ECON207 Session 9

Generalized Least Squares / Intro to Panel Data Models

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Session 9 GLS / Panel Data Models

- Generalized Least Squares
- Applications
    - Heteroskedasticity
    - Clustered Standard Errors
    - Panel Data (Random Effects Models)
- Panel Data Models (Fixed Effects Model)

Generalized Least Squares Theory

You know that in the regression model:

$$y = X\beta + \epsilon\,,\ E(\epsilon \mid X) = 0\,,\ E(\epsilon\epsilon^{\mathrm{T}} \mid X) = \sigma^2 I_n$$

where y is $n \times 1$ and X is a $n \times (k+1)$ matrix of regressors

- OLS estimator is $\hat{\beta}^{ols} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$
  - linear estimator
  - Unbiased and consistent
  - Unbiasedness and consistency requires $y = X\beta + \epsilon\,,\ E(\epsilon \mid X) = 0$

Generalized Least Squares Theory

- Variance-covariance matrix of $\hat{\beta}^{ols}$ is

$$Var(\hat{\beta}^{ols} \mid X) = \sigma^2 (X^{\mathrm{T}} X)^{-1}$$

  - Unbiased estimator for σ^2 is

$$\widehat{\sigma^2} = \frac{\hat{\epsilon}_{ols}^{\mathrm{T}} \hat{\epsilon}_{ols}}{n-k-1} \quad \text{where} \quad \hat{\epsilon}_{ols} = y - X\hat{\beta}^{ols}$$

  - Estimate of $\hat{\beta}^{ols}$ is

$$\widehat{Var}(\hat{\beta}^{ols} \mid X) = \widehat{\sigma^2} (X^{\mathrm{T}} X)^{-1}$$

  - All assumptions required

Generalized Least Squares Theory

- OLS is also *best* linear unbiased estimator
  - if $\tilde{\beta} = Ay$, $A \neq (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$, then for all non-zero $(k+1) \times 1$ vector c,

$$Var(c^{\mathrm{T}}\tilde{\beta} \mid X) \; > \; Var(c^{\mathrm{T}}\hat{\beta}^{ols} \mid X)$$

  - Equivalent description of "best"

$$Var(\tilde{\beta} \mid X) - Var(\hat{\beta}^{ols} \mid X) \text{ is positive-definite}$$

  - "best" requires all assumptions, in particular, $E(\epsilon\epsilon^{\mathrm{T}} \mid X) = \sigma^2 I_n$

Generalized Least Squares Theory

Suppose now that the noise terms ϵ are heteroskedastic or correlated (or both)

$$y = X\beta + \epsilon\,,\;\; E(\epsilon \mid X) = 0\,,\;\; E(\epsilon\epsilon^{\mathrm{T}} \mid X) = \sigma^2\Omega$$

where Ω is an $n \times n$ positive-definite matrix not equal to I_n

- Assume that σ^2 is unknown
- Assume (for the moment) that Ω is **known**

OLS estimator continues to be unbiased / consistent

$$\hat{\beta}^{ols} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}(X\beta + \epsilon) = \beta + (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}\epsilon$$

$$E(\hat{\beta}^{ols} \mid X) = \beta + (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}E(\epsilon \mid X) = \beta$$

Generalized Least Squares Theory

Variance-covariance matrix of $\hat{\beta}^{ols}$ becomes

$$Var(\hat{\beta}^{ols} \mid X) = \sigma^2 (X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} \Omega X (X^{\mathrm{T}} X)^{-1}$$

Proof:

$$\hat{\beta}^{ols} = \beta + (X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} \epsilon$$

$$\begin{aligned} Var(\hat{\beta}^{ols} \mid X) &= Var((X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} \epsilon \mid X) \\ &= (X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} Var(\epsilon \mid X) X (X^{\mathrm{T}} X)^{-1} \\ &= \sigma^2 (X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} \Omega X (X^{\mathrm{T}} X)^{-1} \end{aligned}$$

Generalized Least Squares Theory

But OLS estimator is no longer Best Linear Unbiased,

i.e., we can find another linear estimator that is more "efficient"

- Because Ω is positive definite, we can find non-singular $n \times n$ matrix P such that

$$P \Omega P^{\mathrm{T}} = I_n$$

  Since Ω is known, P is known

- Pre-multiply regression equation by P

$$Py = PX\beta + P\epsilon \quad \text{or} \quad y^* = X^*\beta + \epsilon^*$$

Generalized Least Squares Theory

- The noise term in the modified equation satisfies

$$E(\epsilon^* \mid X) = E(P\epsilon \mid X) = PE(\epsilon \mid X) = 0$$

$$E(\epsilon^* {\epsilon^*}^{\mathrm{T}} \mid X) = E(P\epsilon\epsilon^{\mathrm{T}} P^{\mathrm{T}} \mid X) = PE(\epsilon\epsilon^{\mathrm{T}} \mid X)P^{\mathrm{T}} = \sigma^2 P\Omega P^{\mathrm{T}} = \sigma^2 I_n$$

- Since $y^* = X^*\beta + \epsilon^*$ satisfies all necessary conditions for BLU Estimators
  - $\hat{\beta} = ({X^*}^{\mathrm{T}} X^*)^{-1} {X^*}^{\mathrm{T}} y^* = (X^{\mathrm{T}} P^{\mathrm{T}} P X)^{-1} X^{\mathrm{T}} P^{\mathrm{T}} P y$
  - We refer to this estimator as $\hat{\beta}^{gls}$
- $Var(\beta^{gls} \mid X) = \sigma^2 ({X^*}^{\mathrm{T}} X^*)^{-1} = \sigma^2 (X^{\mathrm{T}} P^{\mathrm{T}} P X)^{-1}$

Generalized Least Squares Theory

Since $P\Omega P^{\mathrm{T}} = I_n$ and P is non-singular, we have

$$\Omega = (P^{-1})(P^{\mathrm{T}})^{-1} \quad \text{and} \quad \Omega^{-1} = P^{\mathrm{T}}P$$

Therefore we can write the GLS estimator and its variance-covariance matrix as

$$\hat{\beta}^{gls} \;=\; (X^{\mathrm{T}}P^{\mathrm{T}}PX)^{-1}X^{\mathrm{T}}P^{\mathrm{T}}Py \;=\; (X^{\mathrm{T}}\Omega^{-1}X)^{-1}X^{\mathrm{T}}\Omega^{-1}y$$

$$Var(\hat{\beta}^{gls} \mid X) \;=\; \sigma^2(X^{\mathrm{T}}P^{\mathrm{T}}PX)^{-1} \;=\; \sigma^2(X^{\mathrm{T}}\Omega^{-1}X)^{-1}$$

An unbiased estimator for σ^2 is

$$\widehat{\sigma^2_*} = \frac{\widehat{\epsilon^*}^{\mathrm{T}}\widehat{\epsilon^*}}{n-k-1} \quad \text{where} \quad \widehat{\epsilon^*} = y^* - X^*\hat{\beta}^{gls}$$

Generalized Least Squares Theory

Remark 1 GLS is equivalent to

$$\widehat{\beta}^{gls} = \arg\min_{\widehat{\beta}} (y - X\widehat{\beta})^{\mathrm{T}} \Omega^{-1} (y - X\widehat{\beta})$$

since

$$\begin{aligned}
\widehat{\beta}^{gls} &= \arg\min_{\widehat{\beta}} (y^* - X^*\widehat{\beta})^{\mathrm{T}} (y^* - X^*\widehat{\beta}) = \arg\min_{\widehat{\beta}} (Py - PX\widehat{\beta})^{\mathrm{T}} (Py - PX\widehat{\beta}) \\
&= \arg\min_{\widehat{\beta}} (P(y - X\widehat{\beta}))^{\mathrm{T}} P(y - X\widehat{\beta}) = \arg\min_{\widehat{\beta}} (y - X\widehat{\beta})^{\mathrm{T}} P^{\mathrm{T}} P (y - X\widehat{\beta}) \\
&= \arg\min_{\widehat{\beta}} (y - X\widehat{\beta})^{\mathrm{T}} \Omega^{-1} (y - X\widehat{\beta})
\end{aligned}$$

Generalized Least Squares Theory

Remark 2

- Since $\hat{\beta}_{gls}$ is OLS on $y^* = X^*\beta + \epsilon^*$ which satisfies the "Gauss-Markov" assumptions, we know that $\hat{\beta}_{gls}$ is BLUE
- Therefore $\hat{\beta}_{gls}$ is more efficient than $\hat{\beta}_{ols}$, since $\hat{\beta}_{ols}$ is OLS on $y = X\beta + \epsilon$ which does not satisfy Gauss-Markov assumptions

The above argument is sufficient to show that

$$Var(\hat{\beta}^{ols} \mid X) - Var(\hat{\beta}^{gls} \mid X) \quad \text{is positive semi-definite}$$

This fact can also be shown directly (exercise)

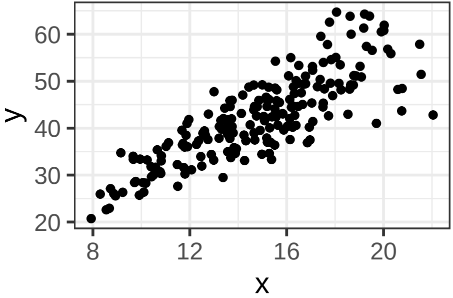
Generalized Least Squares (Heteroskedasticity Example)

Example: A simple heteroskedasticity example

Suppose for all $i, j = 1, \dots, n$

- $Y_i = \beta_0 + \beta_1 X_{i1} + \epsilon_i$
- $E(\epsilon_i \mid X_{11}, \dots, X_{n1}) = 0$
- $E(\epsilon_i^2 \mid X_{11}, \dots, X_{n1}) = \sigma_i^2 = \sigma^2 X_{i1}^2$
- $E(\epsilon_i \epsilon_j \mid X_{11}, \dots, X_{n1}) = 0$ for all $i \neq j$
- $\sum_{i=1}^{n} (X_{i1} - \overline{X_1})^2 > 0$

```
df_het <- read_csv(
  "data\\heterosk.csv", col_types=c("n","n","n"))
ggplot(data=df_het) +
  geom_point(aes(x=x,y=y), size=1) + theme_bw()
```



Generalized Least Squares (Heteroskedasticity Example)

Writing this example in matrix form

$$y = X\beta + \epsilon\,,\; E(\epsilon \mid X) = 0\,,\; E(\epsilon\epsilon^{\mathrm{T}} \mid X) = \sigma^2\Omega \quad \text{where} \quad \Omega = \begin{bmatrix} X_{11}^2 & 0 & 0 & \dots & 0 \\ 0 & X_{21}^2 & 0 & \dots & 0 \\ 0 & 0 & X_{31}^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & X_{n1}^2 \end{bmatrix}$$

What is P such that $P^{\mathrm{T}}P = \Omega^{-1}$? We have

$$\Omega^{-1} = \begin{bmatrix} X_{11}^{-2} & 0 & 0 & \dots & 0 \\ 0 & X_{21}^{-2} & 0 & \dots & 0 \\ 0 & 0 & X_{31}^{-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & X_{n1}^{-2} \end{bmatrix} \implies P = \begin{bmatrix} X_{11}^{-1} & 0 & 0 & \dots & 0 \\ 0 & X_{21}^{-1} & 0 & \dots & 0 \\ 0 & 0 & X_{31}^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & X_{n1}^{-1} \end{bmatrix}$$

Generalized Least Squares (Heteroskedasticity Example)

Then

$$y^* = Py = \begin{bmatrix} X_{11}^{-1} & 0 & 0 & \dots & 0 \\ 0 & X_{21}^{-1} & 0 & \dots & 0 \\ 0 & 0 & X_{31}^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & X_{n1}^{-1} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} Y_1/X_{11} \\ Y_2/X_{21} \\ Y_3/X_{31} \\ \vdots \\ Y_n/X_{n1} \end{bmatrix}$$

$$X^* = PX = \begin{bmatrix} X_{11}^{-1} & 0 & 0 & \dots & 0 \\ 0 & X_{21}^{-1} & 0 & \dots & 0 \\ 0 & 0 & X_{31}^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & X_{n1}^{-1} \end{bmatrix} \begin{bmatrix} 1 & X_{11} \\ 1 & X_{21} \\ 1 & X_{31} \\ \vdots & \vdots \\ 1 & X_{n1} \end{bmatrix} = \begin{bmatrix} 1/X_{11} & 1 \\ 1/X_{21} & 1 \\ 1/X_{31} & 1 \\ \vdots & \vdots \\ 1/X_{n1} & 1 \end{bmatrix}$$

That is, transformed regression is

$$\frac{Y_i}{X_{i1}} = \beta_0 \frac{1}{X_{i1}} + \beta_1 + \frac{\epsilon_i}{X_{i1}} \quad \text{i.e.,} \quad Y_i^* = \beta_1 + \beta_0 X_i^* \epsilon_i^* \,, \; i = 1, \dots, n \,.$$

Generalized Least Squares (Heteroskedasticity Example)

Equivalently,

$$\hat{\beta}^{gls} = \arg\min_{\hat{\beta}} (y - X\hat{\beta})^{\mathrm{T}}\Omega^{-1}(y - X\hat{\beta}) \quad = \quad \arg\min_{\hat{\beta}} \sum_{i=1}^{n} w_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{i1})^2$$

where $w_i = 1/X_{i1}^2$

This form of GLS is called "Weighted Least Squares (WLS)"

- GLS for heteroskedastic, uncorrelated errors
- Impt: Actual form of the weights w_i depends on Ω

Generalized Least Squares (Heteroskedasticity Example)

WLS using `lm()` (option `weights` refer to w_i)

```
df_het$wt <- 1/df_het$x^2
wls <- lm(y~x,data=df_het, weights=wt)
coef(summary(wls)) %>% round(3)
cat("\nR-squared: ", summary(wls)$r.squared,"\n")
```

```
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    4.826      1.407   3.431    0.001
x              2.532      0.102  24.730    0.000

R-squared:  0.755433
```

- WLS and OLS estimates are different, but both consistent
- WLS s.e. smaller (not surprising, since WLS more efficient)
- But why is WLS R^2 larger than OLS R^2? (OLS maximizes R^2)

OLS with Heteroskedasticity-Robust s.e.

```
ols <- lm(y~x, data=df_het)
coeftest(ols, vcov=vcovHC, type="HC") %>%
  round(3)
cat("R-squared: ", summary(ols)$r.squared,"\n")
```

```

t test of coefficients:

            Estimate Std. Error t value Pr(>|t|)
(Intercept)    6.286      1.864   3.372    0.001 ***
x              2.431      0.136  17.841   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'

R-squared:  0.6826877
```

Generalized Least Squares (Heteroskedasticity Example)

R^2 using `lm()` for WLS is a "Weighted R-squared":

$$\text{weighted-}R^2 = \frac{\sum_{i=1}^{n} w_i(\hat{Y}_i^{wls} - \overline{Y}_{wls})^2}{\sum_{i=1}^{n} w_i(Y_i - \overline{Y}_{wls})^2}.$$

where

- $\overline{Y}_{wls}$ is the weighted mean of $\{Y_i\}_{i=1}^{n}$,
- i.e., the WLS estimator of β_0 from the regression $Y_i = \beta_0 + \epsilon_i$ using same weights as in WLS estimator of the main equation

Generalized Least Squares (Heteroskedasticity Example)

For "Unweighted"-R^2, use

$$R^2_{wls} = 1 - \frac{\sum_{i=1}^{n} \hat{\epsilon}^2_{i,wls}}{\sum_{i=1}^{n} (Y_i - \overline{Y})^2}$$

where $\hat{\epsilon}_i^{wls} = Y_i - \hat{Y}_i^{wls} = Y_i - \hat{\beta}_0^{wls} - \hat{\beta}_1^{wls} X_i$

```
ehat <- df_het$y - coef(wls)[1] - coef(wls)[2]*df_het$x
ssr <- sum(ehat^2); sst <- sum((df_het$y - mean(df_het$y))^2)
R2 <- 1 - ssr/sst; cat("R-squared: ", R2,"\n")
```

```
R-squared:  0.6814966
```

- R^2_{wls} is slightly lower than in the OLS regression
- Expected, since OLS minimizes SSR, equivalent to maximizing R^2

Generalized Least Squares (Heteroskedasticity Example)

For more general cases, weighted least squares is not so straightforward

Heteroskedasticity specification may have parameters that need to be estimated, e.g.,

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i \ , \ \ Var(\epsilon_i|X) = \alpha_0 + \alpha_1 X_{i1}^2 + \alpha_2 X_{i2}^2$$

- To do weighted least squares, have to estimate α_0, α_1, α_2
  - Maybe estimate regression by OLS, then estimate α's by regressing

$$\hat{\epsilon}_{i,ols}^2 = \alpha_0 + \alpha_1 X_{i1}^2 + \alpha_2 X_{i2}^2 + u_i$$

  - Then use weights $w_i = (\hat{\alpha}_0 + \hat{\alpha}_1 X_{i1}^2 + \hat{\alpha}_2 X_{i2}^2)^{-2}$
  - Is form of heteroskedasticity correct? Other forms may be preferred/appropriate.

Generalized Least Squares

GLS refers to any method that involves

- Linear transformation of regression equation so that the transformed errors meet Gauss-Markov Criterion
- OLS on transformed equation

In next section, we turn to models for Panel Data

- Random Effects Estimator (GLS)
- Fixed Effects Estimator

Panel Data

Introduction to Panel Data (Longitudinal Data)

Panel Data

- Panel data comprising information on a **fixed** cross section of individual entities (people, firms, cities,...) observed repeatedly over several time periods

$$\{Y_{i,t}, X_{i,t,1}, X_{i,t,2}, \dots, X_{i,t,K}\}\ ,\ i = 1, 2, \dots, N\ ,\ t = 1, 2, \dots T$$

- i refers to the individual entities in the cross-section
- t refers to period of observation
- $X_{i,t,K}$ is period t observation of variable K for individual i

The same individuals are followed over time, e.g., $i = 1$ refers to the same individual for all time periods

Panel Data

- National Longitudinal Surveys of Labor Market Experience (NLS)
    - *Data on attitudes/behaviors/events related to schooling, employment, marriage, fertility, health, ... Follows various cohorts ('older men', 'mature women', 'young men',... annual/biennially since mid-1960s)*
- Panel Study of Income Dynamics (PSID)
    - *Data on employment, income, housing, travel, ... Follows 6000 families, 15000 individuals (and their descendants), still continuing*
- Singapore Life Panel
- German Social Economics Panel, British Household Panel Survey, ...

Panel Data

One benefit of Panel Data is in dealing with certain omitted variables. Suppose

$$
\begin{aligned}
Y_{i,t} &= \beta_0 + \beta_1 X_{i,t,1} + \cdots + \beta_K X_{i,t,K} + \alpha_i + u_{i,t} \\
&= \beta_0 + \beta_1 X_{i,t,1} + \cdots + \beta_K X_{i,t,K} + \epsilon_{i,t}\,, \; i = 1, \ldots, N\,; \; t = 1, \ldots, T
\end{aligned}
$$

where α_i is some unobserved variable or combination of variables, a.k.a. "time invariant individual effect"

- in returns to schooling application, α_i might include ability

If observations are simply pooled, then

- if α_i is correlated with included regressors, OLS estimator for β_1 will be inconsistent

Panel Data

The panel data structure allows us to remove the individual effects

For example, if we "time-demean" the sample, we get

$$
\begin{aligned}
Y_{i,t} &= \beta_0 + \beta_1 X_{i,t,1} + \cdots + \beta_K X_{i,t,K} + \alpha_i + u_{i,t} \\
\overline{Y_i} &= \beta_0 + \beta_1 \overline{X_{i,1}} + \cdots + + \beta_1 \overline{X_{i,K}} + \alpha_i + \overline{u_i} \\
Y_{i,t} - \overline{Y_i} &= \beta_1 (X_{i,t,1} - \overline{X_{i,1}}) + \cdots + \beta_K (X_{i,t,K} - \overline{X_{i,K}}) + u_{i,t} - \overline{u_i} \\
\ddot{Y}_{i,t} &= \beta_1 \ddot{X}_{i,t,1} + \cdots + \beta_K \ddot{X}_{i,t,K} + \ddot{u}_{i,t}
\end{aligned}
$$

The unobserved individual effect has been removed

OLS on the time-demeaned equation gives the "Fixed Effect Estimator"

Panel Data

Assume Large N, Small T; Balanced Panel (observed regularly, no attrition)

- Pooled OLS
- Random Effect Estimator
  - Generalized Least Squares Estimator when α_i **not** correlated with $X_{i,t,k}$
- Fixed Effects Estimator
  - Consistent estimation when α_i is correlated with $X_{i,t,k}$
  - Least Squares Dummy Variable Estimator
  - First-Difference Estimator

Pooled OLS

We begin with the simple case where α_i is **not** correlated with regressors

$$\begin{aligned} Y_{i,t} &= \beta_0 + \beta_1 X_{i,t,1} + \cdots + \beta_K X_{i,t,K} + \alpha_i + u_{i,t} \\ &= \beta_0 + \beta_1 X_{i,t,1} + \cdots + \beta_K X_{i,t,K} + \epsilon_{i,t}\,, \;\; i = 1, \ldots, N\,; \;\; t = 1, \ldots, T \end{aligned}$$

OLS on the "pooled" data set gives consistent estimator

However, observations are no longer uncorrelated across all i and t

- Cannot use default standard errors

Pooled OLS

Setting up using matrix algebra

For any given individual $i = 1, \ldots, N$, let

$$\underset{T\times 1}{y_i} = \begin{bmatrix} Y_{i,1} \\ Y_{i,2} \\ \vdots \\ Y_{i,T} \end{bmatrix}, \quad \underset{T\times(K+1)}{X_i} = \begin{bmatrix} 1 & X_{i,1,1} & \ldots & X_{i,1,K} \\ 1 & X_{i,2,1} & \ldots & X_{i,2,K} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{i,T,1} & \ldots & X_{i,T,K} \end{bmatrix}, \quad \underset{T\times 1}{\epsilon_i} = \begin{bmatrix} \epsilon_{i,1} \\ \epsilon_{i,2} \\ \vdots \\ \epsilon_{i,T} \end{bmatrix}$$

The regression for the ith individual is

$$y_i = X_i\beta + \epsilon_i$$

Pooled OLS

Stacking up the samples for each individual, we get the pooled Regression model

$$y = X\beta + \epsilon$$

where $\underset{NT\times 1}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \quad \underset{NT\times(K+1)}{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}, \quad \underset{NT\times 1}{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \end{bmatrix}$

Pooled OLS estimator:

$$\hat{\beta}^{pols} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$$

Pooled OLS

The pooled OLS estimator is consistent as long as $E(\epsilon \mid X) = 0$

However, usual variance formula invalid because $Var(\epsilon)$ does not have the form $\sigma^2 I$

In $\epsilon_{i,t} = \alpha_i + u_{i,t}$, assume α_i and $u_{i,t}$ independent random variables with

- α_i zero-mean variance σ_α^2 (think of $E(\alpha_i)$ as having been absorbed into β_0)
- $u_{i,t}$ zero-mean variance σ_u^2
  - $Var(\epsilon_{i,t}) = \sigma_\alpha^2 + \sigma_u^2$
  - $Cov(\epsilon_{i,t}, \epsilon_{i,s}) = E((\alpha_i + u_{i,t})(\alpha_i + u_{i,s})) = \sigma_\alpha^2$ for $s \neq t$
  - $Cov(\epsilon_{i,t}, \epsilon_{j,s}) = E((\alpha_i + u_{i,t})(\alpha_j + u_{j,s})) = 0$ for all $i \neq j$, all s, t

Pooled OLS

Then $\underset{(T\times T)}{Var(\epsilon_i \mid X)} = E(\epsilon_i \epsilon_i^{\mathrm{T}} \mid X)$

$$= \begin{bmatrix} \sigma_u^2 + \sigma_\alpha^2 & \sigma_\alpha^2 & \sigma_\alpha^2 & \dots & \sigma_\alpha^2 \\ \sigma_\alpha^2 & \sigma_u^2 + \sigma_\alpha^2 & \sigma_\alpha^2 & \dots & \sigma_\alpha^2 \\ \sigma_\alpha^2 & \sigma_\alpha^2 & \sigma_u^2 + \sigma_\alpha^2 & \dots & \sigma_\alpha^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \sigma_\alpha^2 & \sigma_\alpha^2 & \dots & \sigma_u^2 + \sigma_\alpha^2 \end{bmatrix} = \sigma_u^2 \underbrace{\left(I_T + \frac{\sigma_\alpha^2}{\sigma_u^2} i_T i_T^{\mathrm{T}} \right)}_{"\Omega"}$$

$$\underset{(NT\times NT)}{Var(\epsilon \mid X)} = E(\epsilon \epsilon^{\mathrm{T}} \mid X) = \begin{bmatrix} E(\epsilon_1 \epsilon_1^{\mathrm{T}}) & E(\epsilon_1 \epsilon_2^{\mathrm{T}}) & \dots & E(\epsilon_1 \epsilon_N^{\mathrm{T}}) \\ E(\epsilon_2 \epsilon_1^{\mathrm{T}}) & E(\epsilon_2 \epsilon_2^{\mathrm{T}}) & \dots & E(\epsilon_2 \epsilon_N^{\mathrm{T}}) \\ \vdots & \vdots & \ddots & \vdots \\ E(\epsilon_N \epsilon_1^{\mathrm{T}}) & E(\epsilon_N \epsilon_2^{\mathrm{T}}) & \dots & E(\epsilon_N \epsilon_N^{\mathrm{T}}) \end{bmatrix} = \sigma_u^2 \begin{bmatrix} \Omega & 0 & \dots & 0 \\ 0 & \Omega & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Omega \end{bmatrix}$$

Pooled OLS

"Panel-Robust Estimate" of $Var(\widehat{\beta}^{pols})$:

$$\widehat{Var}(\widehat{\beta}^{pols}) = \left(\sum_{i=1}^{N} X_i^{\mathrm{T}} X_i\right)^{-1} \left(\sum_{i=1}^{N} X_i^{\mathrm{T}} \widehat{\epsilon}_{pols} \widehat{\epsilon}_{pols}^{\mathrm{T}} X_i\right) \left(\sum_{i=1}^{N} X_i^{\mathrm{T}} X_i\right)^{-1}$$

where $\widehat{\epsilon}_{pols} = y - X\widehat{\beta}_{pols}$

This estimator of the variance in fact allows for $Var(\epsilon_i \mid X) = \Omega_i$

Random Effects Estimator

Pooled OLS estimator $\hat{\beta}^{pols}$ is consistent but not efficient

- Panel-Robust variance estimator $\widehat{Var}(\hat{\beta}^{pols})$ accounts for structure of $Var(\epsilon)$
- But $\hat{\beta}^{pols}$ does not make use of this structure

Generalized Least Squares

- Find P such that $Var(P\epsilon \mid X) = E(P\epsilon\epsilon^{\mathrm{T}}P^{\mathrm{T}} \mid X) = \sigma_u^2 I$
- Transform regression $Py = PX\beta + P\epsilon$ or $y^* = X^*\beta + \epsilon^*$
- Apply OLS to transformed equation

GLS estimator $\hat{\beta}^{gls} = ({X^*}^{\mathrm{T}}X^*)^{-1}{X^*}^{\mathrm{T}}y^*$

Random Effects Estimator

What is the appropriate P? Consider sample for just the ith individual:

$$y_i = X_i\beta + \epsilon_i\,,\; Var(\epsilon_i \mid X) = \sigma_u^2 \left(I_T + \frac{\sigma_\alpha^2}{\sigma_u^2} i_T i_T^{\mathrm{T}} \right) = \sigma_u^2 \Omega$$

Let

$$P = M_0 + \psi(I_T - M_0)$$

where $M_0 = I_T - i_T(i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}}$ and $\psi = \dfrac{\sigma_u}{\sqrt{\sigma_u^2 + T\sigma_\alpha^2}}$

Then $E(P\epsilon_i\epsilon_i^{\mathrm{T}}P^{\mathrm{T}}) = \sigma_u^2 I_T$

Random Effects Estimator

Proof: M_0 and $I - M_0$ are both symmetric, idempotent, and $M_0(I - M_0) = (I - M_0)M_0 = 0$

Furthermore, for any non-zero scalar ψ, we have

$$(M_0 + \psi(I_T - M_0))\left(M_0 - \frac{1}{\psi^2}(I - M_0)\right)(M_0 + \psi(I_T - M_0))^{\mathrm{T}} = \; I_T \quad \text{(exercise!)}$$

Finally, we have

$$\begin{aligned}
\Omega &= I_T + \frac{\sigma_\alpha^2}{\sigma_u^2} i_T i_T^{\mathrm{T}} = I_T + \frac{T\sigma_\alpha^2}{\sigma_u^2} i_T (i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}} \\
&= I_T - i_T(i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}} + i_T(i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}} + \frac{T\sigma_\alpha^2}{\sigma_u^2} i_T (i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}} \\
&= M_0 + \left(1 + \frac{T\sigma_\alpha^2}{\sigma_u^2}\right) i_T (i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}} \\
&= M_0 + \left(\frac{\sigma_u^2 + T\sigma_\alpha^2}{\sigma_u^2}\right) i_T (i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}} = M_0 + \frac{1}{\psi^2}(I_T - M_0) \;\; \text{where} \;\; \psi = \sqrt{\frac{\sigma_u^2}{\sigma_u^2 + T\sigma_\alpha^2}}
\end{aligned}$$

Random Effects Estimator

It follows that if $Var(\epsilon_i) = \sigma_u^2 \Omega$ where

$$\Omega = I_T + \frac{\sigma_\alpha^2}{\sigma_u^2} i_T i_T^{\mathrm{T}} = M_0 + \frac{1}{\psi^2}(I_T - M_0) \quad \text{where} \quad \psi = \sqrt{\frac{\sigma_u^2}{\sigma_u^2 + T\sigma_\alpha^2}}$$

Then

$$Var(P\epsilon_i) = P\,Var(\epsilon_i)P^T = \sigma_u^2 P \Omega P^{\mathrm{T}} = \sigma_u^2 I_T$$

Apply the transformation to $y_i = X_i\beta + \epsilon_i$ for every i and then pool the transformed regressions, which gives

$$\begin{bmatrix} Py_1 \\ Py_2 \\ \vdots \\ Py_N \end{bmatrix} = \begin{bmatrix} PX_1 \\ PX_2 \\ \vdots \\ PX_N \end{bmatrix} \beta + \begin{bmatrix} P\epsilon_1 \\ P\epsilon_2 \\ \vdots \\ P\epsilon_N \end{bmatrix} \quad \text{or} \quad y^* = X^*\beta + \epsilon^*, \; Var(\epsilon^* \mid X) = \sigma_u^2 I_{NT}$$

Random Effects Estimator

What is this transformation?

$$\begin{aligned} Py_i &= (M_0 + \psi(I_T - M_0))y_i \\ &= \left(I_T - i_T(i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}} + \psi i_T(i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}}\right) y_i \\ &= \left(I_T - (1-\psi) i_T(i_T^{\mathrm{T}} i_T)^{-1} i_T^{\mathrm{T}}\right) y_i \\ &= y_i - \lambda\, i_T\, \overline{Y_i} \end{aligned}$$

where $\lambda = 1 - \psi = 1 - \sqrt{\dfrac{\sigma_u^2}{\sigma_u^2 + T\sigma_\alpha^2}}$

Random Effects Estimator

i.e., $Py_i = y_i^* = \begin{bmatrix} Y_{i,1} - \lambda\overline{Y_i} \\ Y_{i,2} - \lambda\overline{Y_i} \\ \vdots \\ Y_{i,T} - \lambda\overline{Y_i} \end{bmatrix}$ where $\lambda = 1 - \dfrac{\sigma_u}{\sqrt{\sigma_u^2 + T\sigma_\alpha^2}}$

Same for other regressors $X_i^* = PX_i$

Note:

- To operationalize this theory, we have to estimate σ_u^2 and σ_α^2
- We can obtain these from the Pooled OLS residuals (details omitted)
- Assumptions on Ω more restrictive than in Pooled OLS with Panel-Robust S.E.

Fixed Effects Estimator

We now consider the case where α_i is correlated with regressors

$$
\begin{aligned}
Y_{i,t} &= \beta_0 + \beta_1 X_{i,t,1} + \cdots + \beta_K X_{i,t,K} + \alpha_i + u_{i,t} \\
&= \beta_0 + \beta_1 X_{i,t,1} + \cdots + \beta_K X_{i,t,K} + \epsilon_{i,t}\ , \ \ i = 1, \ldots, N\ ; \ \ t = 1, \ldots, T
\end{aligned}
$$

We have to get deal with the α_i in the $\epsilon_{i,t}$

Fixed Effects Estimator

- subtract time-averages from each cross-sectional observation

Fixed Effects Estimator

i.e., for every i, take $\ddot{y}_i = M_0 y_i$, $\ddot{X}_i = M_0 X_i$, $\ddot{u}_i = M_0 u_i$

$$\underset{(NT\times 1)}{\ddot{y}} = \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_N \end{bmatrix} \quad \underset{(NT\times K)}{\ddot{X}} = \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \vdots \\ \ddot{X}_N \end{bmatrix} \quad \underset{(NT\times 1)}{\ddot{u}} = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_N \end{bmatrix}$$

where $M_0 = I_T - i_T(i_T^{\mathrm{T}} i_T) i_T^{\mathrm{T}}$

The transformed model is

$$\ddot{y} = \ddot{X}\beta + \ddot{u}$$

Estimate by OLS

$$\hat{\beta}^{fe} = (\ddot{X}^{\mathrm{T}}\ddot{X})^{-1}\ddot{X}^{\mathrm{T}}\ddot{y}$$

Fixed Effects Estimator

Remarks:

- Also called “Within Estimator”
- Cannot include any time invariant variables such as race (will get removed along with unobserved individual effect)
- Identical to “Least Squares Dummy Variable” Models (LSDV)

$$Y_{i,t} = \theta_1 d_{1,i,t} + \theta_2 d_{2,i,t} + \cdots + \theta_N d_{N,i.t} + \beta_1 X_{1,i,t} + \ldots \beta_K X_{K,i,t} + u_{i,t}$$

  - proof omitted
  - Intuition: individual dummies capture individual time effects
- Often N is very large, so direct OLS on the LSDV model may be infeasible

Fixed Effects Estimator

Remarks (continued):

- Alternative to FE estimator: First Difference estimator

$$
\begin{aligned}
Y_{i,t} &= \beta_1 X_{1,i,t} + \ldots \beta_K X_{K,i,t} + \alpha_i + u_{i,t} \\
Y_{i,t-1} &= \beta_1 X_{1,i,t-1} + \ldots \beta_K X_{K,i,t-1} + \alpha_i + u_{i,t-1} \\
\Delta Y_{i,t} &= \beta_1 \Delta X_{1,i,t} + \ldots \beta_K \Delta X_{K,i,t} + \Delta u_{i,t}
\end{aligned}
$$

  Estimate by OLS

  - This approach also removes the unobserved individual effects
  - If $T = 2$, then FD = FE
  - If there are missing observations, FD can make many observations become unusable

Example

Data `jtrain` from `wooldridge` library

- Several firms followed over three years (87, 88, 89)
- will use
    - `grant` (whether job training grant was given to firm that year)
    - `grant_1` (whether training grant was given in previous year)
    - `lscrap` (logged scrap rates)
    - `fcode` firm id
    - `year`, `d87`, `d88`, `d89` year and year dummies

Example

```
library(plm); library(lmtest); library(sandwich); library(wooldridge)
dat <- jtrain

# Pooled OLS
pool_mdl <- plm(lscrap~d88+d89+grant+grant_1, data=dat, model="pooling",
                index=c("fcode", "year"))
summary(pool_mdl)$coefficients %>% round(4)
```

```
            Estimate Std. Error t-value Pr(>|t|)
(Intercept)   0.5974     0.2031  2.9421   0.0038
d88          -0.2394     0.3109 -0.7700   0.4424
d89          -0.4965     0.3379 -1.4693   0.1437
grant         0.2000     0.3383  0.5913   0.5552
grant_1       0.0489     0.4361  0.1122   0.9108
```

Example

```
# Pooled OLS with Panel-Robust SE
pool_Var <- vcovHC(pool_mdl, method="arellano", type="HC1", cluster="group")
coeftest(pool_mdl, vcov=pool_Var) %>% round(4)
```

```

t test of coefficients:

            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.5974     0.2184  2.7357   0.0069 **
d88          -0.2394     0.1251 -1.9135   0.0575 .
d89          -0.4965     0.2317 -2.1427   0.0337 *
grant         0.2000     0.3206  0.6239   0.5336
grant_1       0.0489     0.4691  0.1043   0.9171
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example

```
# Random Effects
re_mdl <- plm(lscrap~d88+d89+grant+grant_1, data=dat, model="random",
              index=c("fcode", "year"))
summary(re_mdl)$coefficients %>% round(4)
```

```
            Estimate Std. Error z-value Pr(>|z|)
(Intercept)   0.5974     0.2033  2.9389   0.0033
d88          -0.0935     0.1090 -0.8584   0.3907
d89          -0.2714     0.1315 -2.0643   0.0390
grant        -0.2144     0.1476 -1.4529   0.1463
grant_1      -0.3729     0.2051 -1.8182   0.0690
```

- Results very different from Pooled OLS (which doesn't account for individual fixed effects in any way

Example

Can use Panel-Robust SE after RE estimation

```
# Random Effects with Panel-Robust SE
re_Var <- vcovHC(re_mdl, method="arellano", type="HC1", cluster="group")
coeftest(re_mdl, vcov=re_Var) %>% round(4)
```

```
t test of coefficients:

            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.5974     0.2184  2.7357   0.0069 **
d88          -0.0935     0.0930 -1.0061   0.3159
d89          -0.2714     0.1865 -1.4548   0.1477
grant        -0.2144     0.1303 -1.6463   0.1017
grant_1      -0.3729     0.2659 -1.4021   0.1629
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example

```
# Fixed Effects
fe_mdl <- plm(lscrap~d88+d89+grant+grant_1, data=dat, model="within",
              index=c("fcode", "year"))
summary(fe_mdl)$coefficients %>% round(4)
```

```
        Estimate Std. Error t-value Pr(>|t|)
d88      -0.0802     0.1095 -0.7327   0.4654
d89      -0.2472     0.1332 -1.8556   0.0663
grant    -0.2523     0.1506 -1.6751   0.0969
grant_1  -0.4216     0.2102 -2.0057   0.0475
```

Example

Can also use Panel-Robust SE after FE estimation

```
# Fixed Effects with Panel-Robust SE
fe_Var <- vcovHC(fe_mdl, method="arellano", type="HC1", cluster="group")
coeftest(fe_mdl, vcov=fe_Var) %>% round(4)
```

```
t test of coefficients:

        Estimate Std. Error t value Pr(>|t|)
d88      -0.0802     0.0969 -0.8276   0.4098
d89      -0.2472     0.1949 -1.2681   0.2076
grant    -0.2523     0.1421 -1.7757   0.0787 .
grant_1  -0.4216     0.2798 -1.5067   0.1349
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example

```r
fixef(fe_mdl) %>% round(4)
```

```
 410523  410538  410563  410565  410566  410567  410577  410592  410593  410596
-2.8258  1.0794  1.8915  1.6178  1.7956 -0.5462  0.5973  3.3008  0.1091  1.9360
 410606  410626  410629  410653  410665  410685  418011  418021  418035  418045
-0.7028 -0.1176 -0.2795  0.1821 -2.9002 -1.6470  1.6986  0.3338  1.8337  0.8023
 418051  418054  418065  418076  418083  418091  418097  418107  418118  418125
-0.3945  0.5385  0.5564 -0.2259  0.9310  0.7076  0.6739 -0.1636 -0.6648 -0.2674
 418140  418163  418168  418177  418237  419198  419201  419242  419268  419272
 1.7944  2.3388 -2.6982  3.2422 -1.1806  1.9810  0.8023  1.8211  0.3174  3.2038
 419289  419297  419307  419339  419343  419357  419378  419381  419388  419409
 0.4258 -1.2542  0.1002 -0.5640  1.5580  0.1932  0.7627 -0.4722  2.2829  1.0000
 419432  419459  419482  419483
 1.7180  0.6233  1.1006  3.3144
```

Example

LSDV gives the same result as FE (only part of output shown...)

```
# LSDV
lsdv_mdl <- lm(lscrap~d88+d89+grant+grant_1+factor(fcode), data=dat)
summary(lsdv_mdl)$coefficients %>% round(4)
```

```
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)          -2.8258     0.2962 -9.5407   0.0000
d88                  -0.0802     0.1095 -0.7327   0.4654
d89                  -0.2472     0.1332 -1.8556   0.0663
grant                -0.2523     0.1506 -1.6751   0.0969
grant_1              -0.4216     0.2102 -2.0057   0.0475
factor(fcode)410538   3.9053     0.4064  9.6092   0.0000
factor(fcode)410563   4.7173     0.4064 11.6074   0.0000
factor(fcode)410565   4.4437     0.4064 10.9340   0.0000
factor(fcode)410566   4.6214     0.4064 11.3715   0.0000
factor(fcode)410567   2.2796     0.4064  5.6091   0.0000
factor(fcode)410577   3.4231     0.4064  8.4230   0.0000
factor(fcode)410592   6.1266     0.4064 15.0751   0.0000
factor(fcode)410593   2.9350     0.4064  7.2217   0.0000
factor(fcode)410596   4.7618     0.4064 11.7169   0.0000
```

Example

```r
# First difference
fd_mdl <- plm(lscrap~d88+d89+grant+grant_1, data=dat, model="fd",
              index=c("fcode", "year"))
summary(fd_mdl)$coefficients %>% round(4)
```

```
              Estimate Std. Error t-value Pr(>|t|)
(Intercept)   -0.1387      0.0752 -1.8450   0.0679
d88            0.0481      0.0627  0.7669   0.4449
grant         -0.2228      0.1307 -1.7040   0.0914
grant_1       -0.3512      0.2351 -1.4941   0.1382
```

Example

```
# First difference with Panel Robust SE
fd_Var <- vcovHC(fd_mdl, method="arellano", type="HC1", cluster="group")
coeftest(fd_mdl, vcov=fd_Var) %>% round(4)
```

```
t test of coefficients:

            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.1387     0.0770 -1.8009   0.0746 .
d88           0.0481     0.0459  1.0483   0.2969
grant        -0.2228     0.1063 -2.0957   0.0385 *
grant_1      -0.3512     0.2188 -1.6052   0.1115
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Roadmap

- (Previous) Session 1: Statistics Review
- (Previous) Session 2: Simple Linear Regression
- (Previous) Session 3: Estimator Standard Errors; Multiple Linear Regression
- (Previous) Session 4: Matrix Algebra
- (Previous) Session 5: OLS using Matrix Algebra
- (Previous) Session 6: Hypothesis Testing
- (Previous) Session 7: Prediction
- (Previous) Session 8: Instrumental Variable Regression
- *This Session 9: Panel Data Regressions*
- **Next Session 10: MLE / Limited Dependent Variable Models**
- Session 11-12: Introduction to Time Series / Time Series Regressions