Session 4: Review Exercises

AY2025/26 Term 1

Question 1: Prove that the inverse of a non-singular symmetric matrix is symmetric.

Question 2: Suppose X is an $n \times k$ matrix with $n > k$, and linearly independent columns (we say X has "full column rank") so that $X^{\mathrm{T}}X$ is non-singular. Prove that $(X^{\mathrm{T}}X)^{-1}$ is symmetric.

Question 3: Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad i_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Let

$$M_0 = i_n - i_n(i_n^{\mathrm{T}} i_n)^{-1} i_n^{\mathrm{T}}.$$

(a) Show that M_0 is symmetric (i.e., $M_0^{\mathrm{T}} = M_0$) and "idempotent", meaning $M_0 M_0 = M_0$.

(b) Describe the vector $M_0 x$.

(c) Explain why the sample variance of the elements of x can be written as

$$\frac{x^{\mathrm{T}} M_0 x}{n-1}$$

Question 4: Let x be an $n \times 1$ vector of random variables. In the notes, we defined

$$Var(x) = E((x - E[x])(x - E[x])^{\mathrm{T}}).$$

Show that an alternative expression for $Var(x)$ is

$$Var(x) = E(xx^{\mathrm{T}}) - E(x)E(x)^{\mathrm{T}}.$$