = 0) \right) + o_p(1) \right] \left[ n V_E \left( \bar{\theta}_L \right) + o_p(1) \right] \right\} = \text{tr} \left\{ C_E \left( y, (\tilde{\theta}, \theta_E = 0) \right) V_E \left( \bar{\theta}_L \right) \right\} + o_p(1).
\]
Hence, for \( \tilde{BMT} = BMT + o_p(1) \), we need to obtain an order condition for \( M \) in the original model to ensure (11) and an order condition for \( M_L \) in the expanded model to ensure (12).
2.1 Order condition for $M$ to ensure $\frac{1}{n} \left[ C \left( y, \left( \tilde{\theta}, \theta_E = 0 \right) \right) - C \left( y, (\theta, \theta_E = 0) \right) \right] = o_p(1)$

Let us first assume $\theta$ is a scalar. By the Taylor expansion, we have

$$
\frac{1}{n} C \left( y, \left( \tilde{\theta}, \theta_E = 0 \right) \right)
= \frac{1}{n} \sum_{t=1}^{n} s_t \left( \tilde{\theta}, \theta_E = 0 \right) \sum_{t=1}^{n} s_t \left( \tilde{\theta}, \theta_E = 0 \right)'
= \frac{1}{n} \sum_{t=1}^{n} \left[ s_t \left( \tilde{\theta}, \theta_E = 0 \right) + h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \right] \times \sum_{t=1}^{n} \left[ s_t \left( \tilde{\theta}, \theta_E = 0 \right) + h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \right]'
= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[ s_t \left( \tilde{\theta}, \theta_E = 0 \right) + h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \right] \times \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[ s_t \left( \tilde{\theta}, \theta_E = 0 \right) + h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \right]'
= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} s_t \left( \tilde{\theta}, \theta_E = 0 \right) \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} s_t \left( \tilde{\theta}, \theta_E = 0 \right) \right)'
+ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} s_t \left( \tilde{\theta}, \theta_E = 0 \right) \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \right)'
+ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} s_t \left( \tilde{\theta}, \theta_E = 0 \right) \right)'
+ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \right),
$$

where $\tilde{\theta}_5$ lies between $\tilde{\theta}$ and $\tilde{\theta}$. Under the null hypothesis, we have

$$
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} s_t \left( \tilde{\theta}, \theta_E = 0 \right) \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \right)'
= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} s_t \left( \tilde{\theta}, \theta_E = 0 \right) \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \right) \left( \sqrt{M} \sigma_{1n} \left( \sqrt{\sqrt{M} \sigma_{1n}^{-1} \left( \tilde{\theta} - \tilde{\theta}, 0 \right) } \right) \right)'
= O_p(1) O_p(1) O \left( \frac{\sqrt{n}}{\sqrt{M} \sigma_{1n}} \right) O_p(1) = O_p \left( \frac{\sqrt{n}}{\sqrt{M} \sigma_{1n}} \right),
$$

and

$$
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right) \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \left( \tilde{\theta} - \tilde{\theta}, 0 \right)' \right)'
$$
\[
\begin{align*}
&= \frac{1}{n} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \sqrt{n} \left( \tilde{\theta} - \bar{\theta}, 0 \right) \left( \frac{1}{n} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \sqrt{n} \left( \tilde{\theta} - \bar{\theta}, 0 \right) \right)' \\
&= \frac{1}{n} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \frac{\sqrt{n}}{\sqrt{M}} \sigma_{1n} \left( \sqrt{M} \sigma_{1n}^{-1} \left( \tilde{\theta} - \bar{\theta}, 0 \right) \right) \times \\
&\quad \left( \frac{1}{n} \sum_{t=1}^{n} h_t \left( \tilde{\theta}_5, \theta_E = 0 \right) \frac{\sqrt{n}}{\sqrt{M}} \sigma_{1n} \left( \sqrt{M} \sigma_{1n}^{-1} \left( \tilde{\theta} - \bar{\theta}, 0 \right) \right) \right)' \\
&= O_p(1) O \left( \frac{\sqrt{n}}{\sqrt{M}} \sigma_{1n} \right) O_p(1) O \left( \frac{\sqrt{n}}{\sqrt{M}} \sigma_{1n} \right) O_p(1) = O_p \left( \frac{n^2}{M \sigma_{1n}^2} \right)
\end{align*}
\]

Hence, if \( O \left( \frac{\sqrt{n}}{\sqrt{M}} \sigma_{1n} \right) = o(1) \), that is,
\[
M = O(n^{1+c_3^* \sigma_{1n}^2}), \text{ for any } c_3^* > 0, \tag{13}
\]

then
\[
\frac{1}{n} C_E \left( y, \left( \tilde{\theta}, \theta_E = 0 \right) \right) = \frac{1}{n} C_E \left( y, \left( \tilde{\theta}, \theta_E = 0 \right) \right) + o_p(1).
\]

Again, since \( \theta \) is \( q \)-dimensional, we set
\[
M = n^{1+c_3^* \sigma_{1n}^2}, \text{ for any } c_3^* > 0. \tag{14}
\]

2.2 Order condition for \( M_L \) to ensure \( n \left( \tilde{V}(\tilde{\theta}_L) - V(\bar{\theta}_L) \right) = o_p(1) \)

Since \( \left\{ \theta_{L_n}^{(m)} \right\}_{m=1}^{M_L} \) is a geometrically ergodic Markov chain with stationary distribution as the posterior distribution of \( \theta_L \), the MCMC estimators of posterior mean \( \tilde{\theta}_L \) and posterior variance \( V(\bar{\theta}_L) \) can be given by
\[
\bar{\theta}_L = \frac{1}{M_L} \sum_{m=1}^{M_L} \theta_{L_n}^{(m)}, \quad \tilde{V}(\bar{\theta}_L) = \frac{1}{M_L} \sum_{m=1}^{M_L} \left( \theta_{L_n}^{(m)} - \bar{\theta}_L \right) \left( \theta_{L_n}^{(m)} - \bar{\theta}_L \right)'.
\]

Let \( \sigma_{L_n,b}^2 \) be the long run variance of \( \left\{ \theta_{L_n,b}^{(m)} \right\}_{m=1}^{M_L} \) for \( b = 1, 2, \ldots, r_L \) where \( \theta_L = vech \left[ \left( \theta_L - \bar{\theta}_L \right) \left( \theta_L - \bar{\theta}_L \right)' \right] \). If we choose
\[
M_L = n^{2+c_5^* \sigma_{L_n}^2}, \text{ for any } c_5^* > 0, \tag{15}
\]

with \( \sigma_{L_n}^2 = \max_{b \in \{1,2,\ldots,q_L(q_L+1)/2\}} \sigma_{L_n,b}^2 \), then using the same proof as in Section 1.2, we can show that
\[
n \left( \tilde{V}(\tilde{\theta}_L) - V(\bar{\theta}_L) \right) = o_p(n^{-1/2}),
\]
\[
n \left( \tilde{V}(\tilde{\theta}_L) - V(\bar{\theta}_L) \right) = o_p(1).
\]
3 Order Condition for \( M \) to Ensure \( \tilde{J}_0 - J_0 = o_p(1) \)

\( \hat{\text{BIMT}} = \text{BIMT} + o_p(n^{-1/4}) \) is a sufficient condition to ensure \( \tilde{J}_0 - J_0 = o_p(1) \). This is because if \( \hat{\text{BIMT}} = \text{BIMT} + o_p(n^{-1/4}) \), then we have

\[
\tilde{J}_0 = \sqrt{n} \left( \frac{\hat{\text{BIMT}}}{q} - 1 \right)^2 = \sqrt{n} \left( \frac{\text{BIMT}}{q} + o_p \left( \frac{n^{-1/4}}{q} - 1 \right) \right)^2 = \sqrt{n} \left( \text{BIMT}/q - 1 \right)^2 + \sqrt{n} o_p \left( \frac{n^{-1/2}}{q} - 2 \sqrt{n} (\text{BIMT}/q - 1) o_p \left( \frac{n^{-1/4}}{q} \right) \right)
\]

\[
= J_0 + o_p(1) + o_p \left( \frac{n^{-1/4}}{q} \right) = J_0 + o_p(1).
\]

In Section 1, we have shown that under the null hypothesis, \( \hat{J}_n(\theta) = O_p(1) \) and \( nV(\theta) = O_p(1) \). If \( \hat{J}_n(\theta) - \tilde{J}_n(\theta) = o_p(n^{-1/4}) \) and \( n \left( \tilde{V}(\theta) - V(\theta) \right) = o_p(n^{-1/4}) \), then we have

\[
\hat{\text{BIMT}} = n \text{tr} \left\{ \hat{J}_n(\theta) \tilde{V}(\theta) \right\} = \text{tr} \left\{ \hat{J}_n(\theta) n \tilde{V}(\theta) \right\}
\]

\[
= \text{tr} \left\{ [\hat{J}_n(\theta) + o_p(n^{-1/4})] \left[ n \left( \tilde{V}(\theta) - V(\theta) \right) + nV(\theta) \right] \right\}
\]

\[
= \text{tr} \left\{ [\hat{J}_n(\theta) + o_p(n^{-1/4})] \left[ nV(\theta) + o_p(n^{-1/4}) \right] \right\}
\]

\[
= \text{tr} \left\{ n \hat{J}_n(\theta) V(\theta) \right\} + \text{tr} \left\{ \hat{J}_n(\theta) o_p(n^{-1/4}) \right\} + \text{tr} \left\{ nV(\theta) o_p(n^{-1/4}) \right\} + o_p(n^{-1/2})
\]

\[
= \text{BIMT} + o_p(n^{-1/4}) O_p(1) + o_p(n^{-1/4}) O_p(1) + o_p(n^{-1/2})
\]

\[
= \text{BIMT} + o_p(n^{-1/4}).
\]

According to (8) and (9), to ensure \( n \left( \tilde{V}(\theta) - V(\theta) \right) = o_p(n^{-1/4}) \), we only need

\[
\frac{n}{\sqrt{M}} \sigma^2_{2n} = o(n^{-1/4}),
\]

which is equivalent to

\[
M = O \left( n^{2.5 + c^*_1 \sigma^2_{2n}} \right), \text{ for any } c^*_1 > 0.
\]

This order condition is weaker than that specified in (10).

Furthermore, according to (3) and (4), to ensure \( \hat{J}_n(\theta) = \hat{J}_n(\theta) + o_p(n^{-1/4}) \), we only need

\[
\frac{1}{\sqrt{M}} \sigma^*_1 n = o \left( n^{-1/4} \right),
\]

which is equivalent to

\[
M = O \left( n^{0.5 + c^*_2 \sigma^2_{1n}} \right), \text{ for any } c^*_2 > 0.
\]

This order condition is weaker than that specified in (5).
4 Proof of Theorem 3.2

Combining the order conditions given by (5) and (10) provides the order condition for $M_{BIMT}$ given by Equation (8) in Theorem 3.2 when BIMT is used.

Regarding BMT, according Section 2, if $M = n^{1+c_3^*} \sigma_{1n}^2$ in the original model, then

$$
\frac{1}{n} C_E \left( y, \left( \hat{\theta}, \theta_E = 0 \right) \right) = \frac{1}{n} C_E \left( y, \left( \hat{\theta}, \theta_E = 0 \right) \right) + o_p(1).
$$

If $M_L = n^{2+c_3^*} \sigma_{Ln}^2$ in the expanded model, then

$$
n \left( \bar{V}(\hat{\theta}_L) - V(\hat{\theta}_L) \right) = o_p(1).
$$

Under these two order conditions (one for $M$ and one for $M_L$) we have $J_1 = J_1 + o_p(1)$.

According to Section 3, if $M = \max \left\{ n^{0.5+c_3^*} \sigma_{1n}^2, n^{2.5+c_3^*} \sigma_{2n}^2 \right\}$, we have $J_0 = J_0 + o_p(1)$.

From Section 2, if $M = n^{1+c_3^*} \sigma_{1n}^2$ and $M_L = n^{2+c_3^*} \sigma_{Ln}^2$, we have $J_1 = J_1 + o_p(1)$. Hence, if we set the number of MCMC draws to

$$
M_{BMT} = \max \left\{ n^{0.5+c_3^*} \sigma_{1n}^2, n^{1+c_3^*} \sigma_{1n}^2, n^{2.5+c_3^*} \sigma_{2n}^2 \right\} = \max \left\{ n^{1+c_3^*} \sigma_{1n}^2, n^{2.5+c_3^*} \sigma_{2n}^2 \right\}
$$

in the original model and to

$$
M_L = n^{2+c_3^*} \sigma_{Ln}^2
$$

in the expanded model, then we have

$$
\tilde{BMT} = \tilde{J}_1 + \tilde{J}_0 = BMT + o_p(1).
$$

This proves the asymptotic equivalence of $\tilde{BMT}$ and BMT. Furthermore, under $H_0$, $\tilde{BMT}$ converges to $\chi^2(q_E)$.

To derive the power property of $\tilde{BMT}$, note that $\tilde{BMT} = BMT + o_p(n^{-1/4})$ also holds true for misspecified models when $M = \max \left\{ n^{1+c_3^*} \sigma_{1n}^2, n^{2.5+c_3^*} \sigma_{2n}^2 \right\}$. In addition, when the model is misspecified so that $q^* \neq q$, from Theorem 3.1, we can show that

$$
\tilde{J}_0 &= \sqrt{n} \left( \tilde{BMT} / q - 1 \right)^2 \\
&= \sqrt{n} \left( BMT / q + o_p \left( n^{-1/4} \right) / q - 1 \right)^2 \\
&= \sqrt{n} (BMT / q - 1)^2 + \sqrt{n} o_p \left( n^{-1/2} \right) - 2 \sqrt{n} (BMT / q - 1) o_p \left( n^{-1/4} \right) / q \\
&= J_0 + o_p(1) + o_p \left( n^{-1/4} \right) = J_0 + o_p(1) = O_p(\sqrt{n}).
$$

Hence, the order of the power of $\tilde{BMT}$ is no less than $O_p(\sqrt{n})$. This completes the proofs of Theorem 3.2.

5 Reference
