ECON207 Session 5: Review Exercises

AY2024/25 Term 1

Question 1 Consider the simple linear regression $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$, $i = 1, 2, ..., n$, which can be written $y = X\beta + \epsilon$, where

$$y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \text{ and } \epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}.$$

(a) Show that

$$X^{\mathrm{T}} X = \begin{bmatrix} n & \sum_{i=1}^{n} X_i \\ \sum_{i=1}^{n} X_i & \sum_{i=1}^{n} X_i^2 \end{bmatrix}.$$

(b) The OLS estimator for β is

$$\hat{\beta}^{ols} = \begin{bmatrix} \hat{\beta}_0^{ols} \\ \hat{\beta}_1^{ols} \end{bmatrix} = (X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} y\,.$$

Show *by evaluating the expression above* that $\hat{\beta}_0^{ols} = \overline{Y} - \hat{\beta}_1^{ols}\overline{X}$ and

$$\hat{\beta}_1^{ols} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2}\,.$$

(c) The variance-covariance matrix of $\hat{\beta}^{ols}$ is

$$Var(\hat{\beta} \mid X) = \begin{bmatrix} Var(\hat{\beta}_0 \mid X) & Cov(\hat{\beta}_0, \hat{\beta}_1 \mid X) \\ Cov(\hat{\beta}_0, \hat{\beta}_1 \mid X) & Var(\hat{\beta}_1 \mid X) \end{bmatrix} = \sigma^2 (X^{\mathrm{T}} X)^{-1}.$$

Show that

$$Var(\hat{\beta}_0 \mid X) = \frac{\sigma^2 \sum_{i=1}^{n} X_i^2}{n \sum_{i=1}^{n}(X_i - \overline{X})^2}, \quad Var(\hat{\beta}_1 \mid X) = \frac{\sigma^2}{\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

$$\text{and} \quad Cov(\hat{\beta}_0, \hat{\beta}_1 \mid X) = \frac{-\sigma^2 \overline{X}}{\sum_{i=1}^{n}(X_i - \overline{X})^2}\,.$$

(d) The sign of $Cov(\hat{\beta}_0, \hat{\beta}_1 \mid X)$ is negative if $\overline{X} > 0$ and positive if $\overline{X} < 0$. Can you give any intuition for this result?