Session 11

Maximum Likelihood Estimation / Limited Dependent Variables

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Agenda and Quick Review

- Introduce Maximum Likelihood Estimation using a Simple Example
- Hypothesis Testing / Model Selection
- Application to Limited Dependent Variables

Roughly speaking, ML estimation takes as estimators the values of the parameters that maximize the probability of obtaining the observed sample

R packages for this session:

```
library(tidyverse); library(patchwork); library(stargazer)
library(sandwich); library(lmtest); library(margins);
library(wooldridge); library(AER); library(truncreg)
```

Quick Probability Review

If $Y \sim N(\mu, \sigma^2)$, then its pdf is

$$f_Y(y; \mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}$$

Purpose of pdf is to describe probability of events involving Y

e.g., If $Y \sim N(\mu, \sigma^2)$, then what is $Pr(Y > 1)$?

Ans:

$$\Pr(Y > 1) = \int_1^\infty (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} dy$$

Quick Probability Review

Multivariate Normal Distribution

If $X_k \sim N_k(\mu, \Sigma)$ where μ is $(k \times 1)$ and Σ is $(k \times k)$, then

$$f(x) = (2\pi)^{-k/2}|\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(x-\mu)^{\mathrm{T}}\Sigma^{-1}(x-\mu)\right\}$$

For X, Y bivariate normal distribution, can be shown that

$$Y \mid X \sim \text{Normal}(\mu_{y|x}, \sigma^2_{y|x}) \quad \text{and } X \sim \text{Normal}(\mu_x, \sigma^2_x)$$

where

$$\mu_{y|x} = \mu_y + \frac{\sigma_{xy}}{\sigma^2_x}(x - \mu_x) \quad \text{and} \quad \sigma^2_{y|x} = \sigma^2_y - \frac{\sigma_{xy}}{\sigma^2_x}$$

Quick Probability Review

In general,

- For two random variables X and Y: $f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x)$
- Can extend to three or more variables:

$$f_{X,Y,Z}(x,y,z) = f_{Z|Y,X}(z \mid y,x)f_{Y|X}(y \mid x)f_X(x)$$

- For multivariate normal, all conditionals are normal
- I'll drop the subscripts from now: $f(x,y,z)$ will mean $f_{X,Y,Z}(x,y,z)$, etc.
- If X, Y, Z are independent, then $f(x,y,z) = f(z)f(y)f(z)$

Intro: A Simple Coin Toss Example

Objective: Estimate probability p of observing heads for a certain coin

Observations: $\{Head, Tail, Tail\}$ coded as $\{1, 0, 0\}$

$$Y_i \overset{iid}{\sim} \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Prior to observing outcomes, probability of obtaining $\{1, 0, 0\}$ is

$$\Pr(Y_1 = 1, Y_2 = 0, Y_3 = 0 \mid p) = p(1-p)^2$$

What is the "most likely" value for p?

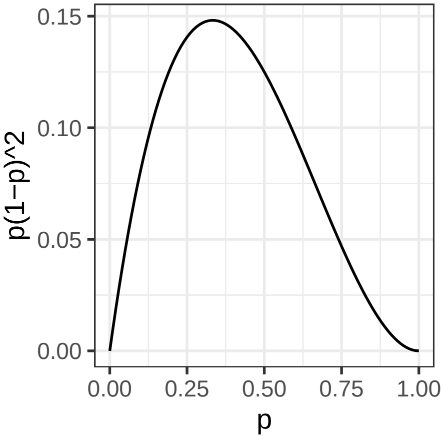
- If $p = 0.1$, then $\Pr(Y_1 = 1, Y_2 = 0, Y_3 = 0) = 0.1(1-0.1)^2 = 0.081$
- If $p = 0.9$, then $\Pr(Y_1 = 1, Y_2 = 0, Y_3 = 0) = 0.9(1-0.9)^2 = 0.009$

Both seem "unlikely" (though certainly possible)

Intro: A Simple Coin Toss Example

Plot of

$$\Pr(Y_1 = 1, Y_2 = 0, Y_3 = 0 \mid p) = p(1-p)^2$$



A reasonable estimate would seem to be in the vicinity of 0.3

In fact $p(1-p)^2$ is maximized at $p = 1/3$

Our maximum likelihood estimate for p is $\hat{p} = 1/3$

Maximum Likelihood Estimation

Maximum Likelihood choose as estimates those parameter values that make your sample observations the most likely, prior to seeing the sample

Maximum likelihood estimators are (subject to certain conditions holding)

- Consistent
- Asymptotically efficient
- Asymptotically normal
- may be biased in finite samples

Maximum Likelihood Estimation

Step 1 Write down a "complete model" describing the data generating process

- Y_i iid with mean $E(Y_i) = \mu$ and $Var(Y_i) = \sigma^2$, $i = 1, 2, \ldots, n$ is not complete
- $Y_i \overset{iid}{\sim}$ Normal(μ, σ^2) , $i = 1, 2, \ldots, n$ is a complete model

With a complete model you can write down the p.d.f. for your data

$$p(Y_1, Y_2, \ldots, Y_n \mid \theta)$$

where θ is a vector containing your parameters

E.g., if $Y_i \overset{iid}{\sim}$ Normal(μ, σ^2) , $i = 1, 2, \ldots, n$, then

$$p(y_1, y_2, \ldots, y_n \mid \mu, \sigma^2) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2}\frac{(y_i - \mu)^2}{\sigma^2}\right\}$$

Maximum Likelihood Estimation

Step 2 Reinterpret your pdf as a function of your parameters given the data, i.e.,

$$L(\theta \mid Y_1, Y_2, \dots, Y_n)$$

and call this the "Likelihood Function"

E.g., if $Y_i \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$, $i = 1, 2, \dots, n$, then

$$L(\mu, \sigma^2 \mid Y_1, Y_2, \dots, Y_n) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2}\frac{(Y_i - \mu)^2}{\sigma^2}\right\}$$

The likelihood function and the pdf have the same form, but different interpretation

Maximum Likelihood Estimation

Step 3 The maximum likelihood estimator is the value of the parameters that maximize the Likelihood Function

E.g., for the $Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $i = 1, 2, \dots, n$ example, we have

$$\hat{\mu}, \widehat{\sigma^2} = \underset{\mu, \sigma^2}{\arg\max} \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2}\frac{(Y_i - \mu)^2}{\sigma^2}\right\}$$

Maximum Likelihood Estimation

We often maximize the log-likelihood instead of the likelihood

e.g., for the $Y_i \overset{iid}{\sim}$ Normal(μ, σ^2), example,

$$L(\mu, \sigma^2 \mid Y_1, Y_2, \dots, Y_n) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left\{ -\frac{1}{2} \frac{(Y_i - \mu)^2}{\sigma^2} \right\}$$

$$\begin{aligned} \ln L(\mu, \sigma^2 \mid Y_1, Y_2, \dots, Y_n) &= \sum_{i=1}^{n} -\frac{1}{2} \ln(2\pi\sigma^2) + \sum_{i=1}^{n} \left\{ -\frac{1}{2} \frac{(Y_i - \mu)^2}{\sigma^2} \right\} \\ &= -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - \mu)^2 \end{aligned}$$

Maximum Likelihood Estimation

(Example continued) Straightforward to show

$$\hat{\mu}_{ml} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \overline{Y}$$

$$\widehat{\sigma^2}_{ml} = \frac{1}{n}\sum_{i=1}^{n} (Y_i - \overline{Y})^2$$

Notice that the ML estimator for μ is unbiased and consistent, but the ML estimator for σ^2 is biased (but consistent)

ML estimator is also asymptotically normal, but what is the estimator variance?

Maximum Likelihood Estimation

Suppose the likelihood function is $L(\theta \mid Y_1, Y_2, \ldots, Y_n)$ (θ may be a vector)

Then

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \text{Normal}\left(0, -nE\left(\frac{\partial^2 \ln L(\theta \mid Y_1, Y_2, \ldots, Y_n)}{\partial\theta\partial\theta^{\mathrm{T}}}\right)^{-1}\right)$$

i.e., we can use the asymptotic approximation

$$\hat{\theta} \overset{a}{\sim} \text{Normal}\left(\theta, -E\left(\frac{\partial^2 \ln L(\theta \mid Y_1, Y_2, \ldots, Y_n)}{\partial\theta\partial\theta^{\mathrm{T}}}\right)^{-1}\right)$$

Maximum Likelihood Estimation

In practice, we further approximate

$$-E\left(\frac{\partial^2 \ln L(\theta \mid Y_1, Y_2, \dots, Y_n)}{\partial\theta\partial\theta^{\mathrm{T}}}\right)^{-1} \quad \text{with} \quad -E\left(\frac{\partial^2 \ln L(\hat{\theta} \mid Y_1, Y_2, \dots, Y_n)}{\partial\theta\partial\theta^{\mathrm{T}}}\right)^{-1}$$

The expression

$$I(\theta) = -E\left(\frac{\partial^2 \ln L(\theta \mid Y_1, Y_2, \dots, Y_n)}{\partial\theta\partial\theta^{\mathrm{T}}}\right)$$

is called the “information matrix”

Maximum Likelihood Estimation

For the $Y_i \overset{iid}{\sim}$ Normal(μ, σ^2) , $i = 1, 2, \dots, n$ example, we can show

$$-E\left(\frac{\partial^2 \ln L(\theta \mid Y_1, \dots, Y_n)}{\partial\theta\partial\theta^{\mathrm{T}}}\right) = -E\begin{bmatrix} \dfrac{\partial^2 \ln L(\theta \mid Y_1, \dots, Y_n)}{(\partial\mu)^2} & \dfrac{\partial^2 \ln L(\theta \mid Y_1, \dots, Y_n)}{\partial\mu\partial\sigma^2} \\ \dfrac{\partial^2 \ln L(\theta \mid Y_1, \dots, Y_n)}{\partial\mu\partial\sigma^2} & \dfrac{\partial^2 \ln L(\theta \mid Y_1, \dots, Y_n)}{(\partial\sigma^2)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{n}{\sigma^2} & 0 \\ 0 & \dfrac{n}{2\sigma^4} \end{bmatrix}$$

therefore $\begin{bmatrix} \hat{\mu} \\ \widehat{\sigma^2} \end{bmatrix} \overset{a}{\sim}$ Normal $\left(\begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix} , \begin{bmatrix} \dfrac{\sigma^2}{n} & 0 \\ 0 & \dfrac{2\sigma^4}{n} \end{bmatrix} \right)$

General Coin Toss Example

$$Y_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases} \qquad \text{iid} \qquad i = 1, 2, \ldots, n$$

PDF: $p(y_1, y_2, \ldots, y_n \mid p) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i}$

Likelihood: $L(p \mid Y_1, Y_2, \ldots, Y_n) = \prod_{i=1}^{n} p^{Y_i}(1-p)^{1-Y_i}$

log-Likelihood: $\ln L(p \mid Y_1, Y_2, \ldots, Y_n) = \ln p \sum_{i=1}^{n} Y_i + \ln(1-p) \sum_{i=1}^{n}(1-Y_i)$

Maximize:

$$\frac{\partial \ln L(\hat{p})}{\partial p} = \frac{1}{\hat{p}} \sum_{i=1}^{n} Y_i - \frac{1}{1-\hat{p}} \sum_{i=1}^{n}(1-Y_i) = 0 \;\Rightarrow\; \hat{p} = \overline{Y}$$

General Coin Toss Example

We also have

$$\frac{\partial^2 \ln L(p)}{\partial p^2} = -\frac{1}{p^2}\sum_{i=1}^{n} Y_i - \frac{1}{(1-p)^2}\sum_{i=1}^{n}(1-Y_i)$$

$$E\left(\frac{\partial^2 \ln L(p)}{\partial p^2}\right) = -\frac{1}{p^2}np - \frac{1}{(1-p)^2}n(1-p) = -\frac{n}{p(1-p)}$$

$$-E\left(\frac{\partial^2 \ln L(p)}{\partial p^2}\right)^{-1} = \frac{p(1-p)}{n}$$

That is,

$$\hat{p} \overset{a}{\sim} \text{Normal}\left(p\,,\ \frac{p(1-p)}{n}\right)$$

OLS

Suppose

$$y = X\beta + \varepsilon\,,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

where X is $n \times k$ and β is $k \times 1$, i.e.,

$$\varepsilon \sim \text{Normal}_n(X\beta, \sigma^2 I_n)$$

$$\begin{aligned} f(y \mid X; \beta, \sigma^2) &= (2\pi)^{-n/2} |\sigma^2 I_n|^{-1/2} \exp\left\{-\frac{1}{2}\varepsilon^{\mathrm{T}}(\sigma^2 I_n)^{-1}\varepsilon\right\} \\ &= (2\pi)^{-n/2}(\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)^{\mathrm{T}}(y - X\beta)\right\} \end{aligned}$$

$$\ln L(\beta, \sigma^2 \mid y, X) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}(y - X\beta)^{\mathrm{T}}(y - X\beta)$$

OLS

Can show ML estimators for β and σ^2 are

$$\hat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y \quad \text{and} \quad \widehat{\sigma^2} = \frac{\hat{\varepsilon}^{\mathrm{T}}\hat{\varepsilon}}{n}$$

where $\hat{\varepsilon} = y - X\hat{\beta}$, and

$$\begin{bmatrix} \hat{\beta} \\ \widehat{\sigma^2} \end{bmatrix} \overset{a}{\sim} \text{Normal}\left(\begin{bmatrix} \beta \\ \sigma^2 \end{bmatrix}, \begin{bmatrix} \sigma^2(X^{\mathrm{T}}X)^{-1} & 0 \\ 0 & \dfrac{2\sigma^4}{n} \end{bmatrix} \right)$$

Remark

In most cases, we are unable to derive a formula for our ML estimators

It seems more common to have to maximize the log-likelihood numerically

Likewise, the information matrix is often also found numerically

We will not concern ourselves with the details

- In the applications of interest to us, the numerical maximization routine is included in the packages we use

Likelihood Ratio Test

Suppose $\ln L(\theta)$ where θ is $(k \times 1)$ vector of parameters

You want to test $R\theta = r_0$ where R is $J \times k$

1. Maximize $\ln L(\theta)$, get $\ln L_{UR} = \ln L(\hat{\theta})$
2. Maximize $\ln L(\theta)$ subject to $R\theta = r_0$, get $\ln L_R = \ln L(\tilde{\theta})$ where $\tilde{\theta}$ satisfies $R\tilde{\theta} = r_0$
3. Under the null hypothesis

$$LR = 2(\ln L_{UR} - \ln L_R) \overset{a}{\sim} \chi^2_{(J)}$$

Likelihood Ratio Test

Example: Jointly test $\beta_1 = 0$ and $\beta_2 = \beta_3$ in:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_k X_{ik} + \epsilon_i \, , \, \epsilon_i \sim N(0, \sigma^2)$$

Get $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_k, \widehat{\sigma^2}$ (unrestricted parameter estimates)

Get $\tilde{\beta}_0, \tilde{\beta}_2 = \tilde{\beta}_3, \tilde{\beta}_4, \ldots, \tilde{\beta}_k, \widetilde{\sigma^2}$ (restricted parameter estimates)

Note that

$$\widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{i1} - \cdots - \hat{\beta}_k X_{ik})^2$$

$$\widetilde{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \tilde{\beta}_0 - \tilde{\beta}_2 (X_{i2} + X_{i3}) - \tilde{\beta}_4 X_{i4} - \cdots - \tilde{\beta}_k X_{ik})^2$$

Likelihood Ratio Test

$$\ln L_{UR} = -\frac{1}{2}\sum_{i=1}^{n}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\ln\widehat{\sigma^2} - \frac{1}{2\widehat{\sigma^2}}\sum_{i=1}^{n}(Y_t - \widehat{\beta}_0 - \widehat{\beta}_1 X_{i1} - \cdots - \widehat{\beta}_k X_{ik})^2$$

$$= -\frac{1}{2}\sum_{i=1}^{n}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\ln\widehat{\sigma^2} - \frac{n}{2}$$

$$\ln L_R = -\frac{1}{2}\sum_{i=1}^{n}\ln(2\pi) - \frac{1}{2}\sum\ln\widetilde{\sigma^2} - \frac{1}{2\widetilde{\sigma^2}}\sum_{i=1}^{n}(Y_i - \widehat{\beta}_0 - \widehat{\beta}_2(X_{i2} + X_{i3}) - \widetilde{\beta}_4 X_{i4} - \ldots - \widetilde{\beta}_k X_{ik})^2$$

$$= -\frac{1}{2}\sum_{i=1}^{n}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\ln\widetilde{\sigma^2} - \frac{n}{2}$$

$$LR = 2(\ln L_{UR} - \ln L_R) = n(\ln\widetilde{\sigma^2} - \ln\widehat{\sigma^2}) \sim \chi^2_{(2)}$$

Information Criteria

Information Criterion Model Selection

- A higher likelihood indicates a better fit, or a “more likely” model
- max value of log-likelihood decreases when restrictions are imposed (like R^2)
- if restrictions are very wrong, log-likelihood will decrease substantially

Information Criteria: choose model with a higher log-likelihood, but with a penalty term to control for excessive parameterization (similar to adjusted-R^2)

Information Criteria

Akaike Information Criterion (AIC)

$$AIC = -2\ln L + 2q$$

where q is number of parameters in the model

- log-likelihood is negated, try to minimize AIC
- attempting to lower AIC by using "larger" model will succeed only if fall in $\ln L$ is greater than increase in q

Bayes Information Criterion (BIC) / Schwarz Information Criterion (SIC):

$$SIC = -2\ln L + q\ln n$$

- Again, choose model with lower SIC

Information Criteria

Remarks:

- Some implementations of AIC/ SIC divide expression throughout by n
  - not a problem if comparing models with same software package
- Models compared must be estimated over the same sample period
- AIC and SIC can be used to compare non-nested models (unlike LR test)
- Dependent variable must be the same across models
  - e.g., cannot compare AIC / SIC of model for Y_i vs those of model for $\ln Y_i$

Information Criteria

The AIC / SIC differ in their asymptotic properties

The SIC is "consistent"

- SIC chooses correct model asymp. if true model is in class of candidate models

AIC is not "consistent"

- positive prob. that AIC does not choose correct model even when true model is in class of candidate models
- However, AIC may perform better than SIC when true model is not in class of candidate models, and in finite samples even when it is

SIC contains a stronger penalty term for additional parameters, and will choose more parsimonious models (fewer parameters) than the AIC

Limited Dependent Variables

Limited Dependent Variables

Situations where dependent variable Y has some special structure

- $Y = 0, 1$
- $Y = 1, 2, 3, \ldots$
- Y is truncated or censored, or there are "corner solutions"

Linear Probability Model

Recall that if Y takes values 1 or 0 only, with probability p and $1-p$ respectively, then $E(Y)=p$ and $Var(Y)=p(1-p)$

This also applies to conditional expectations, e.g.,

$$E(Y \mid X_1,\ldots,X_k)=\Pr(Y=1 \mid X_1,\ldots,X_k)$$

Regression of Y on $X_1,\ldots,X_k$ estimates $E(Y \mid X_1,\ldots,X_k)$ assuming $E(Y \mid X_1 \ldots,X_k)=\beta_0+\beta_1 X_1+\cdots+\beta_k X_k$

If Y is binary, then

$$\widehat{\Pr}(Y=1 \mid X_1,\ldots,X_k)=\hat{\beta}_0+\hat{\beta}_1 X_1+\cdots+\hat{\beta}_k X_k$$

"Linear Probability Model (LPM)"

Logit / Probit Models

Difficulty of LPM is that predicted probabilities may be below 0 or above 1

Logit and Probit models solve this issue by assuming

$$\Pr(Y = 1 \mid X_1, \ldots, X_k) = F(\beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k)$$

where $F(.)$ is a Cumulative Distribution Function

- F is a function such that $\lim_{x \to -\infty} F(z) = 0$ and $\lim_{x \to +\infty} F(z) = 1$
- F is sometimes called the "link" function

Logit / Probit Models

Then

$$\widehat{\Pr}(Y = 1 \mid X_1, \dots, X_k) = F(\hat{\beta}_0 + \hat{\beta}_1 X_1 + \cdots + \hat{\beta}_k X_k) \in (0, 1)$$

Logit: $F(z)$ is the CDF of the *logistic distribution* $\Lambda(z) = \dfrac{e^Z}{1 + e^Z}$, i.e.

$$\Pr(Y = 1 \mid X_1, \dots, X_k) = \frac{e^{\beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k}}{1 + e^{\beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k}}$$

Probit: $F(z)$ is the CDF of the *standard normal distribution* $\Phi(z)$, i.e.

$$\Pr(Y = 1 \mid X_1, \dots, X_k) = \int_{-\infty}^{Z} (2\pi)^{-1/2} \exp(-u^2/2)\, du$$

where $Z = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k$

Logit / Probit Models

A justification for this approach

Suppose $Y^* = \beta_0 + \beta_1 X_1 + \epsilon$, $Y = 1$ if $Y^* > 0$, where ϵ has cdf $F(.)$, then

$$\begin{aligned}
\Pr(Y = 1 \mid X_1) &= \Pr(Y^* > 0 \mid X_1) \\
&= \Pr(\beta_0 + \beta_1 X_1 + \epsilon > 0 \mid X_1) \\
&= \Pr(\epsilon > -\beta_0 - \beta_1 X_1 \mid X_1) \\
&= 1 - \Pr(\epsilon \leq -\beta_0 - \beta_1 X_1 \mid X_1) \\
&= 1 - F(-\beta_0 - \beta_1 X_1 \mid X_1) \\
&= F(\beta_0 + \beta_1 X_1 \mid X_1)
\end{aligned}$$

if pdf is symmetric

Logit / Probit Models

Estimation by Maximum Likelihood (maximized numerically)

$$\mathcal{L}(\beta) = \prod_{i=1}^{n} F(Z_i)^{Y_i}(1 - F(Z_i))^{1-Y_i} \text{ where } Z_i = \beta_0 + \beta_1 X_{i1} + \dots \beta_k X_{ik}$$

$$\text{or} \quad \ln \mathcal{L}(\beta) = \sum_{i=1}^{n} Y_i \ln F(Z_i) + \sum_{i=1}^{n} (1 - Y_i) \ln(1 - F(Z_i))$$

Together with coefficients, often report AIC, SIC, and Pseudo-$R^2 = 1 - \dfrac{\ln \mathcal{L}_{ur}}{\ln \mathcal{L}_0}$

- note that $\ln \mathcal{L}_0 \leq \ln \mathcal{L}_{ur} \leq 0$
- $\ln \mathcal{L}_0$ is log-Likelihood from model with intercept only

LPM, Logit, Probit Example

Dataset: mroz

inlf = in labor force
nwifeinc = non-wife income
educ = years of education

```
library(stargazer)
library(wooldridge)
library(sandwich)
library(lmtest)

data(mroz)
probit_mdl0 <- glm(inlf ~ 1, data=mroz, family=binomial(link="probit"))
probit_mdl1 <- glm(inlf ~ educ, data=mroz, family=binomial(link="probit"))
probit_mdl2 <- glm(inlf ~ nwifeinc + educ + age + kidslt6 + kidsge6, data=mroz,
                   family=binomial(link="probit"))
stargazer(probit_mdl0, probit_mdl1, probit_mdl2,
  type="text",
  add.lines = list(
    c("Pseudo R-sqr",
      round(1 - logLik(probit_mdl0)[1]/logLik(probit_mdl0)[1], 3),
      round(1 - logLik(probit_mdl1)[1]/logLik(probit_mdl0)[1], 3),
      round(1 - logLik(probit_mdl2)[1]/logLik(logit_mdl0)[1], 3))),
  align=TRUE, no.space=TRUE)
```

```
==================================================
                      Dependent variable:
                  --------------------------------
                              inlf
                     (1)        (2)        (3)
--------------------------------------------------
nwifeinc                                -0.021***
                                         (0.005)
educ                         0.108***   0.156***
                             (0.021)    (0.024)
age                                     -0.034***
                                         (0.008)
kidslt6                                 -0.892***
                                         (0.115)
kidsge6                                  -0.038
                                         (0.041)
Constant          0.172***  -1.148***    0.422
                  (0.046)    (0.261)    (0.472)
--------------------------------------------------
Pseudo R-sqr         0        0.026      0.118
Observations        753        753        753
Log Likelihood    -514.873   -501.302   -454.225
Akaike Inf. Crit. 1,031.746 1,006.604   920.451
==================================================
Note:                  *p<0.1; **p<0.05; ***p<0.01
```

Linear Probability Model and Logit / Probit Models

For LPM: $\dfrac{\partial \Pr(Y = 1 \mid X_1, \dots, X_k)}{\partial X_k} = \beta_k$

For Logit / Probit models

$$\begin{aligned}\frac{\partial \Pr(Y = 1 \mid X_1, \dots, X_k)}{\partial X_k} &= \beta_k \frac{\partial F(\beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k)}{\partial X_k} \\ &= \beta_k f(\beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k)\end{aligned}$$

where f is the p.d.f. corresponding to F

- This is the main difference between LPM and Logit/Probit (apart from the fact that LPM can give you probability estimates less than zero or greater than one)
- Care must be taken when comparing LPM / Logit / Probit coefficient estimates

LPM, Logit, Probit Example

Usually compare β_k from LPM with

- Partial Effect at the Average: $\widehat{\beta}_k f(\widehat{\beta}_0 + \widehat{\beta}_1 \overline{X_1} + \cdots + \widehat{\beta}_k \overline{X_k})$
- Average Partial Effect: $\widehat{\beta}_k \left[\frac{1}{n} \sum_{i=1}^{n} f(\widehat{\beta}_0 + \widehat{\beta}_1 X_{i1} + \cdots + \widehat{\beta}_k X_{ik}) \right]$

from Logit / Probit

If $f(\widehat{\beta}_0 + \widehat{\beta}_1 \overline{X_1} + \cdots + \widehat{\beta}_k \overline{X_k})$ or $\frac{1}{n} f(\widehat{\beta}_0 + \widehat{\beta}_1 X_{i1} + \cdots + \widehat{\beta}_k X_{ik})$ is unavailable, can use $\lambda(0) \approx 0.25$ for logit or $\phi(0) \approx 0.4$ for probit

LPM, Logit, Probit Example

```
# Estimate Models
lpm_mdl <- lm(inlf ~ nwifeinc + educ + age + kidslt6 + kidsge6, data=mroz)  # LPM Model
robust_se_lpm <- sqrt(diag(vcovHC(lpm_mdl, type="HC0")))  # Always use HC Standard Errors for LPM
probit_mdl <- glm(inlf ~ nwifeinc + educ + age + kidslt6 + kidsge6, data=mroz, family=binomial(link="probit")) # Probit
logit_mdl <- glm(inlf ~ nwifeinc + educ + age + kidslt6 + kidsge6, data=mroz, family=binomial(link="logit"))   # Logit

# Show LPM, Probit and Logit Output
stargazer(lpm_mdl, probit_mdl, logit_mdl, type="text", se = list(robust_se_lpm, NULL, NULL),
          omit.stat = "all")

# Show APE for Probit and Logit
cat("\n APE Probit")
margins(probit_mdl) %>% summary()
cat("\n APE Logit")
margins(logit_mdl) %>% summary()

# Show PEA for Probit and Logit
at_list <- list(nwifeinc=mean(mroz$nwifeinc), educ=mean(mroz$educ),
  age=mean(mroz$age), kidslt6=mean(mroz$kidslt6), kidsge6=mean(mroz$kidsge6))
cat("\n PEA Probit")
margins(probit_mdl, at = at_list) %>% summary()
cat("\n PEA Logit")
margins(logit_mdl, at = at_list) %>% summary()
```

LPM, Logit, Probit Example

	Dependent variable: inlf		
	OLS	probit	logistic
	(1)	(2)	(3)
nwifeinc	-0.007***	-0.021***	-0.035***
	(0.002)	(0.005)	(0.008)
educ	0.052***	0.156***	0.258***
	(0.007)	(0.024)	(0.041)
age	-0.012***	-0.034***	-0.058***
	(0.002)	(0.008)	(0.013)
kidslt6	-0.297***	-0.892***	-1.484***
	(0.033)	(0.115)	(0.198)
kidsge6	-0.012	-0.038	-0.066
	(0.014)	(0.041)	(0.068)
Constant	0.645***	0.422	0.723
	(0.155)	(0.472)	(0.789)

Very roughly, $LPM \approx 0.4 Probit \approx 0.25 Logit$

APE Probit:

factor	AME	SE	z	p	lower	upper
age	-0.0118	0.0025	-4.7315	0.0000	-0.0167	-0.0069
educ	0.0536	0.0075	7.1019	0.0000	0.0388	0.0684
kidsge6	-0.0130	0.0140	-0.9240	0.3555	-0.0404	0.0145
kidslt6	-0.3067	0.0348	-8.8029	0.0000	-0.3750	-0.2384
nwifeinc	-0.0072	0.0015	-4.6620	0.0000	-0.0102	-0.0042

APE Logit:

factor	AME	SE	z	p	lower	upper
age	-0.0120	0.0025	-4.7500	0.0000	-0.0169	-0.0070
educ	0.0537	0.0076	7.0289	0.0000	0.0388	0.0687
kidsge6	-0.0138	0.0141	-0.9799	0.3271	-0.0415	0.0138
kidslt6	-0.3093	0.0354	-8.7447	0.0000	-0.3786	-0.2400
nwifeinc	-0.0073	0.0016	-4.6454	0.0000	-0.0103	-0.0042

LPM, Logit, Probit Example

PEA Probit:

factor	nwifeinc	educ	age	kidslt6	kidsge6	AME	SE	z	p	lower	upper
age	20.1290	12.2869	42.5378	0.2377	1.3533	-0.0135	0.0030	-4.5458	0.0000	-0.0193	-0.0077
educ	20.1290	12.2869	42.5378	0.2377	1.3533	0.0611	0.0094	6.5063	0.0000	0.0427	0.0795
kidsge6	20.1290	12.2869	42.5378	0.2377	1.3533	-0.0148	0.0160	-0.9227	0.3562	-0.0462	0.0166
kidslt6	20.1290	12.2869	42.5378	0.2377	1.3533	-0.3497	0.0453	-7.7115	0.0000	-0.4386	-0.2608
nwifeinc	20.1290	12.2869	42.5378	0.2377	1.3533	-0.0082	0.0018	-4.4763	0.0000	-0.0118	-0.0046

PEA Logit:

factor	nwifeinc	educ	age	kidslt6	kidsge6	AME	SE	z	p	lower	upper
age	20.1290	12.2869	42.5378	0.2377	1.3533	-0.0141	0.0031	-4.5252	0.0000	-0.0202	-0.0080
educ	20.1290	12.2869	42.5378	0.2377	1.3533	0.0631	0.0100	6.3374	0.0000	0.0436	0.0826
kidsge6	20.1290	12.2869	42.5378	0.2377	1.3533	-0.0162	0.0166	-0.9780	0.3281	-0.0487	0.0163
kidslt6	20.1290	12.2869	42.5378	0.2377	1.3533	-0.3629	0.0486	-7.4668	0.0000	-0.4582	-0.2677
nwifeinc	20.1290	12.2869	42.5378	0.2377	1.3533	-0.0085	0.0019	-4.4203	0.0000	-0.0123	-0.0047

LPM / Logit / Probit Summary

- Linear Probability Model, Logit and Probit Models used for Binary Dependent Variables
- Fitted values are predicted probabilities $\widehat{\Pr}(Y = 1 \mid X_1, \dots, X_k)$
  - Can use for classification: $\widehat{Y} = 1$ if $\widehat{\Pr}(Y = 1 \mid X_1, \dots, X_k) \geq 0.5$
- Predictions from LPM, Logit and Probit are often similar
  - But LPM can sometimes give predicted probabilities greater than 1 or less than 0
- Partial effects from Probit / Logit depend on regressor values, partial effects from LPM are constant (main difference)
- Don't compare coefficients across LPM, Logit and Probit. Compare Average Partial Effects (APE), or Partial Effects at the Average (PEA)

Poisson Regressions

For **Count Dependent Variable** $Y = 0, 1, 2, \ldots$ where most outcomes are low integers

- No. of children, no. of patents filed in a year, etc.

Count data usually modelled using Poisson Distribution with parameter $\lambda > 0$

$$\Pr(Y = h) = \frac{\exp(-\lambda)\lambda^h}{h!}, \ h = 0, 1, 2, \ldots,$$

If $Y \sim$ Poisson, then $E(Y) = \lambda$ and $Var(Y) = \lambda$

Proof omitted, but uses the fact that $\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{h=0}^{\infty} \frac{x^h}{h!}$

Poisson Regressions

A Poisson regression assumes

$$\Pr(Y = h \mid X_1, \ldots, X_k) = \frac{\exp(-\lambda)\lambda^h}{h!}$$

where λ now is a *conditional* expectation, and specifies

$$E(Y \mid X_1 \ \ldots, X_k) = \lambda = \exp(\beta_0 + \beta_1 X_1 + \ldots \beta_k X_k)$$

Interpretations:

- $\dfrac{\partial E(Y \mid X_1, \ldots, X_k)}{\partial X_j} = \exp(\beta_0 + \beta_1 X_1 + \ldots \beta_k X_k)\beta_j$
- $\dfrac{\partial \ln E(Y \mid X_1, \ldots, X_k)}{\partial X_j} = \beta_j$

Poisson Regressions

Estimate by maximum likelihood (numerical maximization)

$$\mathcal{L} = \prod_{i=1}^{n} \frac{\exp(-\exp(\beta_0 + \beta_1 X_{i1} + \ldots \beta_k X_{ik}))\exp(\beta_0 + \beta_1 X_{i1} + \ldots \beta_k X_{ik})^{Y_i}}{Y_i!}$$

$$\ln \mathcal{L} = \sum_{i=1}^{n} Y_i(\beta_0 + \beta_1 X_{i1} + \ldots \beta_k X_{ik}) - \sum_{i=1}^{n} \exp(\beta_0 + \beta_1 X_{i1} + \ldots \beta_k X_{ik}) - \sum_{i=1}^{n} \ln(Y_i!)$$

Can calculate goodness of fit in the usual way

$$R^2 = 1 - \frac{\sum_{i=1}^{n} \hat{u}_i^2}{\sum_{i=1}^{n}(Y_i - \overline{Y})^2} \quad \text{where} \quad \hat{u}_i = Y_i - \widehat{Y_i} = Y_i - (\exp(\hat{\beta}_0 + \hat{\beta}_1 X_{i1} + \ldots \hat{\beta}_k X_{ik}))$$

Incidentally, can show that $\overline{\hat{u}} = 0$

Poisson Regressions

We can continue to use count Y in usual linear regression model

- but noise term obviously not normally distributed
- Coefficients in linear regression not directly comparable to Poisson regression

$$\text{Poisson: } \frac{\partial E(Y \mid X_1, \dots, X_k)}{\partial X_j} = \exp(\beta_0 + \beta_1 X_1 + \dots \beta_k X_k)\beta_j$$

$$\text{Linear Regression: } \frac{\partial E(Y \mid X_1, \dots, X_k)}{\partial X_j} = \beta_j$$

Can compare OLS estimates with "Average Partial Effects" from Poisson

$$(1/n)\sum_{i=1}^{n} \exp(\hat{\beta}_0^{pois} + \hat{\beta}_1 X_{i1}^{pois} + \dots + \hat{\beta}_k^{pois} X_{ik})\hat{\beta}_j^{pois} = \overline{\hat{Y}}\hat{\beta}_j^{pois}$$

Poisson Regressions

One issue with Poisson regression is the assumption that

$$E(Y \mid X_1, \ldots, X_k) = Var(Y \mid X_1, \ldots, X_k)$$

Does not hold in many applications (there is often "overdispersion")

Nonetheless, we often proceed with Poisson regression anyway (we call it "Quasi-Maximum Likelihood"), but assume

$$Var(Y \mid X_1, \ldots, X_k) = \sigma^2 E(Y \mid X_1, \ldots, X_k)$$

- $\widehat{\sigma^2} = \dfrac{1}{n-k+1} \displaystyle\sum_{i=1}^{n} \dfrac{\hat{u}_i^2}{\widehat{Y}_i}$
- Adjust estimator standard errors accordingly

Poisson Regressions (Example)

data: crime1

variables: narr86 (times arrested in 86), pcnv (proportion of prior arrests that led to conviction), qemp86 (quarters employed in 86), inc86 (legal income in 86), black (= 1 if black), hispan (=1 if hispanic)

```
data(crime1)
ols_count <- lm(narr86 ~ pcnv + qemp86 + inc86 + black + hispan, data = crime1)
robust_se_ols <- sqrt(diag(vcovHC(ols_count, type="HC0")))

pois_mle <- glm(narr86 ~ pcnv + qemp86 + inc86 + black + hispan, data = crime1, family = poisson())
pois_qmle <- glm(narr86 ~ pcnv + qemp86 + inc86 + black + hispan , data = crime1,
                 family = quasipoisson())
```

```
stargazer(ols_count, pois_mle, pois_qmle, type="text",
          se = list(robust_se_ols, NULL, NULL), omit.stat = "all", no.space = TRUE)
```

Poisson Regressions (Example)

	Dependent variable: narr86		
	OLS	Poisson	glm: quasipoisson link = log
	(1)	(2)	(3)
pcnv	-0.143***	-0.433***	-0.433***
	(0.033)	(0.085)	(0.106)
qemp86	-0.037***	-0.010	-0.010
	(0.013)	(0.029)	(0.035)
inc86	-0.002***	-0.008***	-0.008***
	(0.0002)	(0.001)	(0.001)
black	0.319***	0.643***	0.643***
	(0.058)	(0.073)	(0.091)
hispan	0.182***	0.472***	0.472***
	(0.041)	(0.074)	(0.091)
Constant	0.536***	-0.667***	-0.667***
	(0.038)	(0.064)	(0.079)

```
summary(margins(pois_mle))[,1:5]
```

factor	AME	SE	z	p
black	0.2599	0.0307	8.4588	0.0000
hispan	0.1910	0.0304	6.2946	0.0000
inc86	-0.0034	0.0004	-7.8449	0.0000
pcnv	-0.1752	0.0349	-5.0191	0.0000
qemp86	-0.0039	0.0115	-0.3385	0.7350

Corner Solutions

E.g. Y "essentially continuous" over strictly positive values, 0 with positive probability

- $Y \sim$ amount spent on alcohol
- $Y \sim$ no. of hours worked

Can use Linear Regression Model, but

- Can get negative predictions
- Estimated conditional expectation might be misleading

Tobit Model (Corner Solutions)

Tobit Model:

$$Y_i^* = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_k X_{ik} + \epsilon_i = X_{i*}\beta + \epsilon_i \, , \;\; \epsilon_i \mid X_{i*} \sim N(0, \sigma^2)$$

$$Y_i = \max\{0, Y_i^*\}$$

Then (all probabilities conditional on X_{i*}):

- $\Pr(Y_i > 0) = \Pr(Y_i^* > 0) = \Pr(\epsilon > -X_{i*}\beta) = \Pr\left(\frac{\epsilon}{\sigma} > \frac{-X_{i*}\beta}{\sigma}\right) = 1 - \Phi(X_{i*}\beta)$
- $\Pr(Y_i = 0) = \Phi(X_{i*}\beta)$

Use this to construct likelihood

Tobit Model (Corner Solutions)

Can show (proof omitted, i dropped) that

$$E(Y \mid Y > 0, X_1, \dots, X_k) = Z + \sigma \underbrace{\frac{\phi(Z/\sigma)}{\Phi(Z/\sigma)}}_{\lambda(Z/\sigma)>0\text{, non-linear in X}}$$

where $Z = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k$, $\lambda(Z/\sigma)$ is the "inverse Mills ratio", $\phi()$ and $\Phi()$ are the pdf and cdf of the standard normal distribution

$$\frac{\partial E(Y \mid Y > 0, X_1, \dots, X_k)}{\partial X_j} = \beta_j \left[1 - \lambda\left(\frac{Z}{\sigma}\right)\left\{\frac{Z}{\sigma} + \lambda\left(\frac{Z}{\sigma}\right)\right\}\right]$$

Use this if interested in population where $Y > 0$

Tobit Model (Corner Solutions)

If interested in entire population

Can show that:

$$E(Y \mid X_1, \dots, X_k) = \Phi(Z/\sigma)Z + \sigma\phi(Z/\sigma)$$

$$\frac{\partial E(Y \mid X_1, \dots, X_k)}{\partial X_j} = \beta_j \Phi(Z/\sigma)$$

Example: data `mroz` from wooldridge package

```
data(mroz)
tobit_mdl <- tobit(hours ~ nwifeinc + educ + exper + expersq + age + kidslt6 + kidsge6,
                   data=mroz, left=0) # from AER package
ols_mdl <- lm(hours ~ nwifeinc + educ + exper + expersq + age + kidslt6 + kidsge6,
                   data=mroz)
```

Tobit Model (Corner Solutions)

```
summary(tobit_mdl)
```

```

Call:
tobit(formula = hours ~ nwifeinc + educ + exper + expersq + age +
    kidslt6 + kidsge6, left = 0, data = mroz)

Observations:
         Total  Left-censored     Uncensored Right-censored
           753            325            428              0

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)  965.30528  446.43614   2.162 0.030599 *
nwifeinc      -8.81424    4.45910  -1.977 0.048077 *
educ          80.64561   21.58324   3.736 0.000187 ***
exper        131.56430   17.27939   7.614 2.66e-14 ***
expersq       -1.86416    0.53766  -3.467 0.000526 ***
age          -54.40501    7.41850  -7.334 2.24e-13 ***
kidslt6     -894.02174  111.87804  -7.991 1.34e-15 ***
kidsge6      -16.21800   38.64139  -0.420 0.674701
Log(scale)     7.02289    0.03706 189.514  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Scale: 1122
```

```
summary(ols_mdl)$coef
```

```
                Estimate  Std. Error   t value     Pr(>|t|)
(Intercept) 1330.4823997 270.7846395  4.913434 1.100937e-06
nwifeinc      -3.4466357   2.5440000 -1.354810 1.758889e-01
educ          28.7611247  12.9545874  2.220150 2.670881e-02
exper         65.6725132   9.9629829  6.591652 8.232264e-11
expersq       -0.7004939   0.3245501 -2.158354 3.121911e-02
age          -30.5116345   4.3638683 -6.991878 6.036470e-12
kidslt6     -442.0899078  58.8466046 -7.512581 1.663354e-13
kidsge6      -32.7792265  23.1762189 -1.414347 1.576777e-01
```

Tobit Model (Corner Solutions)

To compare OLS vs Tobit Coefficients

- Average Partial Effect (APE): $\beta_j \times \frac{1}{n}\sum_{i=1}^{n} \Phi(Z_i/\hat{\sigma})$
- Partial Effect at the Average (PEA): $\beta_j \times \Phi(\overline{Z}/\hat{\sigma})$

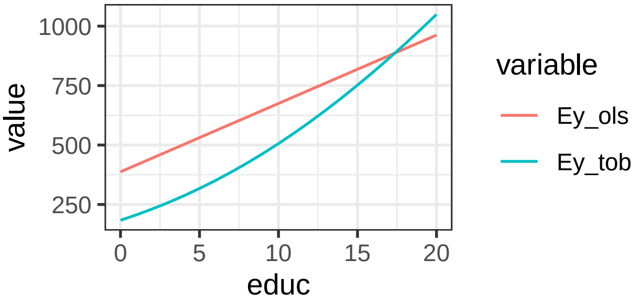
```
APE_factor = mean(pnorm(model.matrix(tobit_mdl) %*% coef(tobit_mdl) / tobit_mdl$scale))
PEA_factor = pnorm(colMeans(model.matrix(tobit_mdl)) %*% coef(tobit_mdl) / tobit_mdl$scale)
```

```
cat("APE factor:", APE_factor, "    PEA factor:", PEA_factor)
```

```
APE factor: 0.5886634    PEA factor: 0.6042994
```

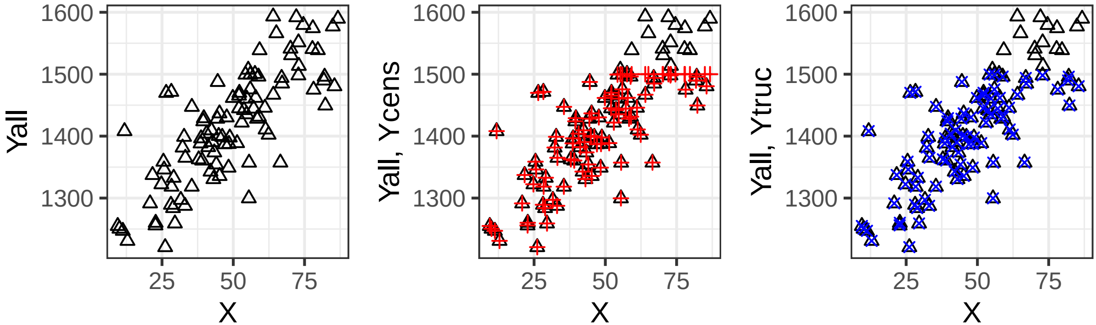
Tobit Model (Corner Solutions)

```
newdata <- data.frame(educ=0:20, nwifeinc=mean(mroz$nwifeinc), exper=mean(mroz$exper), expersq=mean(mroz$expersq),
             age=mean(mroz$age), kidslt6=mean(mroz$kidslt6), kidsge6=mean(mroz$kidsge6))
# Compare predictions Ehat(y \mid educ) between OLS and Tobit, other predictors set at average
ols_corner <- lm(hours ~ nwifeinc + educ + exper + expersq + age + kidslt6 + kidsge6, data=mroz)
Ey_ols <- predict(ols_corner, newdata)
Z <- predict(tobit_mdl, newdata)
Ey_tob <- pnorm(Z/tobit_mdl$scale)*Z + tobit_mdl$scale*dnorm(Z/tobit_mdl$scale)
plotdat <- tibble(educ = 0:20, Ey_ols=Ey_ols, Ey_tob=Ey_tob)
plotdat %>% pivot_longer(cols = c(Ey_ols, Ey_tob), names_to="variable", values_to="value") %>%
  ggplot(aes(x=educ, y=value, color=variable)) + geom_line() + theme_bw()
```



Censored and Truncated Samples

```
df <- read_csv("./data/trunc_censored.csv", show_col_types=FALSE)
p1 <- ggplot(data=df) + geom_point(aes(x=X, y=Yall), pch=2) + theme_bw()
p2 <- p1 + geom_point(aes(x=X, y=Ycens), pch=3, color="red") + ylab("Yall, Ycens")
p3 <- p1 + geom_point(aes(x=X, y=Ytruc), pch=4, color="blue") + ylab("Yall, Ytruc")
p1 | p2 | p3
```



Censored and Truncated Samples

Censored Samples:

$$Y^* = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k + \epsilon\,,\ \ \epsilon \sim N(0, \sigma^2)\,,\ \ Y = \min\{Y^*, c\}$$

Likelihood similar to Tobit

Truncated Samples

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k + \epsilon\,,\ \ \epsilon \sim \text{Truncated Normal}(0, \sigma^2)$$

Both estimate by Maximum Likelihood (Numerical Maximization)

But in both cases: $E(Y \mid X_1, \dots, X_k) = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k$

Censored and Truncated Samples

```
ols_full <- lm(Yall ~ X, data=df)
ols_cens <- lm(Ycens ~ X, data=df)
ols_truc <- lm(Ytruc ~ X, data=df)

# from AER package
tob_cens <- tobit(Ycens ~ X,
  data=df, right=1500)

# from truncreg package
nor_truc <- truncreg(Ytruc ~ X,
  data=df, point=1500,
  direction="right")
```

```
stargazer(ols_full, ols_cens, ols_truc, tob_cens, type="text",
  omit.stat = "all", no.space = TRUE)
```

```
==============================================================
                         Dependent variable:
          ----------------------------------------------------
             Yall         Ycens          Ytruc         Ycens
             OLS          OLS            OLS           Tobit
             (1)          (2)            (3)           (4)
--------------------------------------------------------------
X          3.862***     3.275***       3.134***      3.789***
           (0.290)      (0.262)        (0.329)       (0.311)
Constant 1,228.326*** 1,248.594*** 1,251.424*** 1,230.884***
          (15.144)     (13.686)       (15.668)      (15.607)
==============================================================
==============================================================
Note:                              *p<0.1; **p<0.05; ***p<0.01
```

Censored and Truncated Samples

```
summary(nor_truc)
```

```
Call:
truncreg(formula = Ytruc ~ X, data = df, point = 1500, direction = "right")

BFGS maximization method
56 iterations, 0h:0m:0s
g'(-H)^-1g = 9.96E-05



Coefficients :
              Estimate Std. Error t-value  Pr(>|t|)
(Intercept) 1212.52967   21.55490 56.2531 < 2.2e-16 ***
X              4.39835    0.55319  7.9509 1.776e-15 ***
sigma         57.75881    5.86850  9.8422 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log-Likelihood: -437.36 on 3 Df
```