Session 2 Review Exercises

AY2025/26 Term 1

Question 1. (Econometric Notes Exercise 3.6) Suppose Y and X have the following joint distribution function:

Y \ X	1	2	3	4	5
10	0	0	0	0	0.1
9	0	0	0	0.1	0
8	0	0	0.1	0	0
7	0	0.1	0	0	0
6	0.1	0	0	0	0
5	0.1	0	0	0	0
4	0	0.1	0	0	0
3	0	0	0.1	0	0
2	0	0	0	0.1	0
1	0	0	0	0	0.1

i. Find the marginal distribution of X and of Y.

ii. Find the conditional distribution, conditional mean, and conditional variance of Y given X, and of X given Y.

iii. Find $Cov(X, Y)$.

iv. In what way is the conditional distribution of Y related to X?

Question 2. (Adapted from Econometric Notes Exercise 3.12) For the simple linear regression with intercept, estimated by OLS on the dataset $\{X_i, Y_i\}_{i=1}^n$, define the OLS fitted values $\{\widehat{Y}_i\}_{i=1}^n$ and residuals $\{\hat{\epsilon}_i\}_{i=1}^n$ in the usual way, so that

$$Y_i = \widehat{Y}_i + \hat{\epsilon}_i\,,\ \ i = 1, \ldots, n\,. \tag{1}$$

To simplify notation, we drop the "ols" subscripts from the residuals and fitted values.

(a) Show that $\sum_{i=1}^n Y_i^2 = \sum_{i=1}^n \widehat{Y}_i^2 + \sum_{i=1}^n \hat{\epsilon}_i^2$.

(b) Show that

$$\underbrace{\sum_{i=1}^n (Y_i - \overline{Y})^2}_{TSS} \;=\; \underbrace{\sum_{i=1}^n (\widehat{Y}_i - \overline{\widehat{Y}})^2}_{FSS} \;+\; \underbrace{\sum_{i=1}^n \hat{\epsilon}_i^2}_{RSS}\;. \tag{2}$$

Hint: Subtract $\overline{Y}$ from both sides of (1). Show that $\overline{Y} = \overline{\widehat{Y}}$. Square both sides of the equation and sum both sides from $i = 1$ to n. Show that $\sum_{i=1}^n (\widehat{Y}_i - \overline{\widehat{Y}})\hat{\epsilon}_i = 0$.

Remark: This equation can be described as "Total Sum of Squares = Fitted Sum of Squares + Residual Sum of Squares", or $TSS = FSS + RSS$. Dividing (2) throughout by $n-1$, the equation can also be described as a variance decomposition, specifically:

$$\text{sample var.}(Y_i) = \text{sample var.}(\widehat{Y}_i) + \text{sample var.}(\hat{\epsilon}_i)$$

(c) The R^2 is defined as

$$R^2 = 1 - \frac{RSS}{TSS}\,.$$

It is used as a measure of **goodness-of-fit**, since $RSS = 0$ when you have a perfect fit. Where you don't have a perfect fit, $RSS > 0$ and therefore $R^2 < 1$.

i. When will $R^2 = 0$? Can R^2 fall below zero?

ii. What is R^2 for the case where $Y_1 = \cdots = Y_n = c$ for some constant c?