Financial Mathematics for Actuaries

Chapter 6

Bonds and Bond Pricing

Learning Objectives

1. Types, features and risks of bond investments

2. Formulas for pricing a bond

3. Construction of bond amortization schedules

4. Pricing a bond between two coupon-payment dates

5. Callable bonds and their pricing approaches

6. Price of a bond under a nonflat term structure

6.1 Basic Concepts

- A **bond** is a contract/certificate of debt in which the issuer promises to pay the holder a *definite sequence of interest payments* for a *specified period of time*, and to repay the loan at a specified *terminal date* (called the **maturity** or **redemption date**).

- There are many risks involved in investing in bonds, including

Interest-rate risk: Bond investors receive interest payments periodically before the bond redemption date. Coupon interest payments are usually constant during the life of the bond. After the investor purchased the bond, if the prevailing interest rate rises, the price of the bond will fall as the bond is less attractive. This is called interest-rate risk.

Default risk: Default risk is the risk that the bond issuer is unable to make interest payments and/or redemption repayment. Based on the bond issuer's financial strength, bond rating agencies provide a rating (a measure of the quality and safety) of the bond. Bonds are classified by rating agencies into two categories: **investment-grade bonds** (a safer class) and **junk bonds** (a riskier class).

Reinvestment risk: For coupon-paying bonds, investors receive interest payments periodically before the redemption date. The reinvestment of these interest payments (sometimes referred to as interest-on-interest) depends on the prevailing interest-rate level at the time of reinvestment. **Zero-coupon bonds** do not have reinvestment risk.

Call risk: The issuer of a **callable bond** has the right to redeem the bond prior to its maturity date at a preset **call price** under certain conditions. These conditions are specified at the bond issue date, and are

known to the investors. The issuer will typically consider calling a bond if it is paying a higher coupon rate than the current market interest rate. Callable bonds often carry a **call protection** provision. It specifies a period of time during which the bond cannot be called.

Inflation risk: Inflation risk arises because of the uncertainty in the real value (i.e., purchasing power) of the cash flows from a bond due to inflation. **Inflation-indexed bonds** are popular among investors who do not wish to bear inflation risk.

Other risks in investing in bonds include: **market risk**, which is the risk that the bond market as a whole declines; **event risk**, which arises when some specific events occur; **liquidity risk**, which is the risk that investors may have difficulty finding a counterparty to trade.

6.2 Bond Evaluation

- The following features of a bond are often agreed upon at the issue date.

Face Value: Face value, denoted by F, also known as **par** or **principal** value, is the amount printed on the bond.

Redemption Value: A bond's redemption value or **maturity value**, denoted by C, is the amount that the issuer promises to pay on the redemption date. In most cases the redemption value is the same as the face value.

Time to Maturity: Time to Maturity refers to the length of time before the redemption value is repaid to the investor.

Coupon Rate: The coupon rate, denoted by r, is the rate at which the bond pays interest on its face value at regular time intervals until the redemption date.

- We only consider the financial mathematics of default-free bonds.

- We denote n as the number of coupon payment periods from the **date of purchase** (or the **settlement date**) to the maturity date, P as the purchase price, and i as the market rate of interest (called the **yield rate**).

- The yield rate reflects the current market conditions, and is determined by the market forces, giving investors a fair compensation in bearing the risks of investing in the bond.

- We assume that r and i are measured per coupon-payment period.

Thus, for semiannual coupon bonds, r and i are the rate of interest per half-year.

- The price of a bond is the sum of the present values of all coupon payments plus the present value of the redemption value due at maturity.

- We assume that a coupon has just been paid, and we are interested in pricing the bond after this payment. Figure 6.1 illustrates the cash-flow pattern of a typical coupon bond.

- We shall assume that the term structure is flat, so that cash flows at all times are discounted at the same yield rate i.

- Thus, the *fair price* P of the bond is given by the following basic price formula

$$P = (Fr)a_{\overline{n}|} + Cv^n, \tag{6.1}$$
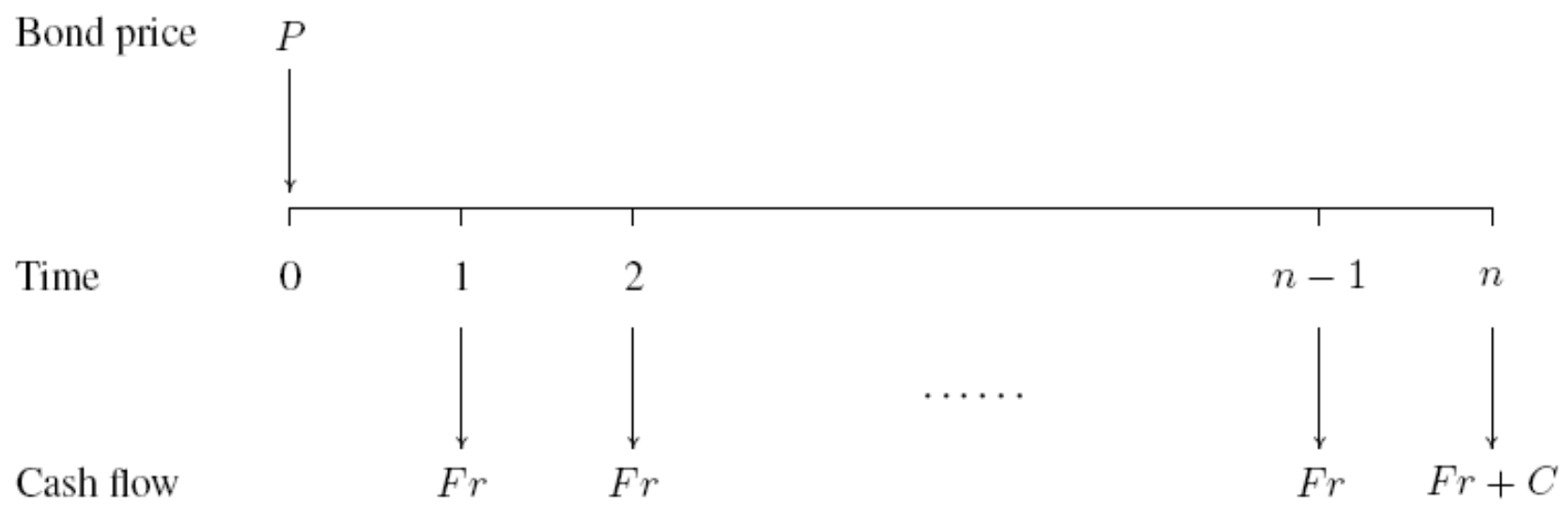
8

Figure 6.1: Cash-flow pattern of a coupon-paying bond with n coupons

where the interest and annuity functions are calculated at the yield rate i.

Example 6.1: The following shows the results of a government bond auction:

Type of Bond	*Government bond*
Issue Date	*February 15, 2010*
Maturity Date	*February 15, 2040*
Coupon Rate	*4.500% payable semiannually*
Yield Rate	*4.530% convertible semiannually*

Assume that the redemption value of the bond is the same as the face value, which is \$100. Find the price of the bond.

Solution: This is a 30-year bond and we can use the basic price formula (6.1) with

$$F = 100,$$

$$
\begin{aligned}
C &= 100, \\
r &= \frac{0.045}{2} = 0.0225, \\
i &= \frac{0.0453}{2} = 0.02265, \\
n &= 60,
\end{aligned}
$$

to calculate the price of the bond, which is

$$
\begin{aligned}
P &= (Fr)a_{\overline{n}|} + Cv^n \\
&= 2.25a_{\overline{60}|0.02265} + 100(1.02265)^{-60} \\
&= \$99.51.
\end{aligned}
$$

$\square$

Example 6.2: The following shows the information of a government bond traded in the secondary market.

Type of Bond	*Government bond*
Issue Date	*June 15, 2005*
Date of Purchase	*June 15, 2009*
Maturity Date	*June 15, 2020*
Coupon Rate	*4.2% payable semiannually*
Yield Rate	*4.0% convertible semiannually*

Assume that the redemption value of the bond is the same as the face value, which is $100. Find the purchase price of the bond immediately after its 8th coupon payment on June 15, 2009.

Solution: This is a 15-year bond at issue but there are only 22 remaining coupon-payment periods after its 8th coupon payment on June 15, 2009. We use the basic price formula (6.1) with

$$F = 100,$$
$$C = 100,$$

$$r = \frac{0.042}{2} = 0.021,$$

$$i = \frac{0.040}{2} = 0.020,$$

$$n = 22,$$

to obtain

$$
\begin{aligned}
P &= (Fr)a_{\overline{n}|} + Cv^n \\
&= 2.1a_{\overline{22}|0.02} + 100(1.02)^{-22} \\
&= \$101.77.
\end{aligned}
$$

$\square$

Example 6.4: Show that

$$P = K + \frac{g}{i}(C - K),$$

where $K = Cv^n$ is the present value of the redemption payment and $g = Fr/C$ is the **modified coupon rate**.

Solution: We start with (6.1) to obtain

$$
\begin{aligned}
P &= (Fr)a_{\overline{n}|} + Cv^n \\
&= (Fr)\left(\frac{1 - v^n}{i}\right) + K \\
&= Cg\left(\frac{1 - v^n}{i}\right) + K \\
&= \frac{g}{i}(C - Cv^n) + K \\
&= \frac{g}{i}(C - K) + K. \tag{6.2}
\end{aligned}
$$

This formula is called the **Makeham formula**. $\square$

6.3 Bond Amortization Schedule

- From the basic bond price formula, we have:

$$\begin{aligned} P &= (Fr)a_{\overline{n}|} + Cv^n \\ &= (Fr)a_{\overline{n}|} + C(1 - ia_{\overline{n}|}) \\ &= C + (Fr - Ci)a_{\overline{n}|}, \end{aligned} \tag{6.3}$$

which is called the **premium/discount formula** because

$$P - C = (Fr - Ci)a_{\overline{n}|} \tag{6.4}$$

represents the bond **premium** (when it is positive) or the bond **discount** (when it is negative).

- In other words, if the selling price of a bond is larger than its redemption value, the bond is said to be traded at a premium of value $P - C = (Fr - Ci)a_{\overline{n}|}$.

- On the other hand, if the selling price of a bond is less than its redemption value, the bond is said to be traded at a discount of amount $C - P = (Ci - Fr)a_{\overline{n}|}$.

- In most cases the redemption value is the same as the face value (i.e., $C = F$), so that the bond is traded at

Premium:	$P - F = F(r - i)a_{\overline{n}|}$,	if $r > i$,
Par:	$P - F = 0$,	if $r = i$,
Discount:	$F - P = F(i - r)a_{\overline{n}|}$,	if $i > r$.

- In the bond premium situation the bondholder pays more than the face value. The premium represents an amount that the investor will not receive at maturity.

- This paid-in-advance premium (which is an asset for the bondholder) will be refunded (amortized) periodically from the coupon payments over the life of the bond.

- We consider the **effective interest method** for the amortization of a bond.

- As an illustrative example, we consider a $1,000 face value (same as the redemption value) 3-year bond with semiannual coupons at the rate of 5% per annum. The current required rate of return i for the bond is 4% convertible semiannually. The price of the bond is

$$
\begin{aligned}
P &= (1{,}000 \times 0.025)a_{\overline{6}|0.02} + 1{,}000(1.02)^{-6} \\
&= \$1{,}028.01,
\end{aligned}
$$

and the premium is $28.01.

- In each half-year period a $25 coupon payment is received. This amount is larger than the interest earned at the yield rate based on the book value, which is $1,028.01 \times 0.02 = \$20.56$ for the first half-year.

- The remaining part of the coupon payment is then used to reduce the unamortized premium. We can construct a bond premium amortization schedule based on this principle. The result is shown in Table 6.1.

Table 6.1: A bond premium amortization schedule

Half-year	Coupon payment	Effective interest earned	Amortized amount of premium	Book value
0				$1{,}028.01^a$
1	25.00^b	20.56^c	4.44^d	$1{,}023.57^e$
2	25.00	20.47	4.53	1,019.04
3	25.00	20.38	4.62	1,014.42
4	25.00	20.29	4.71	1,009.71
5	25.00	20.19	4.81	1,004.90
6	25.00	20.10	4.90	1,000.00
Total	150.00	121.99	28.01	

The details of the computation are as follows:

[a] The purchase price is the beginning balance of the book value.

[b] The coupon payment is $1{,}000(0.05/2) = \$25.00$.

[c] Effective interest earned in the first half-year is $1{,}028.01(0.04/2) = \$20.56$.

[d] Amortized amount of premium in the first half-year is $25.00 - 20.56 = \$4.44$.

[e] The book value after amortization is $1{,}028.01 - 4.44 = \$1{,}023.57$.

- The bond amortization schedule is useful to bondholders for accounting and taxation purposes. In some countries the bond interest income is taxable. In the above example, however, the $25 coupon payment received each period should not be fully taxable.

- In the bond discount situation the bondholder purchases the bond

for less than the face value. The discount represents an amount that must be paid by the issuer to the investor at the time of maturity.

- The discount (a form of pre-paid interest) will be amortized periodically from the coupon payments over the life of the bond.

- We revisit the illustrative example. For the $1,000 face value 3-year bond with semiannual coupons at the rate of 5% per annum, we now assume that the required rate of return is 6% convertible semiannually. The purchase price of this bond is $972.91, and the discount is $27.09.

- We construct a bond discount amortization schedule in Table 6.2. The computational steps follow closely those of Table 6.1.

Table 6.2:   A bond discount amortization schedule

Half-year	Coupon payment	Effective interest earned	Amortized amount of discount	Book value
0				972.91
1	25.00	29.19	–4.19	977.10
2	25.00	29.31	–4.31	981.41
3	25.00	29.44	–4.44	985.86
4	25.00	29.58	–4.58	990.43
5	25.00	29.71	–4.71	995.15
6	25.00	29.85	–4.85	1,000.00
Total	150.00	177.09	–27.09	

From equation (6.4), we obtain

$$\text{Premium:} \quad P - C = C(g - i)a_{\overline{n}|}, \quad \text{if } g > i,$$
$$\text{Discount:} \quad C - P = C(i - g)a_{\overline{n}|}, \quad \text{if } i > g,$$

where $g = (Fr)/C$ is the modified coupon rate as defined in Example 6.4.

- A general bond amortization schedule is given in Table 6.3.

Table 6.3:   Formulas for a general bond amortization schedule

Half-year	Coupon payment	Effective interest earned	Amortized amount	Book value
0				$P = C + C(g-i)a_{\overline{n}|}$
1	Cg	$i[C + C(g-i)a_{\overline{n}|}]$	$C(g-i)v^n$	$C + C(g-i)a_{\overline{n-1}|}$
2	Cg	$i[C + C(g-i)a_{\overline{n-1}|}]$	$C(g-i)v^{n-1}$	$C + C(g-i)a_{\overline{n-2}|}$
.	.	.	.	.
.	.	.	.	.
t	Cg	$i[C + C(g-i)a_{\overline{n-t+1}|}]$	$C(g-i)v^{n-t+1}$	$C + C(g-i)a_{\overline{n-t}|}$
.	.	.	.	.
.	.	.	.	.
n	Cg	$i[C + C(g-i)a_{\overline{1}|}]$	$C(g-i)v$	$C + C(g-i)a_{\overline{0}|} = C$
Total	nCg	$nCg - C(g-i)a_{\overline{n}|}$	$C(g-i)a_{\overline{n}|}$	

Example 6.5: Find the price of a \$1,000 face value 15-year bond with redemption value of \$1,080 and coupon rate of 4.32% payable semiannually. The bond is bought to yield 5.00% convertible semiannually. Show the first 5 entries, as well as entries of the 20th and 30th half-year periods, of its bond amortization schedule.

Solution: Using the bond price formula (6.3), we have

$$
\begin{aligned}
P &= C + (Fr - Ci)a_{\overline{n}|} \\
&= 1{,}080 + (21.6 - 27.0)a_{\overline{30}|0.025} \\
&= \$966.98.
\end{aligned}
$$

Entries of the amortization schedule, using formulas in Table 6.3, can be computed as in Table 6.4.

Table 6.4: Amortization schedule for Example 6.5

Half-year	Coupon payment	Effective interest earned	Amortized amount	Book value
0				966.98
1	21.60	24.17	−2.57	969.55
2	21.60	24.24	−2.64	972.19
3	21.60	24.30	−2.70	974.89
4	21.60	24.37	−2.77	977.67
.	.	.	.	.
20	21.60[a]	25.72[b]	−4.12[c]	1,032.74[d]
.	.	.	.	.
30	21.60	26.87	−5.27	1,080.00

The details of the computation are as follows:

[a] The modified coupon rate is $g = Fr/C = 2\%$ per half-year period. Coupon payment is $Cg = 1{,}080(0.02) = \$21.60$, and it is the same as $Fr = 1{,}000(0.0216) = \$21.60$, where r is $0.0432/2 = 2.16\%$ per half-year period.

[b] Effective interest earned in the 20th half-year is $0.025[1{,}080+1{,}080(0.02-0.025)a_{\overline{11}|0.025}] = \$25.72$.

[c] Amortized amount in the 20th half-year is $1{,}080(0.02-0.025)(1.025)^{-11} = -\$4.12$. A negative amortized amount represents a discount situation. Alternatively, this is equal to $21.60 - 25.72$.

[d] The ending balance (after the 20th coupon payments) of the book value is $1{,}080 + 1{,}080(0.02 - 0.025)a_{\overline{10}|0.025} = \$1{,}032.74$.

The rest of the computations follows similarly.

$\square$

6.4 Valuation between Coupon-Payment Dates

- The pricing formulas discussed are applicable to a bond at its issue date or at a date immediately after a coupon payment.

- Bond transactions may occur any time before maturity.

- We now consider the pricing formula of a bond traded between coupon-payment dates.

- In practice the market price of a bond is stated as a percentage (e.g., 90.125, or 90.125%) of its face value, which is called the **quoted price** (or the **clean price**).

- When a bond is purchased between the coupon-payment dates, interest is earned by the seller of the bond from the last coupon-payment date, which is referred to as the **accrued interest**.

- The purchaser of the bond has to pay the accrued interest to the seller since the seller will not be entitled to any amount of the next coupon payment.

- The accrued interest is added to the quoted price to determine the **purchase price** (also called the **dirty price** or the **invoice price**), i.e.,

$$\text{Purchase price} = \text{quoted price} + \text{accrued interest.} \qquad (6.5)$$

- Let us denote k as the time immediately after the kth coupon payment.

- We consider a bond traded in between two coupon-payment dates (say, between time k and $k+1$) at time $k+t$, where t measures the length of time since the last coupon payment as a fraction of the time between k and $k+1$.

- While there are different market practices in defining t, we adopt the **actual/actual day count** convention and define t as

$$t = \frac{\text{Number of days since time } k}{\text{Number of days between } k \text{ and } k+1}, \qquad (6.6)$$

so that $0 < t < 1$.

- We denote the bond price at time k and $k+1$ by P_k and P_{k+1}, respectively.

- These prices are the bond values after the coupon payments. For an investor who purchased the bond after the coupon payment at time k and held it till time $k+1$, the value of her investment in the period is illustrated in Figure 6.2.

- If the yield rate remains unchanged from time k to $k+1$, we have

Bond price after
coupon payment

P_k

P_{k+1}

Time   0 $\cdots\cdots$ k $k+t$ $k+1$

Value of investment P_k P_{k+t} $P_{k+1}+Fr$
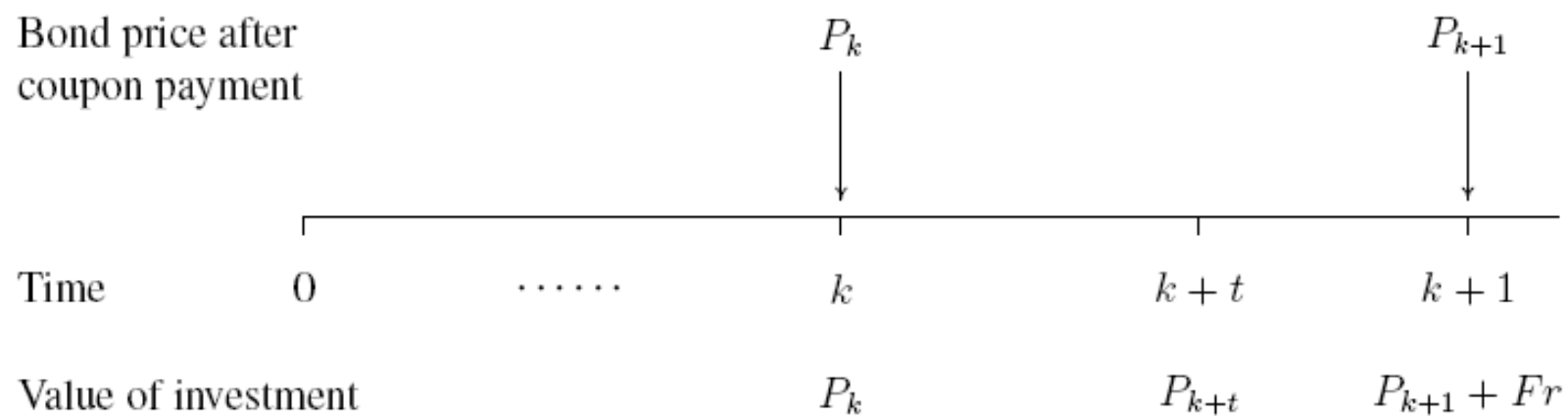
Figure 6.2:   Cash-flow pattern of a coupon-paying bond with n coupons

the following relationship

$$P_{k+t} = P_k(1+i)^t = (P_{k+1} + Fr)(1+i)^{-(1-t)}, \qquad (6.7)$$

where i is the yield rate in the coupon-payment period.

- We decompose P_{k+t} into two components. In common practice, the simple-interest method is used to calculate the accrued interest at time $k + t$, denoted by I_{k+t}.

- Using the simple-interest formula, we have

$$I_{k+t} = t(Fr). \qquad (6.8)$$

The book value (quoted price) at time t, denoted by B_{k+t}, is then given by

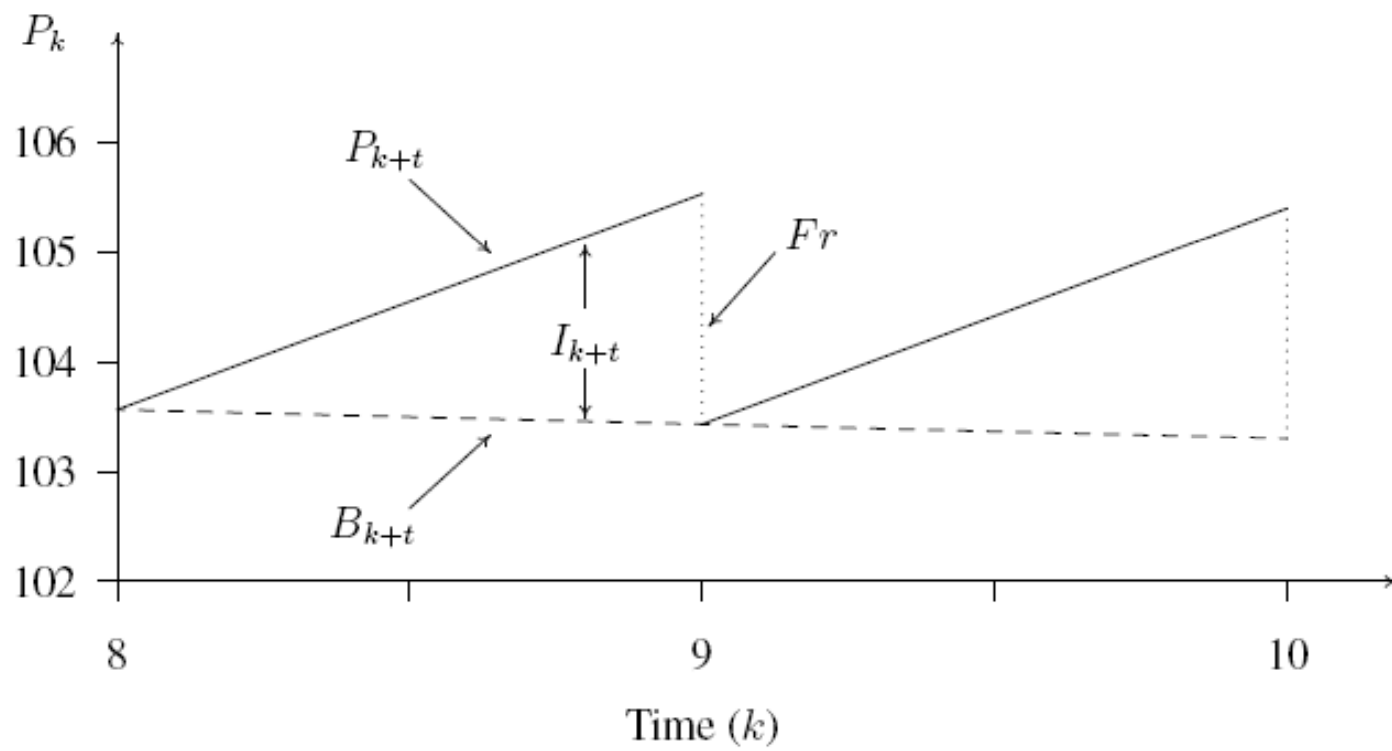$$B_{k+t} = P_{k+t} - I_{k+t}. \qquad (6.9)$$

Figure 6.3:   Bond price between coupon-payment dates

Example 6.6: Assume that the coupon dates for the government bond in Example 6.2 are June 15 and December 15 of each year. On August 18, 2009, an investor purchased the bond to yield 3.8% convertible semiannually. Find the purchase price, accrued interest and the quoted price of the bond at the date of purchase, based on a face value of 100.

Solution: We have $i = 0.038/2 = 1.9\%$, so that the values of P_k and P_{k+1} are

$$
\begin{aligned}
P_k &= 2.1a_{\overline{22}|0.019} + 100(1.019)^{-22} = 103.5690, \\
P_{k+1} &= 2.1a_{\overline{21}|0.019} + 100(1.019)^{-21} = 103.4368.
\end{aligned}
$$

The number of days between June 15, 2009 and August 18, 2009 is 64 and the number of days between the two coupon dates in 2009 is 183. Therefore $t = 64/183$. Hence,

$$
P_{k+t} = P_k(1 + i)^t = 103.5690(1.019)^{\frac{64}{183}} = 104.2529,
$$

or equivalently,

$$P_{k+t} = v^{1-t}(P_{k+1} + Fr) = (1.019)^{-\frac{119}{183}}(103.4368 + 2.1) = 104.2529.$$

The accrued interest is

$$I_{k+t} = t(Fr) = 0.7344,$$

and the quoted price (book value) is:

$$B_{k+t} = P_{k+t} - I_{k+t} = 104.2529 - 0.7344 = 103.5185.$$

- The Excel function PRICE can be used to compute the quoted price of a bond between coupon-payment dates. Its specification is as follows:

> **Excel function: PRICE(smt,mty,crt,yld,rdv,frq,basis)**
>
> **smt** = settlement date
>
> **mty** = maturity date
>
> **crt** = coupon rate of interest per annum
>
> **yld** = annual yield rate
>
> **rdv** = redemption value per 100 face value
>
> **frq** = number of coupon payments per year
>
> basis = day count, 30/360 if omitted (or set to 0) and actual/actual if set to 1
>
> Output = quoted price of bond at settlement date per 100 face value

- The default for the input "basis" is the 30/360 convention. Under this convention, every month has 30 days and every year has 360 days.

	A	B	C	D	E
	A5 ▼	fx	=Price(A1,A2,A3,A4,100,2,1)		
1	8/18/2009				
2	6/15/2020				
3	0.042				
4	0.038				
5	103.51852				
6					

Exhibit 6.1: Excel solution of Example 6.6

6.5 Callable Bonds

- **Callable bonds** are bonds that can be redeemed by the issuer prior to the bonds' maturity date.

- Investors whose bonds are called are paid a specified **call price**, which was fixed at the issue date of the bond.

- The call price may be the bond's face value, or it may be a price somewhat higher. The difference between the call price and the face value is called the **call premium**.

- If a bond is called between coupon dates, the issuer must pay the investor accrued interest in addition to the call price.

- Investors of callable bonds often require a higher yield to compensate for the call risk as compared to non-callable bonds.

- Some callable bonds offer a **call protection period**. The issuer is not allowed to call the bond before the ending date of the protection period. The **first call date** is the date after which the bond is fully callable.

- Theoretically, there is an optimal call date for the issuer to maximize the call benefit.

- However, in addition to the call benefit, there are many other factors (such as transaction costs, possible negative impact on the company's reputation and competition, etc.) affecting the decision of the issuer to make a call. Therefore, it is difficult to predict when the issuer will call.

- Here we consider a defensive pricing approach for the investor. Under this approach, an investor assumes that the issuer will call the

bond at a date which will maximize the call benefit.

- If a callable bond is traded at a discount and the call price is fixed at the bond's redemption value, the optimal call date for the issuer is at maturity.

- On the other hand, if a callable bond is traded at a premium and the call price is fixed at the bond's redemption value, the optimal call date for the issuer is the first call date.

- In general, when the call price is not a constant but varies in a pre-fixed relation with the possible call dates, it is not easy to determine the issuer's optimal call date.

- In this situation, we can apply the strategy in which the investor pays the lowest price among the prices calculated by assuming all possible call dates.

Example 6.7: Assume that the bond in Table 6.1 is callable and the first call date is the date immediately after the 4th coupon payment. Find the price of the bond for the yield to be at least 4% compounded semiannually.

Solution: Treating the first call date as the maturity date of the bond, the price of the bond is $25a_{\overline{4}|0.02} + 1{,}000(1.02)^{-4} = \$1{,}019.04$. The prices of the bond assuming other possible call dates, namely, after the 5th and 6th coupon payments, are, respectively,

$$25a_{\overline{5}|0.02} + 1{,}000(1.02)^{-5} = \$1{,}023.57,$$

and

$$25a_{\overline{6}|0.02} + 1{,}000(1.02)^{-6} = \$1{,}028.01.$$

Thus, the price of the bond is the lowest among all possible values, i.e., $\$1{,}019.04$. $\qquad\square$

Example 6.8: Consider a $1,000 face value 15-year bond with coupon rate of 4.0% convertible semiannually. The bond is callable and the first call date is the date immediately after the 15th coupon payment. Assume that the issuer will only call the bond at a date immediately after the nth coupon $(15 \leq n \leq 30)$ and the call price (i.e., redemption value) is

$$C = \begin{cases} 1,000, & \text{if } 15 \leq n \leq 20, \\ 1,000 + 10(n - 20), & \text{if } 20 < n \leq 30. \end{cases}$$

Find the price of the bond if the investor wants to achieve a yield of at least 5% compounded semiannually.

Solution: First, we compute the prices of the bond assuming all possible call dates, with $r = 0.02$, $i = 0.025$, $F = 1,000$ and the value of C following the given call price formula. The results are given in Table 6.5.

Table 6.5:   Results for Example 6.8

n	Redemption value C	Price
15	1,000	938.09
16	1,000	934.72
17	1,000	931.44
18	1,000	928.23
19	1,000	925.11
20	**1,000**	**922.05**
21	1,010	925.03
22	1,020	927.79
23	1,030	930.34
24	1,040	932.69
25	1,050	934.85
26	1,060	936.82
27	1,070	938.62
28	1,080	940.25
29	1,090	941.71
30	1,100	943.02

Thus, the minimum price is $922.05, which is the price a defensive investor is willing to pay for a yield of at least 5% compounded semiannually.

6.6 Bond Pricing under a General Term Structure

- The basic price formula in equation (6.1) assumes a flat term structure such that the yield rate for all cash flows (coupons and redemption payment) is the same.

- In the case when the term structure is not flat the pricing formula has to be modified.

- Following the principle that the fair price of the bond is the present value of the cash flows, we can compute the price of the bond as the sum of the present values of the coupon and redemption payments.

- We consider a bond with n coupon payments of amounts $C_1, \cdots, C_n$, and redemption at time n with redemption value C. Let i_j^S be the

spot rate of interest for cash flows due in j years and r be the rate of interest per coupon payment.

- For the case of an annual level-coupon bond, the price P is given by

$$
\begin{aligned}
P &= \sum_{j=1}^{n} \frac{C_j}{(1 + i_j^S)^j} + \frac{C}{(1 + i_n^S)^n} \\
&= Fr \sum_{j=1}^{n} \frac{1}{(1 + i_j^S)^j} + \frac{C}{(1 + i_n^S)^n}.
\end{aligned} \tag{6.10}
$$

- For a semiannual level-coupon bond, the pricing formula is

$$
P = Fr \sum_{j=1}^{n} \frac{1}{\left(1 + \dfrac{i_{\frac{j}{2}}^S}{2}\right)^j} + \frac{C}{\left(1 + \dfrac{i_{\frac{n}{2}}^S}{2}\right)^n}. \tag{6.11}
$$

Example 6.9: Consider a $100 face value bond with coupon rate of 4.0% per annum paid semiannually. The spot rates of interest in the market are given by

j (year)	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
i_j^S (%)	3.0	3.0	3.5	3.5	4.0	4.0	4.5	4.5	5.0	5.0

Find the price of the bond if it matures in (a) 3 years, and (b) 5 years.

Solution: For (a), the price of the bond is

$$
P = 2\left[\frac{1}{1.015} + \frac{1}{1.015^2} + \frac{1}{1.0175^3} + \frac{1}{1.0175^4} + \frac{1}{1.02^5} + \frac{1}{1.02^6}\right] + \frac{100}{1.02^6}
$$
$$
= 100.0608.
$$

Similarly, for (b), the price of the bond is

$$
P = 2\left[\frac{1}{1.015} + \frac{1}{1.015^2} + \cdots \frac{1}{1.0225^8} + \frac{1}{1.025^9} + \frac{1}{1.025^{10}}\right] + \frac{100}{1.025^{10}}
$$
$$
= 95.9328.
$$

Note that we have a premium bond for the case of (a), with the price higher than the redemption value. This is due to the fact that the spot rates of interest are less than the coupon rate of interest for cash flows of maturity less than 3 years.

On the other hand, for the case of (b) we have a discount bond, with the price lower than the redemption value. As the spot rates of interest for cash flows of maturity of more than 3 years are higher than the coupon rate, the present value contributions of the cash flows beyond 3 years are lower, causing the price of the bond to drop.

Hence, when the term structure is not flat the premium/discount relationship of a bond with respect to its redemption value may vary with the time to maturity even for bonds with the same coupon rate of interest. □