An Empirical Study of Pricing and Hedging Collateralized Debt Obligation (CDO)

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Abstract

This paper studies the pricing of collateral debt obligation (CDO) using Monte Carlo and analytic methods. Both methods are developed within the framework of the reduced form model. One-factor Gaussian Copula is used for treating default correlations amongst the collateral portfolio. Based on the two methods, the portfolio loss, the expected loss in each CDO tranche, tranche spread and the default delta sensitivity are analyzed with respect to different parameters such as maturity, default correlation, default intensity or hazard rate, and recovery rate. We provide a careful study of the effects of different parametric impact. Our results show that Monte Carlo method is slow and not robust in the calculation of default delta sensitivity. The analytic approach has comparative advantages for pricing CDO. We also employ empirical data to investigate the implied default correlation and base correlation of the CDO. The implication of extending the analytical approach to incorporating Levy processes is also discussed.

Keywords: Collateral debt obligation (CDO) pricing, Monte Carlo, Default correlation, Copula

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1. Introduction to Collateralized Debt Obligation

In recent years, due to the burgeoning credit derivatives market, there has been much research work on the Collaterized Debt Obligation (CDO). A CDO is an asset-backed security whose payment depends on the collateral portfolio. There are different types of CDOs. A CDO whose collateral is made up of cash assets such as corporate bonds or loans is called cash CDO, while a CDO whose collateral is made up of credit default swaps is called a synthetic CDO. The structure of a CDO consists of partitions of the collateral portfolio into different tranches of increasing seniority. The CDO in effect transfers credit risk from the portfolio holder to investors. Investors of CDO are called protection sellers, while the issuer of CDO is called protection buyer. A particular tranche of a CDO is defined by its lower and an upper attachment point. The tranche with a lower attachment point L and a higher attachment point H will bear all the losses in the collateral portfolio in excess of L and up to H percent of the initial value of the portfolio. The portfolio loss is absorbed in ascending order of tranches, starting with the Equity tranche, then the Mezzanine tranche, and eventually the Senior tranche.

As compensation for taking potential loss, the protection seller receives a periodic premium payment from the issuer of CDO until the maturity of the CDO or at the time when the tranche is expended through loss. The premium is paid from the interest income of the collateral portfolio. Interest is distributed to the tranches starting with the Senior tranche, then the Mezzanine tranche, and eventually the Equity tranche. As the Equity tranche absorbs the first layer of loss, the premium of this tranche is the largest among all the tranches. An example of CDO is illustrated in Figure 1, where the portfolio is composed of 100 loans. Each loan has \$10 million notional amount. The Equity tranche absorbs the first losses within [0%, 3%] of the initital portfolio notional amount. The Mezzanine tranche absorbs losses within [3%, 14%]. The Senior tranche absorbs the remaining loss within [14%, 100%]. The premium of the tranches is paid as a percentage of the outstanding notional amounts of the corresponding tranches. For example, if there is a 1% loss in the portfolio, the 3% portfolio value of the Equity tranche is then reduced to 2% due to the loss. This amounts to 1/3 or a tranche loss of 33.3% in value. Consequently, the Equity tranche pays only the pre-determined interest rate on a remaining 66.7% of tranche capital. For practical details of a CDO, see for example Elizalde (2004). Finger (2003), as well as Bluhm et.al. (2004), discusses the standard pricing model framework for synthetic CDO and some of the outstanding implementation and application issues.

The major risk in a CDO is default risk of the entitites of the portfolio collateral. Such default risk can be modeled in two primary types of models that describe default processes in the credit risk literature: structural models and reduced form models. Structural models determine the time of default using the evolution of firms' structural variables such as asset and debt values. Reduced form models determine the default process as a stochastic Poisson process with random default intensity. Empirically, the results in the literature show that the structural models under-predict the default probability while the reduced form models could predict the default process well.

More specifically, the problem of pricing CDO is equivalent to determining the premium of each tranche. There are three important components in the pricing of CDO: modeling credit risk, handling default correlations among collateral portfolio, and calculating the portfolio loss. The last two components are not common to simpler credit derivatives such as single-name credit default swaps. For an understanding of the background of default correlation and portfolio loss in the context of a CDO, we provide a brief summary of the literature in Table 1 where existing literature is surveyed in terms of the methodology for calculating default correlation and portfolio loss, and whether default delta sensitivity is studied or not. Most of the existing studies used Copula to treat default correlation, except for Duffie and Garleanu (2001) where default correlation is treated by using dependent default intensity. Based on Copula, different analytic methods are proposed for calculating the portfolio loss. Most of the works demonstrated that the analytic methodologies constitute a powerful tool for evaluating CDO.

In the framework of reduced form models, there are basically three methodologies for treating default correlation among multiple assets in the collateral portfolio: conditionally independent default model of Duffie and Garleanu (2001), contagion model of Jarrow and Yu (2001), and Copula method by Li (2000). Conditionally independent default model handles default correlation by simulating correlated default intensities based on a common set of state variables. The major disadvantage of the conditionally independent default model is that the correlation generated by the model is often too small in empirical data with high default correlation. In the contagion model, the default of one firm triggers the default of the other related firms, and the default times tend to be concentrated in certain time periods. The disadvantage of the contagion method is that it is difficult to calibrate the parameters of the model. The resulting model is thus hard to implement. The other method of treating default correlation is the Copula method. Using Copula in default correlation modeling is originally proposed by Li (2000). The Copula function is actually a correlated multivariate function defined by the marginal default probability distribution**.** A variety of functions can be used as Copula, such as $t-$student and Gaussian. The Copula method is simple and easy to implement.

Apart from its simplicity, another advantage of using Copula is that the portfolio loss in a CDO can be analytically computed without relying on Monte Carlo simulations that can be computationally intensive and time-consuming. Sidenius and Basu (2003) describe an analytic method of calculating the portfolio loss based on a one factor Gaussian Copula. Gibson (2004) describes the analytic method more explicitly. Besides the portfolio loss, Sidenius and Basu (2003) also propose an analytic method for calculating the default delta sensitivity.

This paper studies the pricing and hedging of CDO by comparing the analytic method of Gibson (2004) with the Monte Carlo method. There has been few studies performing such comparisons, and it is important to be able to decide which models to use in practice. The CDO data employed in Peixoto (2004) are used in the empirical investigation in this paper. In addition, this paper provides a careful study of the effects of different parametric impact. The portfolio loss, the expected loss allocated to each tranche, the tranche spread, and the default delta sensitivity are analyzed with respect to different parameters such as maturity, default correlation, default intensity or hazard rate and recovery rate. In the current literature, the default delta sensitivity is discussed only in Sidenius and Basu (2003), Gibson (2004), Mina and Stern (2003), and Andersen and Sidenius (2004). By providing a more thorough study of the delta sensitivity with respect to some key parameters, this study will help in the hedging performance of CDO. Furthermore, the implied default correlation and base correlation are also empirically investigated.

The remainder of the paper is organized as follows. Section 2 describes the methodology of pricing CDO using the analytic method and the Monte Carlo method. The methodology of calculating default hedge ratio is described in Section 3. Section

4 presents the empirical results. The implication of extending the analytical approach to incorporating Levy processes is also discussed. Section 5 contains the conclusions.

2. Methodology of Pricing CDO

The model is set up in a filtered probability space $(\Omega, F, (F_t)_{(t\geq 0)}, P)$, where P is a pre-specified martingale measure. The filtration $(F_t)_{(t\geq 0)}$ satisfies the usual conditions and the initial filtration F_0 is trivial. There is also a finite time horizon T with $F = F_T$. The remaining notations used in this paper are described as follows.

I_k	Notional amount for asset k, $k = 1, 2, \cdots, K$
R_k	Recovery rate for asset k
λ_k	Default intensity for asset k
τ_k	Default time for asset k
T_i	The payment date in CDO, $i = 1, 2, \cdots, n$. We assume that for a standard CDO, all tranches are paid interest at the same time points.
l_i	The total amount of loss in the portfolio at time T_i
e_i	The total amount of loss allocated to the tranche at T_i
B_i	The price of a default-free zero coupon bond with maturity T_i and face value of \$1 at present time

The pricing of a CDO consists of pricing the single tranches that make up the CDO structure. For a single tranche with attachment points $[L, H]$, the cash flows can be described as follows. The seller of a CDO pays a periodic coupon to the investor of the CDO at each payment date T_i, $i = 1, 2, \cdots, n$. The coupon paid at T_i for a tranche is calculated based on the outstanding notional amount in that tranche. Obviously, the initial dollar value of notional amount for the tranche is equal to $H - L$. When default occurs in the portfolio and the portfolio loss exceeds L, the investor of this CDO tranche has to pay to the seller of CDO the amount of loss in excess of L. The loss between $[T_{i-1}, T_i]$ is assumed[1] to be paid at T_i. The maximal value of the total amount payable by the investor is equal to $H - L$. Thus, the pricing of a tranche consists of calculating the premium leg corresponding to the payment by the seller and the default leg corresponding to the payment by the investor when there is default in excess of L.

In the reduced form model of Li (2004), the risk-neural default probability of an asset at T_i is calculated by

$$p(\tau_k \le T_i) = 1 - e^{-\int_0^{T_i} \lambda_k(u)du} \quad . \tag{1}$$

If the default time τ_k in each k[th] asset is known, the portfolio loss l_i at T_i can be calculated by

$$l_i = \sum_{k=1}^{K} I_k (1 - R_k) 1_{\{\tau_k \le T_i\}}, \; i = 1, 2, \cdots, n \tag{2}$$

and $$1_{\{\tau_k \le T_i\}} = \begin{cases} 1 & \tau_k \le T_i \\ 0 & \tau_k > T \end{cases} .$$

Given l_i, the total amount of dollar loss allocated to the single tranche of $[L, H]$ at T_i is equal to

$$e_i = Max\,(\,Min\,(\,l_i, H\,) - L, 0\,)\,, i = 1, 2, \cdots, n \; . \tag{3}$$

Thus the present value of the default leg (denoted as DL) is equal to the sum of the present values of the expected values of the loss paid by the investor of tranche to the seller of tranche at the various T_i's, that is calculated by

[1] This assumption effectively puts any loss at the beginning of the CDO as zero, since any loss within $[T_0, T_1]$ is paid at T_1.

$$E(DL) = \sum_{i=0}^{n} B_i \left(E(e_i) - E(e_{i-1})\right) \tag{4}$$

where by definition $E(e_{-1}) = 0$. The expected loss between $[T_{i-1}, T_i]$ is equal to $(E(e_i) - E(e_{i-1}))$, which is assumed to be paid at T_i. Obviously $E(e_0) = 0$ as well if the earliest first loss payment is at T_1. In this case, the summation indices in E(DL) could also be written without loss of generality to start at i equal to 1 instead of 0.

Let s denote the tranche spread, which is the annualized interest charge or coupon rate on the tranche. The expected value of the premium leg (denoted as PL) can be expressed as the sum of the present values of the expected values of the amount paid by the seller of the tranche to the investor, which is calculated by

$$E(PL) = \sum_{i=1}^{n} s\, \Delta t_i\, B_i\, (H - L - E(e_i)) \tag{5}$$

where $\Delta t_i = T_i - T_{i-1}$ is denoted as a fraction of a year. $H - L - E(e_i)$ denotes the expected value of outstanding notional amount at T_i.

The equilibrium pricing of the tranche under risk-neutrality implies that s is found by setting $E(DL) = E(PL)$. Thus,

$$s = \frac{\sum_{i=0}^{n} B_i (E(e_i) - E(e_{i-1}))}{\sum_{i=1}^{n} \Delta t_i B_i (H - L - E(e_i))} \; . \tag{6}$$

From Equations (4)-(6), it can be observed that the crucial task of pricing is to calculate the expected value of the tranche loss $E(e_i)$ for each T_i. The methodologies of calculating $E(e_i)$ thus obtaining s in the analytic method of Gibson (2004) and Monte Carlo method are described as below.

Analytic Method

The analytic method uses a continuous state variable X_k taking values in (-∞,∞) to represent the default status of an asset k. When X_k approaches -∞ from the right, the probability of default approaches 0. When X_k approaches +∞ from the left, the probability of default approaches 1. The (cumulative) probability distribution function of X_k is the (unconditional) probability of default. A single (common) factor model of X_k consists of a common factor M and an individual factor Z_k.

$$X_k = a_k M + \sqrt{1-a_k^2} Z_k \ , k = 1, 2, \cdots, K \tag{7}$$

where a_k represents the fraction of the common factor relative to Z_k. The value of a_k is between $[0,1]$ and the variances of the X_k's are ones. $a_k a_j$ thus denotes the correlation of X_j and X_k. It represents the default correlation between asset k and j. For simplicity, X_k, M, and Z_k are assumed to follow standard normal distributions. M and Z_k are independent variables. Equation (7) is often termed the one-factor Gaussian Copula.

The probability distribution of X_k is equal to the risk-neural probability of default in Equation (1). Thus, the conditional default probability $p(\tau_k \le T_i \mid M)$ at T_i for asset k can be calculated by

$$p(\tau_k \le T_i \mid M) = N\left(\frac{N^{-1}(p(\tau_k \le T_i)) - a_k M}{\sqrt{1-a_k^2}}\right) \ . \tag{8}$$

Let $p_i^K(k \mid M)$ denote the conditional probability of k defaults up to T_i in a reference portfolio of size K. The analytic method calculates $p_i^K(k \mid M)$ using the following recursive algorithm:

$$p_i^{K+1}(0 \mid M) = p_i^{K}(0 \mid M)(1 - p(\tau_{k+1} \le T_i \mid M)) \tag{9}$$

$$p_i^{K+1}(k \mid M) = p_i^{K}(k \mid M)(1 - p(\tau_{k+1} \le T_i \mid M)) +$$

$$+\, p_i^{K}(k-1 \mid M)\, p(\tau_{k+1} \le T_i \mid M) \tag{10}$$

and $$p_i^{K+1}(k+1 \mid M) = p_i^{K}(k \mid M)\; p(\tau_{k+1} \le T_i \mid M) \quad \text{for } k = 1, \cdots, K. \tag{11}$$

For $K = 0$, $p_i^{0}(0 \mid M) = 1$. After computing $p_i^{K}(k \mid M)$, for k = 0,1,….K, the unconditional portfolio number of defaults distribution $p_i^K(k)$ is calculated by

$$p_i^K(k) = \int_{-\infty}^{\infty} p_i^K(k \mid M)\, g(M)\, dM \tag{12}$$

where $g(.)$ is the probability density function of M. The integral can be computed using numerical integration. Note that in the above computation, conditional on M, the probabilities of default for the different assets in the portfolio are independent. The conditional probabilities of default for the different asset k, $p(\tau_{k+1} \le T_i \mid M)$, are different as the assets' characteristics are different.

Let $p(l_i)$ be the portfolio loss probability. Suppose this is a discrete distribution, then we can write the expected tranche loss as

$$E(e_i) = \sum_{l_i \ge L}^{l_i \le H} (l_i - L)\, p(l_i) + \sum_{l_i \ge H} (H - L)\, p(l_i)\,. \tag{13}$$

From $E(e_i)$, Equations (4)-(6) are then used to calculate the spread s of each tranche.

Suppose the collateral portfolio is a large homogeneous portfolio made up of small similar assets. Homogeneity is with respect to the terms of I_k, R_k, and λ_k, resulting in the same I, R, and λ. Then, instead of the above algorithm, $p_i^K(k \mid M)$ can be calculated simply as a binomial function:

$$p_i^K(k \mid M) = \binom{K}{k} \times \left(N\left(\frac{N^{-1}(p(\tau \;\le T_i)) - a\; M}{\sqrt{1-a^2}}\right) \right)^{k} \times \left(1 - N\left(\frac{N^{-1}(p(\tau \;\le T_i)) - a M}{\sqrt{1-a^2}}\right) \right)^{K-k} \tag{14}$$

where all a_i's equal a, and τ denotes the default time of any one of the assets. In this case, we treat each credit entity in the portfolio as identical. The unconditional portfolio number of defaults distribution $p_i^K(k)$ is then similarly computed using Equation (12). Under homogeneity, the portfolio loss l_i can be simplified from Equation (2) to I (1-R) times the number of defaults. Hence in this case, the probability distribution of portfolio loss $p(l_i)$ is represented by the distribution of the number of defaults. The expected portfolio loss is then I (1-R) times the expected number of default by a certain time.

Monte Carlo Method

The Monte Carlo method takes into account the default correlation using the Copula function. The following Gaussian Copula is most commonly used.

$$C(u_1, u_2, \cdots, \cdots u_K) = N(N^{-1}(u_1), N^{-1}(u_2), \cdots, N^{-1}(u_K)) \qquad (15)$$

where u_k is equal to $p(\tau_k \le T_i)$ in Equation (1), and $N(v_1, v_2, \ldots, v_K)$ on the right-side of Equation (15) denotes a multivariate normal probability distribution function with mean zero and correlation matrix ($\rho_{i,j}$), where i,j = 1,2,…,K. As in typical applications, we employ a constant correlation matrix with a single parameter $\rho \in [0,1]$ for all $\rho_{i,j}$. If (15) is specialized to the single-factor analytical model, the value of ρ in (15) would be related to a_k in (7) by the following formulae: $\rho_{k,j} = a_k a_j$, $k \neq j$ and $\rho_{k,j} = 1$, $k = j$.

Based on Equation (15) with a constant correlation matrix, the Monte Carlo method calculates e_i using the following algorithm. Perform N number of simulations each of which takes the following steps.

(1) Employ the multivariate normal distribution $N(v_1, v_2, \ldots, v_K)$ to generate for a given ρ, the K random variables. Calculate the default time of each asset by

$$\tau_k = -\frac{\ln(1-\Phi(v_k))}{\lambda_k}, \; k = 1, 2, \cdots, K \tag{16}$$

where $\Phi(.)$ denotes the univariate standard normal probability distribution function.

(2) From the computed τ_k for k = 1,2,….,K, we can determine if for asset k, default has occurred by time T_i, whether $1_{\{\tau_k \le T_i\}}$ takes value 1 in the event of default by time T_i, or 0 otherwise in the event of no-default. Then we can calculate the portfolio loss l_i at each payment date T_i, $i = 1, 2, \cdots, n$ according to

$l_i = \sum_{k=1}^{K} I_k (1-R_k) 1_{\{\tau_k \le T_i\}}$ in Equation (2).

(3) Next calculate the tranche loss e_i at each payment date T_i, $i = 1, 2, \cdots, n$ according to Equation (3).

(4) Then calculate $DL = \sum_{i=0}^{n} B_i (e_i - e_{i-1})$ and $Q \equiv \frac{PL}{s} = \sum_{i=1}^{n} \Delta t_i \, B_i \, (H - L - e_i)$.

Finally, the N number of simulations each involving terms DL and Q in step (4) are averaged to obtain the spread s by the following

$$s = \frac{\sum_{q=1}^{N} DL(q)}{\sum_{q=1}^{N} Q(q)} \tag{17}$$

where q denotes the q^{th} simulation. The expected portfolio loss at each time T_i can also be computed by averaging across the portfolio loss values at each step (2). The number of simulations N may be 50,000 or less depending on the complication of the model and the allocated computing time.

3. Methodology of Calculating Default Delta Sensitivity

In Monte Carlo, the brute force method is used for calculating the sensitivity of the price of the [L,H] CDO tranche to default intensity λ_k . The approach is described as follows. Firstly, λ_k for the k-th asset is increased by a small amount $\Delta\lambda_k$ to re-calculate price of the tranche. Secondly, the ratio of the price difference to $\Delta\lambda_k$ is calculated as the default delta sensitivity. The mathematical formula is written as

$$\frac{\partial V}{\partial \lambda_k} = \frac{V(\lambda_k + \Delta\lambda_k) - V(\lambda_k)}{\Delta\lambda_k} \tag{18}$$

and
$$V = \sum_{i=0}^{n} B_i (E(e_i) - E(e_{i-1})) - \sum_{i=1}^{n} s\Delta t_i B_i (H - L - E(e_i)) \tag{19}$$

where V is the market value of the tranche to the CDO issuer or the protection buyer.

In the analytic method, the analytic methodology proposed in Andersen and Sidenius (2004) is used for calculating the default delta sensitivity. The methodology is described below.

From (19), it can be observed that $\frac{\partial V}{\partial \lambda_k}$ is equal to

$$\frac{\partial V}{\partial \lambda_k} = \sum_{i=0}^{n} B_i \left(\frac{\partial E(e_i)}{\partial \lambda_k} - \frac{\partial E(e_{i-1})}{\partial \lambda_k} \right) - \sum_{i=1}^{n} s\Delta t_i B_i \left(H - L - \frac{\partial E(e_i)}{\partial \lambda_k} \right). \tag{20}$$

According to (13), $\frac{\partial E(e_i)}{\partial \lambda_k}$ is equal to

$$\frac{\partial E(e_i)}{\partial \lambda_k} = \sum_{l_i \geq L}^{l_i \leq H} (l_i - L) \frac{\partial p(l_i)}{\partial \lambda_k} + \sum_{l_i \geq H} (H - L) \frac{\partial p(l_i)}{\partial \lambda_k} \tag{21}$$

From (12), by assuming λ_k and M are independent, $\frac{\partial p(l_i)}{\partial \lambda_k}$ is calculated by

$$\frac{\partial p(l_i)}{\partial \lambda_k} = \int_{-\infty}^{\infty} \frac{\partial p_i{}^K (k \mid M)}{\partial \lambda_k} g(M) dM \tag{22}$$

where we have used the result that the probability distribution of portfolio loss $p(l_i)$ is equivalent to the unconditional probability distribution of the number of defaults. According to Equations (9)-(11), $\frac{\partial p_i^K(k \mid M)}{\partial \lambda_k}$ is calculated by

$$\frac{\partial p_i^K(0 \mid M)}{\partial \lambda_k} = -p_i^{K-1}(0 \mid M)\frac{\partial p(\tau_k \leq T_i \mid M)}{\partial \lambda_k} \tag{23}$$

$$\frac{\partial p_i^K(k \mid M)}{\partial \lambda_k} = \frac{\partial p(\tau_k \leq T_i \mid M)}{\partial \lambda_k}(-p_i^{K-1}(k \mid M) + p_i^{K-1}(k-1 \mid M)) \tag{24}$$

and

$$\frac{\partial p_i^K(k \mid M)}{\partial \lambda_k} = \frac{\partial p(\tau_k \leq T_i \mid M)}{\partial \lambda_k} p_i^{K-1}(k-1 \mid M) \quad \text{for } k = 1, \cdots, K. \tag{25}$$

From Equations (1) and (8), we can derive

$$\frac{\partial p(\tau_k \leq T_i \mid M)}{\partial \lambda_k} = e^{-0.5\left(\frac{N^{-1}(p(\tau_k \leq T_i)) - a_k M}{\sqrt{1-a_k^2}}\right)^2} \frac{1}{\sqrt{1-a_k^2}} e^{0.5(N^{-1}(p(\tau_k \leq T_i)))^2} T_i e^{-\lambda_k T_i} \tag{26}$$

From Equations (21)-(26), the default delta sensitivity in Equation (20) can be calculated.

4. Empirical Results

The CDO data studied in Peixoto (2004) are employed in our study. The collateral portfolio of CDO is composed of 100 loans each with equal face value. The maturity of the CDO is 5 years. The default intensity or the hazard rate of each loan is 0.03. The recovery rate of each asset is 0.4. The premium and default loss is paid quarterly. The risk-free interest rate is 5% with continuous compounding. As illustrated in Table 2, the attachment points of Equity tranche are $[0, 3\%]$. Those of Mezzanine tranche are $[3\%, 14\%]$. Those of Senior tranche are $[14\%, 100\%]$. The expected loss, the spread and default delta sensitivity in each tranche are analyzed.

4.1 Expected loss (EL)

In the first set of empirical results, the portfolio loss and the loss distributed in each tranche are analyzed with respect to different parameters. Figures 2(a) and 2(b) show the portfolio EL with different maturity T and default correlation ρ in the analytic and the Monte Carlo method. It can be observed that both methods give close results, while Monte Carlo method generates the EL surface that is smoother than that of the analytic method. For a fixed value of ρ, the portfolio EL increases with maturity T as more defaults are likely to happen at larger T. When T is fixed, the portfolio EL increases is less sensitive to increase in ρ. The difference between the maximum and minimum values of the portfolio EL for different values of ρ is within 20 basis points or 0.2% of portfolio value.

Figures 3(a) and 3(b) show that the EL's allocated to the Equity tranche in the analytic and the Monte Carlo methods agree closely. Equity EL increases with T for fixed ρ due to the larger default probability at larger T. When T is fixed, Equity EL decreases with the increase of ρ. The result can be explained as follows. As ρ increases, there is a higher probability that either many obligors default together, resulting in larger losses, or many do not default together resulting in smaller overall losses. The latter obviously has a more weighted impact on the Equity tranche that takes the first loss, resulting in an overall lower expected loss.

Figures 4(a) and 4(b) are produced from the analytical method and show the portfolio loss distribution of $\rho = 0$ and $\rho = 0.9$ with $T = 5$. For large losses in the range 0.19 to 0.58, it can be observed from Figure 4(b) that the probability of occurrence in the case $\rho = 0.9$ is much higher than that in the case $\rho = 0$. For small losses in the range 0.08 to 0.11, it can be observed from Figure 4(a) that the probability of occurrence in the case $\rho = 0.9$ is much smaller than that in the case $\rho = 0$. Thus, higher default correlation ρ leads to higher chances of large portfolio loss and lower chances of small portfolio loss. Contagion effect or high default correlation is therefore risky from the point of view of the tranche buyers. The larger losses make the tranche with higher seniority suffer from more loss. For higher ρ, the lower probability of the smaller portfolio losses hitting mainly the Equity tranche means also that expected loss on the latter is lower. This concurs with the result expressed in Figures 3(a) and 3(b).

Figures 5(a) and 5(b) illustrate the loss distributions of the Equity and Senior tranches respectively for cases $\rho = 0$ and $\rho = 0.9$. These are computed from the analytical method. Figure 5 (a) shows that in Equity tranche the probability of losing the tranche notional amount for case $\rho = 0.9$ is less than that for case $\rho = 0$ once the loss amount gets into the non-trivial range above 0.01. This results in smaller Equity EL for large ρ as discussed in the last two sets of figures. Figure 5(b) shows that in the Senior tranche the probability of large loss is much higher in the case $\rho = 0.9$ compared to the case $\rho = 0$.

Figures 6(a) and 6(b) show EL allocated to the Mezzanine tranche in the analytic and the Monte Carlo methods. Mezzanine EL increases as T increases. For long maturity $T = [4, 5]$, Mezzanine EL decreases with the increase of ρ, having the same characteristics as that of Equity. However, for short maturity $T = [1, 4]$, Mezzanine EL firstly increases and then decreases with the increase of ρ. The latter decrease is due to the rapid increase in the probability of loss for Senior tranche at high levels of ρ, in which the loss impact on the Mezzanine tranche would be reduced. In effect, larger values of ρ reduce the chance of absorbing loss in the Mezzanine tranche.

Figures 7(a) and (b) show EL allocated to the Senior tranche in the analytic and the Monte Carlo methods. The largest EL occurs for higher T and ρ values, for example when $T = 5$ and $\rho = 0.9$. Senior tranche does not absorb loss at smaller values of T and ρ. This is consistent with the characteristic of Senior tranche that it is the last tranche to take losses in the portfolio. When either ρ or T increases, Senior EL increases.

Table 3 summarizes the expected losses of the analytic and the Monte Carlo methods for the case of $T = 5$. It can be observed that the difference in expected losses between the two methods is within 20 basis points.

4.2 Tranche Spread and Default Delta Sensitivity

Tranche spread and default delta sensitivity in each tranche are analyzed in the second set of empirical results. Figures 8(a) and 8(b) show the spread of Equity tranche with different T and ρ in the analytic and Monte Carlo methods. For a fixed value of T Equity spread decreases with the increase of ρ, due to the decreased expected loss. When ρ is fixed Equity spread increase is not sensitive to increase in T.

Figures 9(a) and 9(b) show the default delta sensitivity in Equity tranche for the analytic and the Monte Carlo methods respectively. For calculating sensitivity, the spreads of Equity, Mezzanine and Senior tranches are arbitrarily set as 1000 bp, 500 bp and 1 bp respectively. In the Monte Carlo method, $\Delta\lambda_k$ is set as 10 bp. The Monte Carlo results do not converge as fast as the analytical ones in calculating sensitivity. Thus Monte Carlo method is not robust in the calculation of default delta sensitivity. Figure 9(a) shows that the sensitivity is highest at $T = 1$ and $\rho = 0$ in the Equity tranche. This is consistent with the fact that Equity is mostly sensitive to small losses occurring at an early time. The results show that when T or ρ is small, the value of delta sensitivity decreases with increases of ρ or T. When T or ρ is large the value of delta sensitivity increases initially and then later decreases with increases of ρ or T.

Figures 10(a) and 10(b) show the Mezzanine tranche spread at different T and ρ. It can be observed that there the largest spread occurs at $T = 5$ and $\rho = 0$, corresponding to the maximum values of Mezzanine EL in Figures 6(a) and 6(b). When maturity is large where $T = [4, 5]$, the Mezzanine spread decreases with the increase of ρ, having the same characteristic of the Equity tranche. When maturity is small where $T = [1, 4]$, the Mezzanine spread increases initially and then decreases with the increase of ρ, showing the same characteristic as the Mezzanine EL in Figures 6(a) and 6(b). For a fixed value of ρ, Mezzanine spread increases with T.

Figures 11(a) and 11(b) show the default delta sensitivity in the Mezzanine tranche for the analytic and the Monte Carlo methods. The largest value of sensitivity occurs at $T = 5$ and $\rho = 0$. The same relationship between the Mezzanine spread (EL) and ρ can be applied here. That is, where $T = [4, 5]$, the Mezzanine sensitivity decreases with the increases of ρ. When T is small where , the Mezzanine sensitivity increases initially and then decreases with the increase of ρ. For a fixed value of ρ, Mezzanine default delta sensitivity increases with T.

Figures 12(a), 12(b), 13(a), and 13(b) respectively illustrates the spread and the default delta sensitivity of the Senior tranche in the analytic and the Monte Carlo methods. The largest values of spread and default delta sensitivity for the Senior tranche occur at $T = 5$ and $\rho = 0.9$, corresponding to the maximum values of the

Senior EL. When T or ρ is fixed, both the spread and sensitivity increase with the increase of ρ or T.

Table 4 compares the spread difference between the analytic and the Monte Carlo methods for the case of $T = 5$ and across various values of ρ. It can be observed that the spread difference between the two methods falls within 30 bp.

Table 5 compares the difference of the default delta sensitivity values between the analytic and the Monte Carlo methods for the case of $T = 5$ and across various values of ρ. It can be observed that the difference between the analytic and the Monte Carlo results is generally small at approximately less than 10 bp. However, but for a few cases, the difference is larger as the Monte Carlo result is sensitive to the choice of $\Delta\lambda$.

The spread and default delta sensitivity are further examined by using different values of recovery rates R and default intensity or hazard rate λ, with maturity and default correlation set at $T = 5$ and $\rho = 0.4$. Figures 14(a), 14(b), and 14(c) respectively illustrate the spreads of the Equity, Mezzanine and Senior tranches with respect to different values of R and λ. All the figures show that spread increases with increase of λ when R is fixed, and decreases with R when λ is fixed .

Figures 15(a), 15(b), and 15(c) respectively illustrate the default delta sensitivities of the Equity, Mezzanine and Senior tranches with respect to different values of R and λ. Figure 15(a) shows that the delta sensitivity decreases with the increase of λ and slightly increases with the increase of R in the Equity tranche. Figure 15(b) shows that the delta sensitivity for the Mezzanine tranche is non-monotone. For small values of R and λ, the delta sensitivity decreases with increase in λ and increases with increase in R, having the same characteristic of the Equity tranche. For large values of R and λ, the delta sensitivity decreases with the increases of R and λ.

Figure 15(c) shows that in the Senior tranche, the delta sensitivity increases with λ and decreases with increase in R.

4.3 Implied Correlation and Base Correlation

The implied correlation for each tranche is calculated as the correlation which makes the spread of the tranche equal to its market price. It is sometimes referred as "compound correlation" in the CDO literature. The implied correlation is usually calculated based on trial and error. One major disadvantage of the implied correlation is that it exhibits a "smile". For overcoming this problem, the base correlation is proposed by JP Morgan – see McGinty and Ahluwalia (2004). The base correlation is calculated by defining a series of hypothetical equity tranches.

The first Equity tranche remains unchanged at detachment points [0%, 3%]. The Mezzanine tranche is now replaced conceptually by a hypothetical equity tranche at detachment points [0%, 14%] that combines the original Equity tranche [0%, 3%] and Mezzanine tranche [3%, 14%]. The base correlation for the new tranche at "Mezzanine" level is calculated as the correlation which makes the spread of this hypothetical tranche equal to its market price that would be the sum of the market prices of the original Equity [0%, 3%] tranche and the original Mezzanine [3%, 14%] tranche. For computing the model price, the expected losses of the new hypothetical tranche is the sum of the expected losses in the original Equity [0%, 3%] tranche and the original Mezzanine [3%, 14%] tranche.

In the same way, the Senior tranche is now replaced conceptually by a hypothetical equity tranche at detachment points [0%, 100%] that combines the original Equity tranche [0%, 3%], Mezzanine tranche [3%, 14%], and Senior tranche [14%, 100%]. The base correlation for the new tranche at "Senior" level is calculated as the correlation which makes the spread of this hypothetical tranche equal to its market price that would be the sum of the market prices of the original Equity [0%, 3%] tranche, the original Mezzanine [3%, 14%] tranche, and the Senior tranche [14%, 100%]. For computing the model price, the expected losses of the new hypothetical tranche is the sum of the expected losses in the original Equity [0%, 3%] tranche, the original Mezzanine [3%, 14%] tranche, and the original Senior tranche [14%, 100%].

Figure 16 shows the implied correlation and the base correlation calculated in the Equity, Mezzanine and Senior tranches. It can be observed that the implied correlation is larger for Equity and Senior tranches and smaller for Mezzanine tranche, exhibiting a “smile” characteristic. In contrast, the base correlation does not display the smile though it increases slightly with the seniority of tranches.

4.4 Extending to Lévy Processes

From the empirical results it is clear that the spreads in the various tranches are sensitive to the default probabilities. In particular, in the analytic model, the default probabilities are represented by X_k which follows a distribution, for example, in the single factor Gaussian copula model. With a single factor approach, one can extend the default modeling to encompass more complicated situations with modeling correlated defaults or introducing fat-tailed distribution to X_k. Examples of the latter include the Variance Gamma, the Normal Inverse Gaussian (NIG), the Meixner, and other distributions. These generically belong to the class of Lévy processes described in Sato (2000). NIG processes are also discussed in detail in Rydberg (1996). Using the NIG process for example could lead to a more accurate pricing of all the tranches within a CDO structure. This is because by more accurately modeling the default probabilities at different loss levels, the spread in each tranche is more accurately priced. An NIG process could be used as follows.

Following Equation (7), suppose there is a single (common) factor model of X_k comprising a common factor M and an individual factor Z_k where

$$X_k = a_k M + \sqrt{1-a_k^2} Z_k \ , k = 1, 2, \cdots, K$$

but where *M* and Z_k now follow independent NIG-processes. In particular, the density of the NIG($X_{k;}$ α, β, δ, μ) is given by

$$f(X_k) = \frac{\alpha\delta}{\pi} \frac{\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(X_k-\mu)\right)}{\sqrt{\delta^2+(X_k-\mu)^2}} K_1\left(\alpha\sqrt{\delta^2+(X_k-\mu)^2}\right)$$

where $\alpha^2 > \beta^2 > 0$, and $K_1(\omega) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\omega\left(\theta+\theta^{-1}\right)\right) d\theta$ is modified Bessel function of the third kind. Then the probability distribution of default can be modeled by the distribution of the NIG process as described. Simulations can be performed according to the density function.

An immediate outcome of Lévy process modeling of the default processes would be the more accurate pricing of individual tranches within a CDO. Fatter tails allocated to the probabilities of default modeling would provide for higher default intensities at the Equity tranche and also at the Senior tranche. This would imply that compared to the Gaussian copula method, an NIG method is likely to produce higher theoretical spreads for Equity and Senior tranches, and lower spread for the Mezzanine tranche. By matching to the market price, this would in turn imply that the implied correlations for the Equity and the Senior tranche under Lévy processes would be lower than those in the Gaussian process. In the latter, the implied correlation has to work harder and be pumped up in order to reflect a higher market price due to the higher default probabilities. Since the latter are captured by the Lévy process, the implied correlation becomes flatter. Indeed, many research undertaken at this time all attempt to bring about a flat implied correlation curve using fat-tailed processes. In this sense, the correlation bias or smile can be explained away.

5. Conclusions

This paper studies the pricing of CDO using Monte Carlo and analytic methods. The portfolio loss, the expected loss in each CDO tranche, the tranche spread, and the default delta sensitivity are analyzed with respect to maturity, default correlation, default intensity and recovery rate. The results are summarized as follows.

Maturity

The portfolio loss and the loss distributed in each tranche increases with the increase in time to maturity T, due to the higher probability of default for larger T. The spread of Equity tranche is not sensitive to T. However, in the Mezzanine and Senior tranches, the spreads increase with a larger T. As for default delta sensitivity, the Equity tranche showed mixed characteristics with respect to T. Both these sensitivities of the Mezzanine and Senior tranches increase with increase in T.

Default correlation

The portfolio expected loss EL is not sensitive to the default correlation ρ once the intensity λ is fixed. Equity tranche EL appears to decrease with increase of ρ. In contrast, Senior tranche EL increases with increase of ρ. Mezzanine EL displays both possibilities. Similar to EL, the spread as well as default correlation sensitivity of Equity tranche decreases and that of Senior tranche increases with the increase of ρ. Mezzanine tranches show mixed results. For sensitivity, both Equity and Mezzanine tranches have mixed results with respect to ρ, while Senior tranche sensitivity increases with increase of ρ.

Intensity

The spreads in all tranches increase with increase of λ. Regarding default delta sensitivity, Equity tranche's decreases and Senior tranche's increases with increase of λ. Mezzanine tranche shows mixed results.

Recovery

Contrary to the case of intensity, recovery rate has an reverse relationship with the spreads of all tranches. The spreads in all tranches decrease with increase of recovery rate R. The sensitivity of Equity increases and that of Senior tranche decreases with increase of R. Mezzanine tranche shows mixed results.

The analysis of default correlation shows that the implied default correlation has a "smile" characteristic, while the base correlation increases slightly with the seniority of tranches.

Our results also show that the Monte Carlo method is slower in terms of computational time than the analytic method. Monte Carlo does not appear to be a satisfactory approach for calculating default delta sensitivity as the sensitivity values computed under Monte Carlo vary widely.

Considering the disadvantages of the current Monte Carlo methods, future work should explore improved Monte Carlo methods. The performance of the analytic approach can also be further improved in future work. The Likelihood and Pathwise methods used in Joshi and Kainth (2004) can be explored for calculating the default delta sensitivity of CDO.

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Figure 1: Illustration of a CDO structure

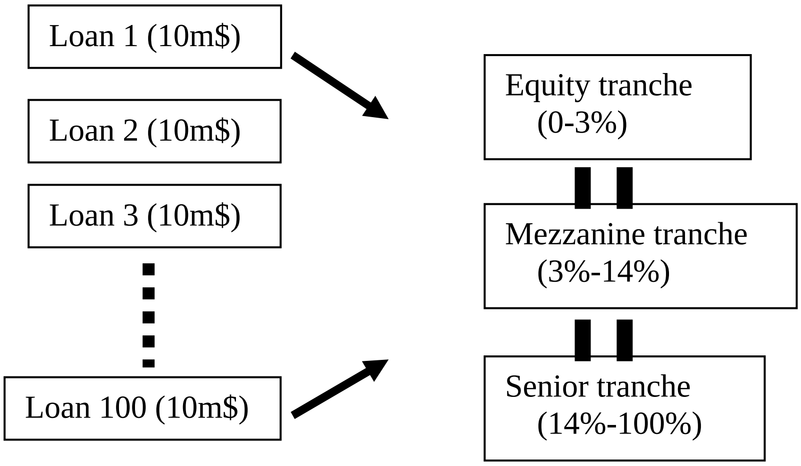
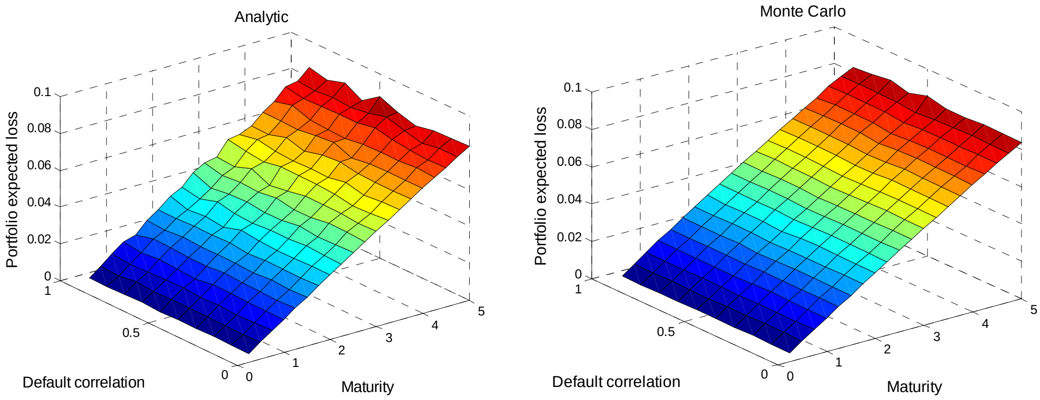
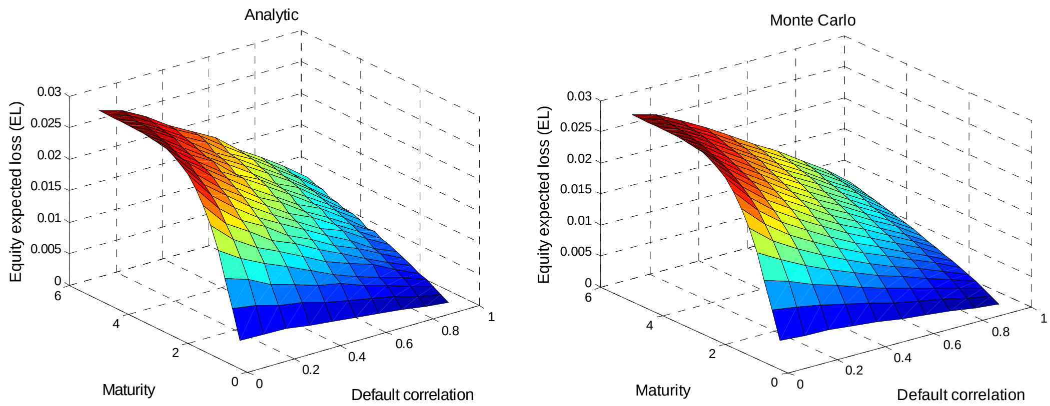


Figure 2: The portfolio expected loss (EL) with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)



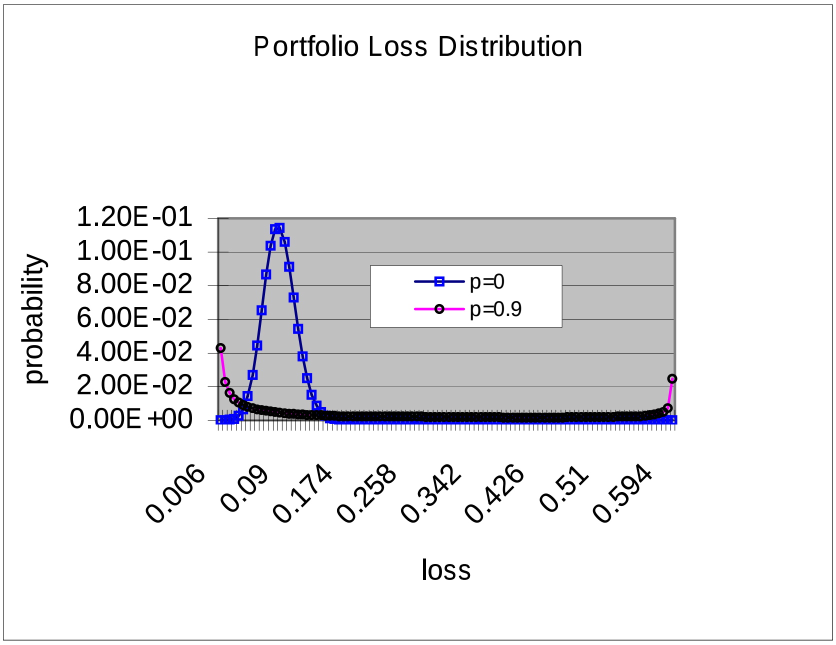
(a) **(b)**

Figure 3: The Equity tranche expected loss (EL) with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)

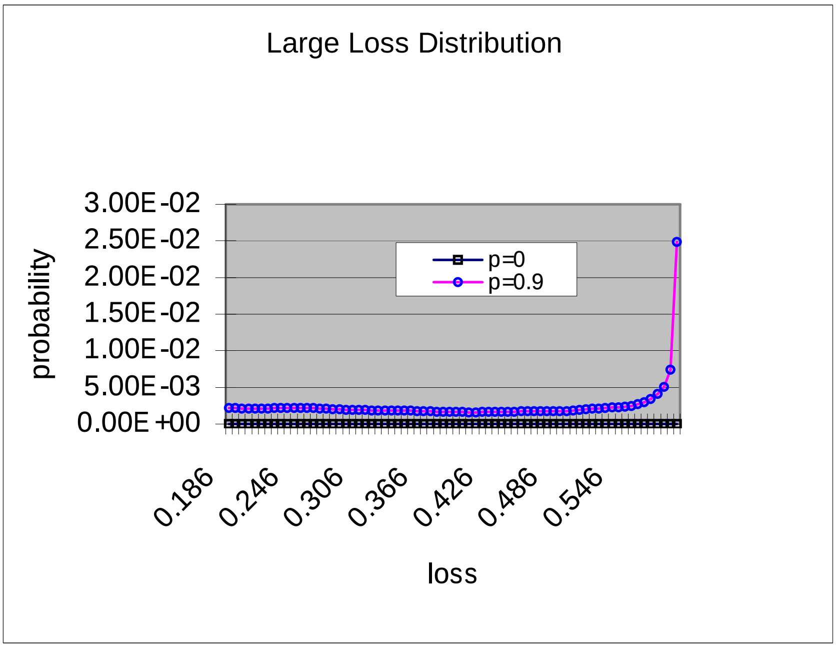


(a) **(b)**

Figure 4: Portfolio loss distribution in the entire loss range (a) and large losses (b). Large loss is defined as loss over 0.186. Two cases of default correlation ρ = 0 and ρ = 0.9 are considered. Time horizon is T = 5 years.

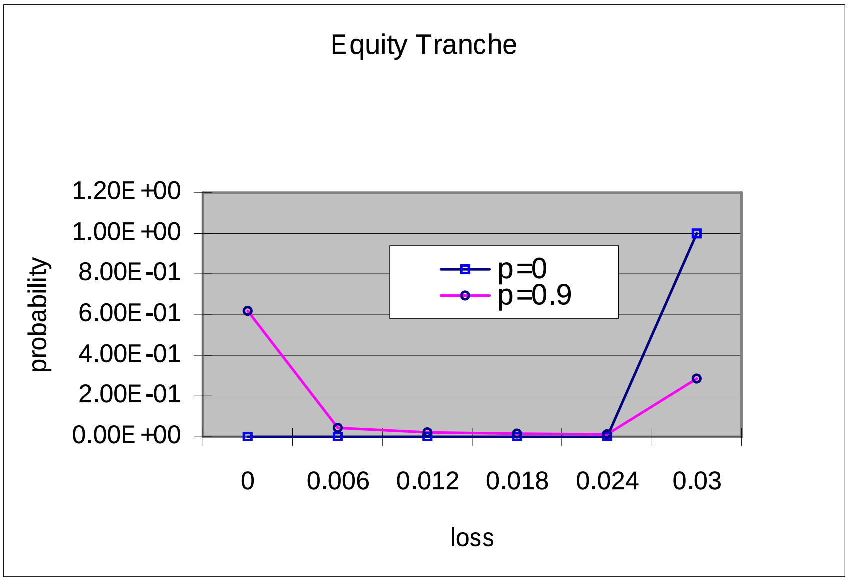


(a)

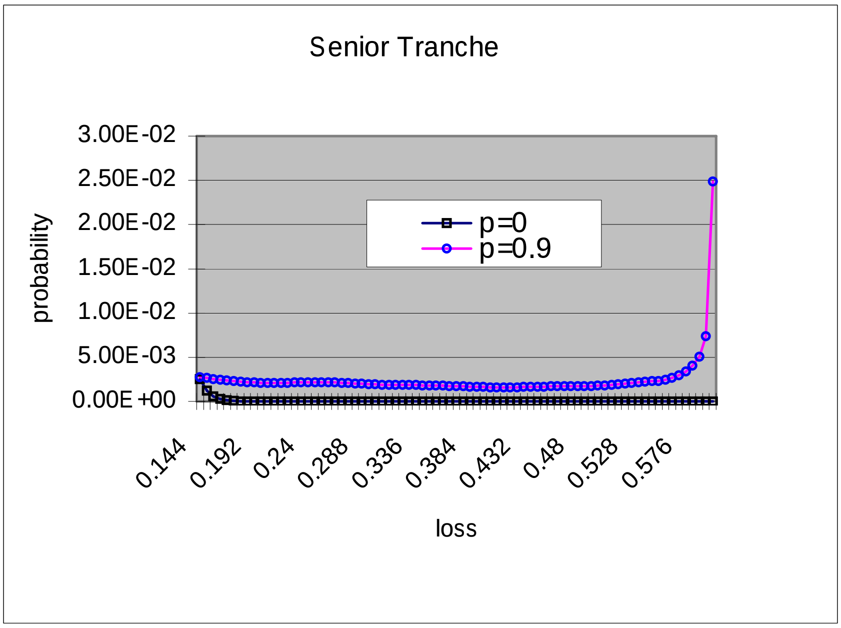


(b)

Figure 5: The portfolio loss distribution of Equity tranche (a) and Senior tranche (b). Two cases of default correlation ρ = 0 and ρ = 0.9 are considered.

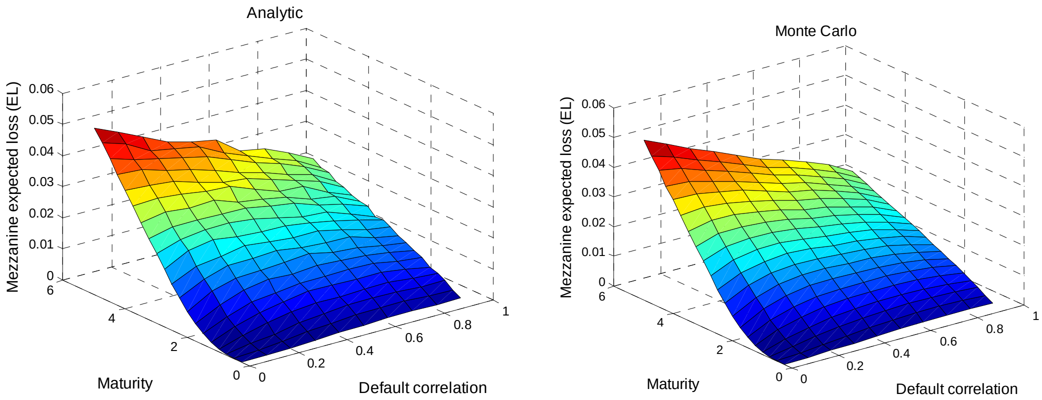


(a)



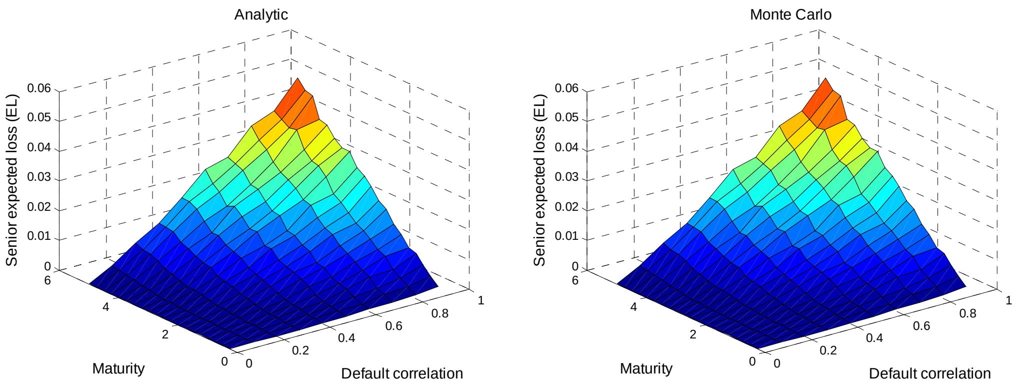
(b)

Figure 6: The Mezzanine tranche expected loss (EL) with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)



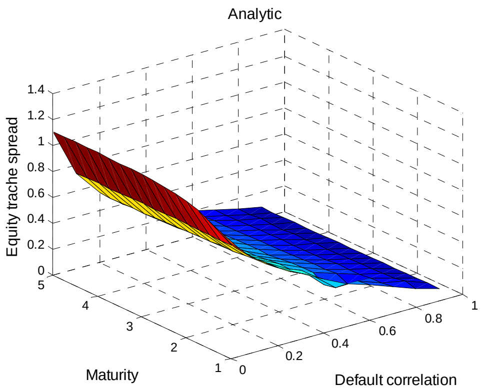
(a) **(b)**

Figure 7: The Senior tranche expected loss (EL) with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)

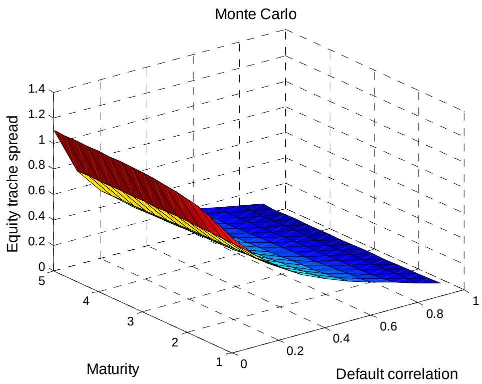


(a) **(b)**

Figure 8: The Equity tranche spread with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)

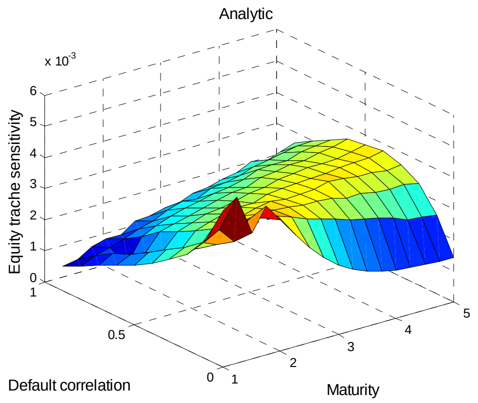


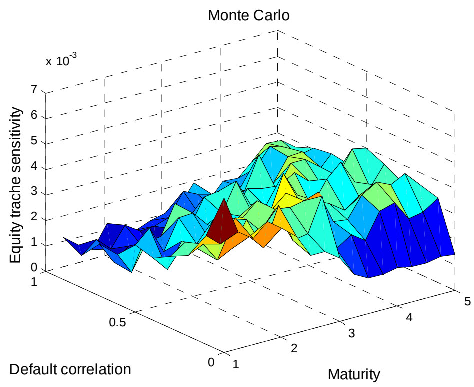
(a)



(b)

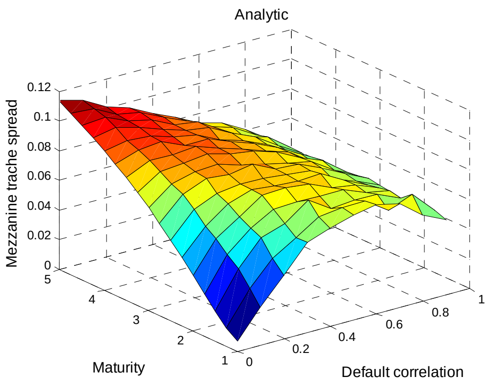
Figure 9: The delta sensitivity of Equity tranche with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)

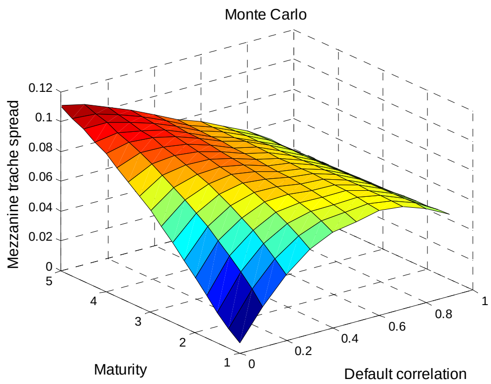




(a) **(b)**

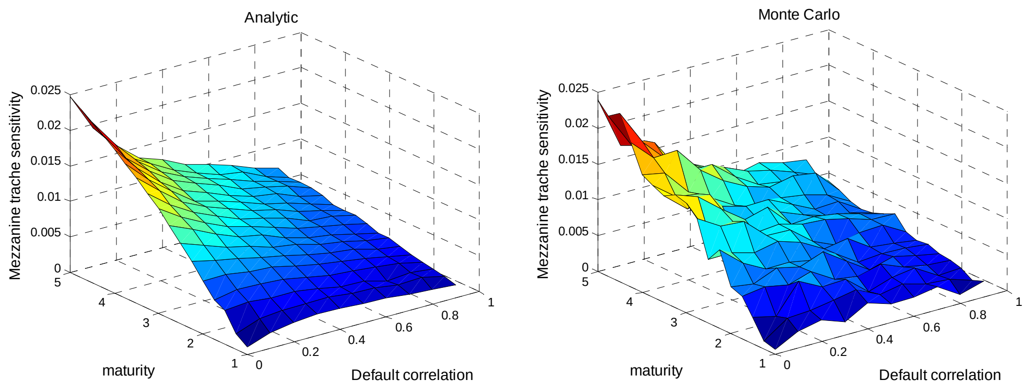
Figure 10: The Mezzanine tranche spread with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)





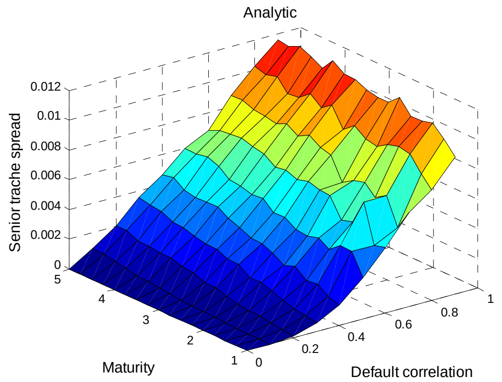
(a) **(b)**

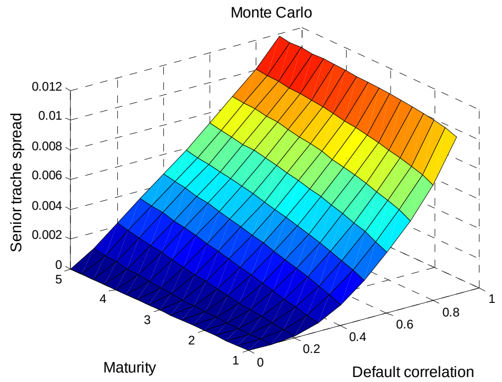
Figure 11: The delta sensitivity of Mezzanine tranche with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)



(a) **(b)**

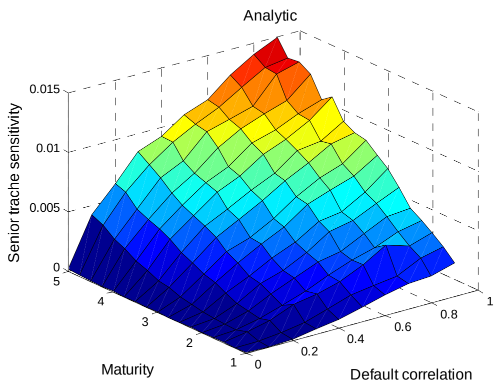
Figure 12: The Senior tranche spread with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)



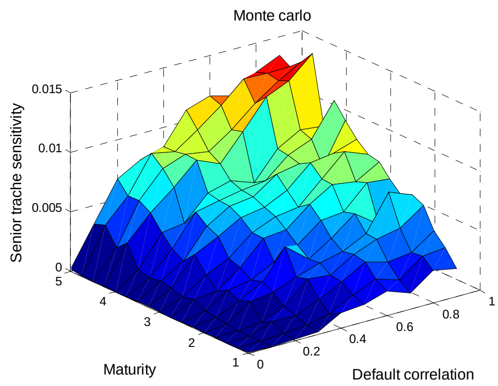


(a) **(b)**

Figure 13: The delta sensitivity of Senior tranche with different maturity and default correlation in the analytic method (a) and Monte Carlo method (b)

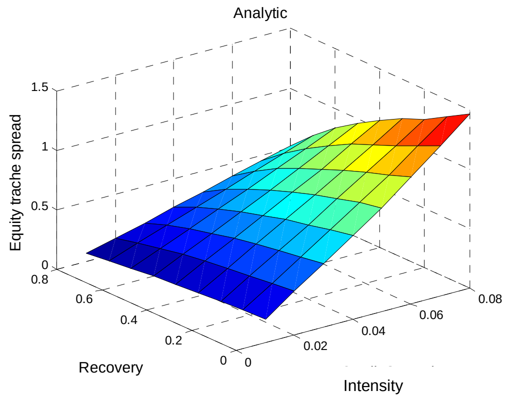


(a)

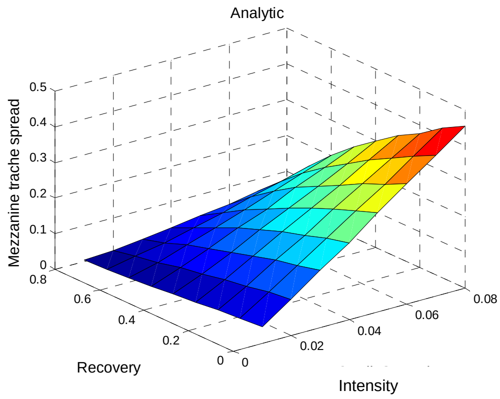


(b)

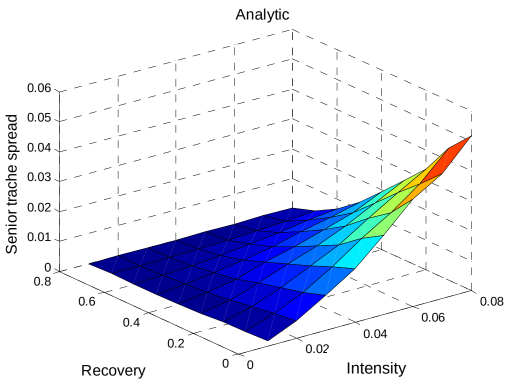
Figure 14: Tranche spread with respect to different default intensities and recovery rates in the Equity tranche (a), Mezzanine tranche (b), and Senior tranche (c)



(a)

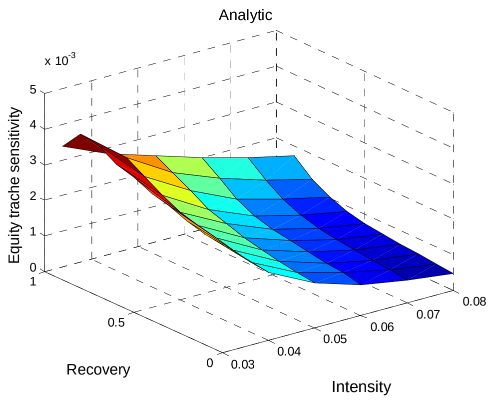


(b)

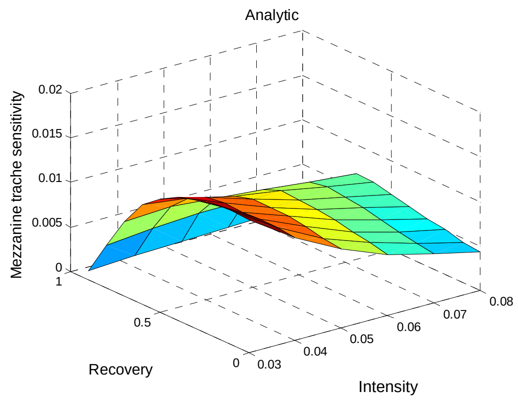


(c)

Figure 15: Default Delta sensitivity with respect to different default intensities and recovery rates in the Equity tranche (a), Mezzanine tranche (b), and Senior tranche (c)



(a)



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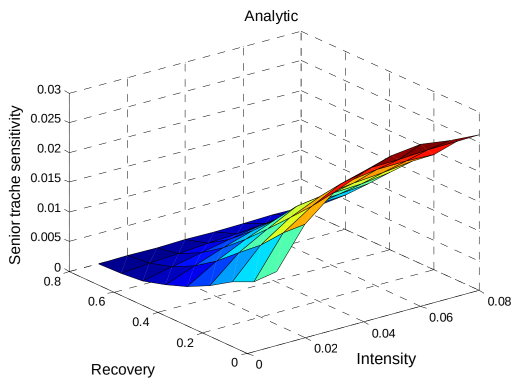

(c)

Figure 16: The implied correlation and base correlation

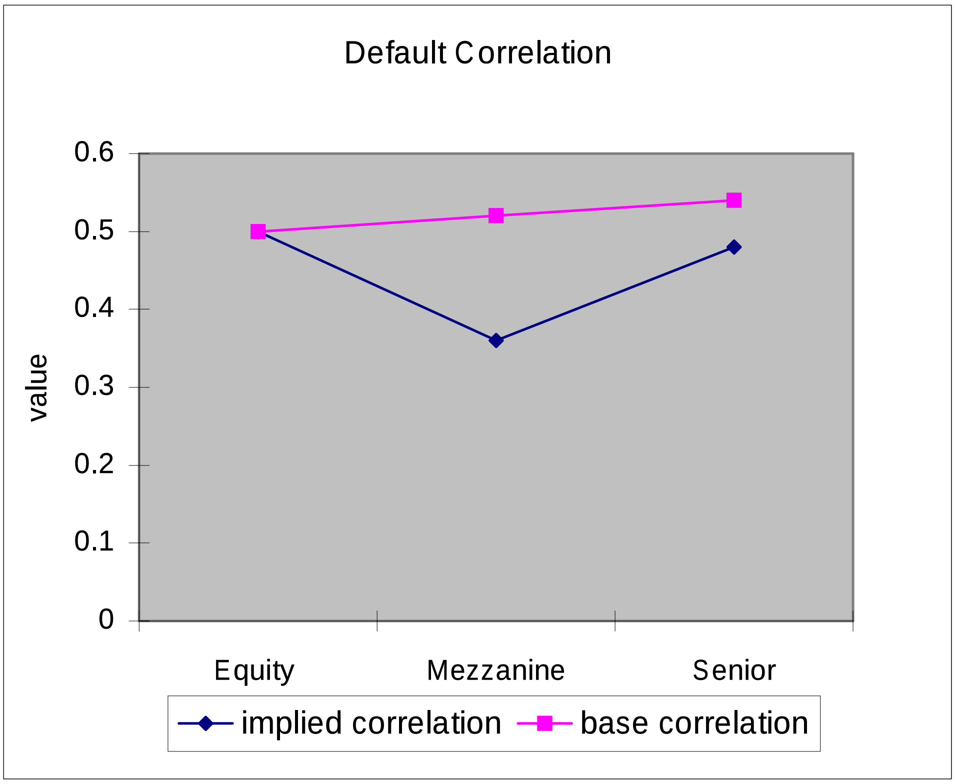

Table 1: A summary of literature review in CDO

Publications*	Default correlation methods	Portfolio loss methods	Default delta sensitivity	Conclusions
Sidenius and Basu (2003)	One-factor Copula	Analytical approach by a recursion-based probability calculation	Uses Brute-force method and an analytical method	Proposed a number of techniques to improve the efficiency with which prices and hedge parameters can be calculated for credit basket derivatives.
Gibson (2004)	Same as above	Same as above	Brute-force method	The value of the senior tranche decreases as correlation increases. In contrast, the equity tranche value increases as default correlation increases. CDO tranches are sensitive to the business cycle.
Laurent and Gregory (2003)	One-factor Copula	Analytic approach based on Fourier method	N.A.	Proposed an analytic approach based on Fourier method to calculate the conditional loss distribution on a portfolio as a convolution of the conditional loss distributions of each entity in the portfolio.
Burtschell etc. (2005)	Copulas	Laurent and Gregory's analytical approach	N.A.	Compare some popular Copula functions such as Gaussian Copula model, stochastic correlation extension to Gaussian Copula, Student t Copula model, double t factor model, clayton and Marshall-Olkin Copula.
Peixoto (2004)	Copula	Monte Carlo and analytical methods	N.A.	Compare Monte Carlo and analytical method in the pricing of CDO. Both prices are within one standard deviation.
Mina and Stern (2003)	One-factor Copula	Analytical approach based on Fourier transform method	Analytic method	Senior Tranche price depends on the best while Equity Tranche on the worst names in a portfolio. Mezzanine behavior varies over time. Loan-equivalent hedges depend on the entire portfolio. Equity Tranche value rises while Senior Tranche value drops when correlation increases.
Chen and Zhang (2003)	One-factor Copula	Analytical approach based on Fourier transform method	N.A.	FFT/FI generates loss distributions more accurate than those by the Monte Carlo simulations.
Duffie and Garleanu (2001)	Dependent default intensity	Monte Carlo	N.A.	Illustrated the effects of correlation and prioritization for the market valuation, diversity score and risk of CDO in a simple jump-diffusion setting for correlated default intensities.
Hull and White (2004)	One-factor Copula	Two analytic methods: recursive approach and iterative numerical procedure	N.A.	The procedures are attractive alternatives to Monte Carlo simulation and have advantages over the fast Fourier transform approach. Implied correlations are typically not the same for all tranches.
Andersen and Sidenius (2004)	One-factor Copula	Analytical recursive method	Analytic method	This paper extends the standard Gaussian Copula model by using random recovery rates and random systematic factor loadings. It is capable of producing correlation skews similar to those observed in the market.
Kalemanova, etc. (2005)	Copula with Normal inverse Gaussian distribution	Analytic approach based on Large Homogenous portfolio (LHP) approach	N.A.	Proposed a modification of the LHP model replacing the Student t distribution with the Normal inverse Gaussian (NIG). The employment of the NIG distribution does not only speed up the computation time significantly but also brings more flexibility into the dependence structure.
Blum and Overbeck (2004)	One-factor Copula	Analytic approach	N.A.	Analytic techniques constitute a powerful tool for the evaluation of CDO.
Morokoff (2003)	Copula	Monte Carlo	N.A.	This paper describes a multiple-time step simulation approach that tracks cash flows over the life of a CDO deal to determine the risk characteristics of CDO tranches.
Hurd and Kuznetsov (2005)	N.A.	Affine Markov Chain model	N.A.	Combined a continuous time Markov Chain with an independent set of affine processes that yield a flexible framework for which computations are very efficient.

* Note: The publication name can be found in the Reference section of this paper.

Table 2: The characteristics of CDO used in this study

Maturity	5 years
Hazard rate	0.03
Recovery rate	0.4
Risk-free rate	5% with continuous compounding
Payment frequency	quarterly payment
Equity tranche	$[0, 3\%]$
Mezzanine tranche	$[3\%, 14\%]$
Senior tranche	$[14\%, 100\%]$

Table 3: Portfolio expected loss (EL), Equity EL, Mezzanine EL and Senior EL with different values of ρ for $T = 5$

Default correlation ρ	Portfolio EL			Equity EL			Mezzanine EL			Senior EL		
	Analytic	Monte Carlo	Difference	Analytic	Monte Carlo	Difference	Analytic	Monte Carlo	Difference	Analytic	Monte Carlo	Difference
0	0.0837	0.0836	0.0001	0.0300	0.0300	0.0000	0.0537	0.0535	0.0001	0.0000	0.0000	0.0000
0.1	0.0838	0.0831	0.0008	0.0290	0.0290	0.0000	0.0504	0.0500	0.0004	0.0045	0.0041	-0.0003
0.2	0.0839	0.0829	0.0010	0.0270	0.0270	0.0000	0.0467	0.0463	0.0004	0.0102	0.0096	-0.0006
0.3	0.0830	0.0811	0.0019	0.0248	0.0246	0.0002	0.0435	0.0426	0.0009	0.0157	0.0140	-0.0017
0.4	0.0835	0.0839	-0.0003	0.0225	0.0226	-0.0001	0.0402	0.0407	-0.0004	0.0208	0.0206	-0.0002
0.5	0.0857	0.0876	-0.0019	0.0201	0.0204	-0.0004	0.0370	0.0387	-0.0017	0.0257	0.0275	0.0018
0.6	0.0829	0.0812	0.0016	0.0178	0.0175	0.0003	0.0346	0.0336	0.0009	0.0315	0.0301	-0.0014
0.7	0.0851	0.0859	-0.0009	0.0153	0.0153	0.0000	0.0316	0.0321	-0.0005	0.0372	0.0385	0.0013
0.8	0.0833	0.0828	0.0005	0.0127	0.0128	-0.0001	0.0286	0.0286	0.0000	0.0429	0.0414	-0.0016
0.9	0.0836	0.0831	0.0005	0.0300	0.0300	0.0000	0.0242	0.0247	-0.0005	0.0498	0.0505	0.0007

Table 4: Spreads of Equity, Mezzanine and Senior tranches with different values of ρ for $T=5$

Default correlation ρ	Equity			Mezzanine			Senior		
	Analytic	Monte Carlo	Difference	Analytic	Monte Carlo	Difference	Analytic	Monte Carlo	Difference
0	1.1065	1.1036	0.0029	0.1133	0.1107	0.0026	0.0000	0.0000	0.0000
0.1	0.7372	0.7375	-0.0003	0.1099	0.1077	0.0021	0.0009	0.0010	0.0000
0.2	0.5347	0.5327	0.0020	0.1012	0.1015	-0.0003	0.0021	0.0023	-0.0002
0.3	0.4033	0.4009	0.0024	0.0964	0.0953	0.0010	0.0035	0.0035	0.0000
0.4	0.3210	0.3185	0.0025	0.0885	0.0880	0.0004	0.0047	0.0047	-0.0001
0.5	0.2493	0.2470	0.0023	0.0818	0.0803	0.0015	0.0060	0.0059	0.0002
0.6	0.1973	0.1944	0.0028	0.0739	0.0749	-0.0011	0.0068	0.0073	-0.0005
0.7	0.1557	0.1537	0.0019	0.0692	0.0678	0.0014	0.0088	0.0087	0.0001
0.8	0.1152	0.1156	-0.0003	0.0606	0.0604	0.0001	0.0103	0.0101	0.0002
0.9	0.0808	0.0794	0.0015	0.0505	0.0500	0.0005	0.0116	0.0119	-0.0002

Table 5: Default delta sensitivity of Equity, Mezzanine and Senior tranches with different values of ρ for $T = 5$

Default correlation ρ	Equity			Mezzanine			Senior		
	Analytic	Monte Carlo	Difference	Analytic	Monte Carlo	Difference	Analytic	Monte Carlo	Difference
0	0.0014	0.0014	0.0000	0.0248	0.0239	0.0009	0.0001	0.0002	0.0000
0.1	0.0025	0.0028	-0.0003	0.0195	0.0167	0.0028	0.0042	0.0034	0.0008
0.2	0.0033	0.0035	-0.0002	0.0161	0.0165	-0.0004	0.0063	0.0067	-0.0004
0.3	0.0036	0.0033	0.0003	0.0135	0.0133	0.0002	0.0085	0.0078	0.0007
0.4	0.0038	0.0037	0.0001	0.0124	0.0115	0.0009	0.0092	0.0084	0.0008
0.5	0.0037	0.0040	-0.0003	0.0108	0.0100	0.0008	0.0105	0.0109	-0.0004
0.6	0.0036	0.0034	0.0002	0.0099	0.0074	0.0025	0.0113	0.0116	-0.0003
0.7	0.0033	0.0031	0.0002	0.0089	0.0090	-0.0001	0.0128	0.0121	0.0027
0.8	0.0030	0.0030	0.0000	0.0076	0.0082	-0.0006	0.0140	0.0115	0.0025
0.9	0.0026	0.0027	-0.0001	0.0068	0.0077	-0.0008	0.0150	0.0144	0.0026