Session 5: Review Exercises

AY2025/26 Term 1

Question 1: (Exercise 6.1 from Econometric Notes) Let $y = X\beta + \varepsilon\,,\; E(\varepsilon \mid X) = 0\,,\; E(\varepsilon\varepsilon^{\mathrm{T}}) = \sigma^2 I$ represent the simple linear regression

$$Y_i = \beta_0 + \beta_1 X_{i1} + \epsilon_i\,,\; i = 1, 2, ..., n\,.$$

(a) Use

$$\hat{\beta}^{ols} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$$

to find expressions for $\hat{\beta}_0$ and $\hat{\beta}_1$ and show that they are the same as those obtained in Chapter 3, i.e.,

$$\hat{\beta}_0^{ols} = \overline{Y} - \hat{\beta}_1^{ols}\overline{X}$$

$$\hat{\beta}_1^{ols} = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})X_i}{\sum_{i=1}^{n}(X_i - \overline{X})X_i} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2}\,.$$

(b) Use

$$Var(\hat{\beta} \mid X) = \begin{bmatrix} Var(\hat{\beta}_0 \mid X) & Cov(\hat{\beta}_0, \hat{\beta}_1 \mid X) \\ Cov(\hat{\beta}_0, \hat{\beta}_1 \mid X) & Var(\hat{\beta}_1 \mid X) \end{bmatrix} = \sigma^2 (X^{\mathrm{T}}X)^{-1}$$

to find expressions for $Var(\hat{\beta}_0 \mid X)$ and $Cov(\hat{\beta}_0, \hat{\beta}_1 \mid X)$ in the simple linear regression. What is the sign of $Cov(\hat{\beta}_0, \hat{\beta}_1 \mid X)$?

Remark: For intuition regarding the sign of $Cov(\hat{\beta}_0, \hat{\beta}_1 \mid X)$, consider the fact that estimated regression lines always pass through the point $(\overline{X}, \overline{Y})$.