12. Kronecker Product and the Vec Operator

The Kronecker Product

We use partitioned matrices to prove a few results involving the Kronecker product and the Vec operator.

Suppose $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B}$ is a $p\times q$ matrix. Then the Kronecker product $\otimes$ is defined to be the $mp\times nq$ matrix

$$\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

Example If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$, then $\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} 5 & 2 & 10 & 4 & 0 & 0 \\ 1 & 3 & 2 & 6 & 0 & 0 \\ 15 & 6 & 0 & 0 & 5 & 2 \\ 3 & 9 & 0 & 0 & 1 & 3 \end{bmatrix}$

If $\mathbf{b}' = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}$, then $\mathbf{I}_2\otimes\mathbf{b}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\otimes\begin{bmatrix} 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 1 \end{bmatrix}$

Properties

1. In general $\mathbf{A}\otimes\mathbf{B} \neq \mathbf{B}\otimes\mathbf{A}$; the two products will in general not even have the same dimensions.

2. $(\mathbf{A}\otimes\mathbf{B})' = \mathbf{A}'\otimes\mathbf{B}'$

   Proof

$$(\mathbf{A}\otimes\mathbf{B})' = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} a_{11}\mathbf{B}' & a_{21}\mathbf{B}' & \cdots & a_{m1}\mathbf{B}' \\ a_{12}\mathbf{B}' & a_{22}\mathbf{B}' & \cdots & a_{m2}\mathbf{B}' \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}\mathbf{B}' & a_{2n}\mathbf{B}' & \cdots & a_{mn}\mathbf{B}' \end{bmatrix} = \mathbf{A}'\otimes\mathbf{B}'$$

3. $\mathbf{A}\otimes(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\otimes\mathbf{B}) + (\mathbf{A}\otimes\mathbf{C})$

   Proof: Left as as exercise.

4. $(\underset{m\times n}{\mathbf{A}}\otimes\underset{p\times q}{\mathbf{B}})(\underset{n\times r}{\mathbf{C}}\otimes\underset{q\times s}{\mathbf{D}}) = \mathbf{AC}\otimes\mathbf{BD}$

   Exercise: check that the dimensions of the matrices are such that all products involved exist.

   Proof:

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}\begin{bmatrix} c_{11}\mathbf{D} & c_{12}\mathbf{D} & \cdots & c_{1r}\mathbf{D} \\ c_{21}\mathbf{D} & c_{22}\mathbf{D} & \cdots & c_{2r}\mathbf{D} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}\mathbf{D} & c_{n2}\mathbf{D} & \cdots & c_{nr}\mathbf{D} \end{bmatrix}$$

   The (i,j)th block of this product is $\sum_{k=1}^{n} a_{ik}c_{kj}\mathbf{BD} = [\mathbf{AC}]_{ij}\,\mathbf{BD}$, which gives the required result.

5. $(\mathbf{A}\otimes\mathbf{B})^{-1} = \mathbf{A}^{-1}\otimes\mathbf{B}^{-1}$ where $\mathbf{A}$ and $\mathbf{B}$ are $n\times n$ and $m\times m$ invertible matrices.

   Proof: From (4), we have $(\mathbf{A}^{-1}\otimes\mathbf{B}^{-1})(\mathbf{A}\otimes\mathbf{B}) = \mathbf{I}_n\otimes\mathbf{I}_m = \mathbf{I}_{nm}$.

6. $tr(\mathbf{A}\otimes\mathbf{B}) = tr(\mathbf{A})tr(\mathbf{B})$ where $\mathbf{A}$ and $\mathbf{B}$ are square matrices.

Proof: Left as an exercise.

7. For the square matrices $\mathbf{A}_{(m\times m)}$ and $\mathbf{B}_{(n\times n)}$, we have $|\mathbf{A}\otimes\mathbf{B}| = |\mathbf{A}|^n|\mathbf{B}|^m$.

Proof: Omitted.

*Remark: We can use eigenvalues to prove this result. Specifically, if λ_A and λ_B are eigenvalues of **A** and **B** respectively, then $\lambda_A\lambda_B$ is an eigenvalue of $\mathbf{A}\otimes\mathbf{B}$. If λ_A and λ_B are eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ respectively, with associated eigenvectors $\mathbf{a}$ and $\mathbf{b}$, then we have $\mathbf{Aa} = \lambda_A\mathbf{a}$ and $\mathbf{Bb} = \lambda_B\mathbf{b}$. This means that*

$$\begin{aligned}(\mathbf{A}\otimes\mathbf{B})(\mathbf{a}\otimes\mathbf{b}) &= (\mathbf{Aa}\otimes\mathbf{Bb})\\ &= (\lambda_A\mathbf{a}\otimes\lambda_B\mathbf{b})\\ &= \lambda_A\lambda_B(\mathbf{a}\otimes\mathbf{b})\end{aligned}$$

which says that $\lambda_A\lambda_B$ is an eigenvalue of $\mathbf{A}\otimes\mathbf{B}$. *Furthermore, the determinant of a square matrix is the product of its eigenvalues. Applying this to the m eigenvalues of* **A** *and the n eigenvalues of* **B** *completes the proof.*

<u>The vec operator</u>

The vec operator is used to 'vectorize' matrices. For any $m\times n$ matrix $\mathbf{A} = (a_{ij})_{m\times n} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$, where $\mathbf{a}_i$ is the ith column of **A**, $i = 1,2,...,n$, define $vec(\mathbf{A})$ to be the $mn\times 1$ vector

$$vec(\mathbf{A}) = \begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_n\end{bmatrix}.$$

<u>Example</u>: if $\mathbf{A} = \begin{bmatrix}1 & 2 & 0\\ 3 & 0 & 1\end{bmatrix}$, then $vec(\mathbf{A}) = \begin{bmatrix}1\\3\\2\\0\\0\\1\end{bmatrix}$.

<u>Properties</u>

1. $vec(\mathbf{A}+\mathbf{B}) = vec(\mathbf{A}) + vec(\mathbf{B})$

Proof: Left as an exercise.

2. $vec(\underset{m\times n}{\mathbf{A}}\ \underset{n\times p}{\mathbf{B}}) = (\mathbf{I}_p\otimes\mathbf{A})vec(\mathbf{B}) = (\mathbf{B}'\otimes\mathbf{I}_m)vec(\mathbf{A})$

Proof: The columns of the product $\mathbf{AB}$ can be written as $\mathbf{AB} = \mathbf{A}[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p]$. Therefore,

$$vec(\mathbf{AB}) = \begin{bmatrix}\mathbf{Ab}_1\\ \mathbf{Ab}_2\\ \vdots\\ \mathbf{Ab}_p\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{A} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}\end{bmatrix}\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \vdots\\ \mathbf{b}_p\end{bmatrix} = (\mathbf{I}_p\otimes\mathbf{A})vec(\mathbf{B}).$$

Since $\mathbf{Ab}_i = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = b_{1i}\mathbf{a}_1 + b_{2i}\mathbf{a}_2 + \cdots + b_{ni}\mathbf{a}_n$, we have

$$vec(\mathbf{AB}) = \begin{bmatrix} \mathbf{Ab}_1 \\ \mathbf{Ab}_2 \\ \vdots \\ \mathbf{Ab}_p \end{bmatrix} = \begin{bmatrix} b_{11}\mathbf{I}_m & b_{21}\mathbf{I}_m & \cdots & b_{n1}\mathbf{I}_m \\ b_{12}\mathbf{I}_m & b_{22}\mathbf{I}_m & \cdots & b_{n2}\mathbf{I}_m \\ \vdots & \vdots & \ddots & \vdots \\ b_{1p}\mathbf{I}_m & b_{2p}\mathbf{I}_m & \cdots & b_{np}\mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} = (\mathbf{B}' \otimes \mathbf{I}_m) vec(\mathbf{A}).$$

3. $tr(\underset{m\times n}{\mathbf{A}}\ \underset{n\times m}{\mathbf{B}}) = vec(\mathbf{A}')' vec(\mathbf{B})$

Proof: Let $\tilde{\mathbf{a}}_i$ denote the *i*th column of $\mathbf{A}'$ (i.e., the transpose of the *i*th row of $\mathbf{A}$).

Then

$$tr[\mathbf{AB}] = \sum_{i=1}^{m} \tilde{\mathbf{a}}_i' \mathbf{b}_i = \begin{bmatrix} \tilde{\mathbf{a}}_1' & \tilde{\mathbf{a}}_2' & \cdots & \tilde{\mathbf{a}}_m' \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{bmatrix} = vec(\mathbf{A}')' vec(\mathbf{B}).$$