ECON207 Session 9: Review Exercises

Question 1 It is shown in Session 9 slides that the variance-covariance matrices of the OLS and GLS estimators for the linear regression model

$$y = X\beta + \epsilon\,,\;\; E(\epsilon \mid X) = 0\,,\;\; Var(\epsilon \mid X) = \sigma^2\Omega$$

are

$$Var(\hat{\beta}^{ols} \mid X) = \sigma^2(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}\Omega X(X^{\mathrm{T}}X)^{-1} \text{ and } Var(\hat{\beta}^{gls} \mid X) = \sigma^2(X^{\mathrm{T}}\Omega^{-1}X)^{-1}$$

respectively. Show that

$$Var(\hat{\beta}^{ols} \mid X) - Var(\hat{\beta}^{gls} \mid X)$$

is positive-definite by showing that

$$Var(\hat{\beta}^{ols} \mid X) - Var(\hat{\beta}^{gls} \mid X) = \sigma^2 A\Omega A^{\mathrm{T}} \tag{1}$$

where $A = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}} - (X^{\mathrm{T}}\Omega^{-1}X)^{-1}X^{\mathrm{T}}\Omega^{-1}$.

Remarks:

- Simply multiply out $A\Omega A^{\mathrm{T}}$ to obtain the left-hand side of (1).
- Positive semi-definiteness follows because we can write $\Omega = (P^{-1})(P^{\mathrm{T}})^{-1}$, so

  $$A\Omega A^{\mathrm{T}} = A(P^{-1})(P^{\mathrm{T}})^{-1}A^{\mathrm{T}} = A(P^{-1})(P^{-1})^{\mathrm{T}}A^{\mathrm{T}}$$

  and so $c^{\mathrm{T}}A(P^{-1})(P^{-1})^{\mathrm{T}}A^{\mathrm{T}}c$ is a sum of squared terms which cannot be negative.