ECON207 Session 3: Review Exercise (Replacement)

AY2024/25 Term 1

Question 1 The argument for consistency of $\hat{\beta}_1^{ols}$ in the simple linear regression model

$$Y = \beta_0 + \beta_1 X + \epsilon$$

is that

$$\hat{\beta}_1^{ols} = \beta_1 + \frac{\sum_{i=1}^{n}(X_i - \overline{X})\epsilon_i}{\sum_{i=1}^{n}(X_i - \overline{X})^2} \xrightarrow{p} \beta_1 + \frac{Cov(X, \epsilon)}{Var(X)}$$

so that $\hat{\beta}_1^{ols} \xrightarrow{p} \beta_1$ as long as $Cov(X, \epsilon) = 0$. On the other hand, the OLS estimator will converge to a value less than β_1 if $Cov(X, \epsilon) < 0$, and will converge to a value greater than β_1 if $Cov(X, \epsilon) > 0$. Similar arguments carry over to the bias of the estimator, although the arguments are easier using the consistency concept.

Suppose $Y = \beta_0 + \beta_1 X + \epsilon$ but X is only observed with error, i.e., you only observe $X^* = X + u$. Assume that the measurement error u is independent of X. Then

$$\begin{aligned} Y &= \beta_0 + \beta_1 X + \epsilon \\ &= \beta_0 + \beta_1 (X^* - u) + \epsilon \\ &= \beta_0 + \beta_1 X^* + (\epsilon - \beta_1 u) \\ &= \beta_0 + \beta_1 X^* + v \end{aligned}$$

where $v = \epsilon - \beta_1 u$. Let $\hat{\beta}_1^{ols}$ be the OLS estimator for the slope coefficient in the regression of Y_i on X_i^*.

(a) Explain informally why $\hat{\beta}_1^{ols}$ will converge to a value less than β_1 if $\beta_1 > 0$, and why it will converge to a value greater than β_1 if $\beta_1 < 0$. This is why measurement error bias in the simple linear regression model is called an attenuation[1] bias, or a bias toward zero.

(b) Will the measurement error bias ever be so bad that $\hat{\beta}_1^{ols}$ converges to a value with a different sign than β_1? Hint: Calculate $cov(X^*, v)$ and $Var(X^*)$ from $X^* = X + u$ and $v = \epsilon - \beta_1 u$, assuming that X and u are independent, with variances σ_X^2 and σ_u^2 respectively.

(c) Will there be any measurement error bias if the measurement error was on Y instead of X?

[1] Attenuation: the reduction of a force or effect or amplitude of a signal.