Mathematics for Economics: Linear Algebra **Anthony Tay**

10. A Formula for the Inverse

In this chapter, we develop a formula for the inverse of an $(n \times n)$ matrix, based on cofactors. We will not be using this formula for computing inverses – for that the elementary row operations approach is the most efficient. The objective in studying the formula for the inverse is, for us, to understand where Cramer's Rule comes from.

Recall that the (i, j)th "minor" of an $(n \times n)$ matrix $\mathbf{A}$ is the determinant of the $(n-1)\times(n-1)$ matrix after removing the i th row and j th column of $\mathbf{A}$. We will sometimes refer to this as the minor associated with the (i, j) th element of $\mathbf{A}$. The (i, j) th times $(-1)^{i+j}$ gives us the (i, j)th "cofactor" of $\mathbf{A}$, or the cofactor associated with the (i, j) th element of $\mathbf{A}$.

For example, let $\mathbf{A} = \begin{bmatrix} 3 & 5 & 6 & 7 \\ 5 & 4 & 7 & 3 \\ 1 & 2 & 9 & 10 \\ 2 & 8 & 2 & 1 \end{bmatrix}$.

$$\mathbf{A} = \begin{bmatrix} \cancel{3} & 5 & 6 & 7 \\ \boxed{\cancel{5}} & \cancel{4} & \cancel{7} & \cancel{3} \\ \cancel{1} & 2 & 9 & 10 \\ \cancel{2} & 8 & 2 & 1 \end{bmatrix}$$

The $(2,1)$ th minor is $M_{21} = \begin{vmatrix} 5 & 6 & 7 \\ 2 & 9 & 10 \\ 8 & 2 & 1 \end{vmatrix}$.

The $(2,1)$ th cofactor is $C_{21} = (-1)^{2+1} \begin{vmatrix} 5 & 6 & 7 \\ 2 & 9 & 10 \\ 8 & 2 & 1 \end{vmatrix}$.

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & \cancel{6} & 7 \\ 5 & 4 & \cancel{7} & 3 \\ \cancel{1} & \cancel{2} & \boxed{\cancel{9}} & \cancel{10} \\ 2 & 8 & \cancel{2} & 1 \end{bmatrix}$$

The $(3,3)$ th minor is $M_{33} = \begin{vmatrix} 3 & 5 & 7 \\ 5 & 4 & 3 \\ 2 & 8 & 1 \end{vmatrix}$.

The $(3,3)$ th cofactor is $C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 5 & 7 \\ 5 & 4 & 3 \\ 2 & 8 & 1 \end{vmatrix}$.

$$\mathbf{A} = \begin{bmatrix} \cancel{3} & \cancel{5} & \cancel{6} & \boxed{\cancel{7}} \\ 5 & 4 & 7 & \cancel{3} \\ 1 & 2 & 9 & \cancel{10} \\ 2 & 8 & 2 & \cancel{1} \end{bmatrix}$$

The $(1,4)$ th minor is $M_{14} = \begin{vmatrix} 5 & 4 & 7 \\ 1 & 2 & 9 \\ 2 & 8 & 2 \end{vmatrix}$.

The $(1,4)$ cofactor is $C_{14} = (-1)^{1+4} \begin{vmatrix} 5 & 4 & 7 \\ 1 & 2 & 9 \\ 2 & 8 & 2 \end{vmatrix}$.

We can collect all the cofactors of a matrix -- one cofactor for each element of the matrix – into a single cofactor matrix. For example, the cofactor matrix of $\mathbf{A}$, denoted $C(\mathbf{A})$, is the (4×4) matrix

$$C(\mathbf{A}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} & -M_{14} \\ -M_{21} & M_{22} & -M_{23} & M_{24} \\ M_{31} & -M_{32} & M_{33} & -M_{34} \\ -M_{41} & M_{42} & -M_{43} & M_{44} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} 4 & 7 & 3 \\ 2 & 9 & 10 \\ 8 & 2 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 7 & 3 \\ 1 & 9 & 10 \\ 2 & 2 & 1 \end{vmatrix} & \begin{vmatrix} 5 & 4 & 3 \\ 1 & 2 & 10 \\ 2 & 8 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 4 & 7 \\ 1 & 2 & 9 \\ 2 & 8 & 2 \end{vmatrix} \\ -\begin{vmatrix} 5 & 6 & 7 \\ 2 & 9 & 10 \\ 8 & 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 6 & 7 \\ 1 & 9 & 10 \\ 2 & 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 5 & 7 \\ 1 & 2 & 10 \\ 2 & 8 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 5 & 6 \\ 1 & 2 & 9 \\ 2 & 8 & 2 \end{vmatrix} \\ \begin{vmatrix} 5 & 6 & 7 \\ 4 & 7 & 3 \\ 8 & 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 6 & 7 \\ 5 & 7 & 3 \\ 2 & 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 5 & 7 \\ 5 & 4 & 3 \\ 2 & 8 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 5 & 6 \\ 5 & 4 & 7 \\ 2 & 8 & 2 \end{vmatrix} \\ -\begin{vmatrix} 5 & 6 & 7 \\ 4 & 7 & 3 \\ 2 & 9 & 10 \end{vmatrix} & \begin{vmatrix} 3 & 6 & 7 \\ 5 & 7 & 3 \\ 1 & 9 & 10 \end{vmatrix} & -\begin{vmatrix} 3 & 5 & 7 \\ 5 & 4 & 3 \\ 1 & 2 & 10 \end{vmatrix} & \begin{vmatrix} 3 & 5 & 6 \\ 5 & 4 & 7 \\ 1 & 2 & 9 \end{vmatrix} \end{bmatrix}$$

Exercise

1. Compute the cofactor matrix of the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 2 & 0 \\ 4 & 6 & 9 \end{bmatrix}$.

2. Compute the cofactor matrix of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 & 4 & 2 \\ 5 & 2 & 0 & 9 & 5 \\ 4 & 6 & 9 & 0 & 8 \\ 10 & -3 & 2 & -7 & 1 \\ 0 & 2 & 4 & 1 & 11 \end{bmatrix},$$

leaving each element as a determinant of a (4×4) matrix – don't compute them!

A Property of Cofactors

Recall the Laplace expansion for the determinant of a matrix:

$$|\mathbf{A}| = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{for any column } j$$

where we have given only the expansion along a column (all of the following statements works if we replace 'column' by 'row). That is, the determinant can be computed as the sum of the product of the cofactors of a column with the corresponding elements of that column. What happens if we were to take the sum of the product of the cofactors of a column with the corresponding elements of a <u>different</u> column? To take a specific example, consider a (4×4) matrix $\mathbf{A}$ and its corresponding cofactor matrix $C(\mathbf{A})$, and take an expansion along the second column

$$\mathbf{A} = \begin{bmatrix} a_{11} & \boxed{a_{12}} & a_{13} & a_{14} \\ a_{21} & \boxed{a_{22}} & a_{23} & a_{24} \\ a_{31} & \boxed{a_{32}} & a_{33} & a_{34} \\ a_{41} & \boxed{a_{42}} & a_{43} & a_{44} \end{bmatrix}, \qquad C(\mathbf{A}) = \begin{bmatrix} C_{11} & \boxed{C_{12}} & C_{13} & C_{14} \\ C_{21} & \boxed{C_{22}} & C_{23} & C_{24} \\ C_{31} & \boxed{C_{32}} & C_{33} & C_{34} \\ C_{41} & \boxed{C_{42}} & C_{43} & C_{44} \end{bmatrix}$$

The expansion $a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} + a_{42}C_{42}$ gives the determinant of $\mathbf{A}$. What happens if we take, for example, the sum of the column 2 cofactors multiplied by the column 1 elements:

$$a_{11}C_{12} + a_{21}C_{22} + a_{31}C_{32} + a_{41}C_{42}\,?$$

To answer this question, take the following matrix $\widetilde{\mathbf{A}}$ and the corresponding cofactor matrix

$$\widetilde{\mathbf{A}} = \begin{bmatrix} a_{11} & \boxed{a_{11}} & a_{13} & a_{14} \\ a_{21} & \boxed{a_{21}} & a_{23} & a_{24} \\ a_{31} & \boxed{a_{31}} & a_{33} & a_{34} \\ a_{41} & \boxed{a_{41}} & a_{43} & a_{44} \end{bmatrix} \quad \text{and} \quad C(\widetilde{\mathbf{A}}) = \begin{bmatrix} \widetilde{C}_{11} & \boxed{\widetilde{C}_{12}} & \widetilde{C}_{13} & \widetilde{C}_{14} \\ \widetilde{C}_{21} & \boxed{\widetilde{C}_{22}} & \widetilde{C}_{23} & \widetilde{C}_{24} \\ \widetilde{C}_{31} & \boxed{\widetilde{C}_{32}} & \widetilde{C}_{33} & \widetilde{C}_{34} \\ \widetilde{C}_{41} & \boxed{\widetilde{C}_{42}} & \widetilde{C}_{43} & \widetilde{C}_{44} \end{bmatrix}$$

and make the following observations:

(i) the determinant of $\widetilde{\mathbf{A}}$ can be computed as

$$|\widetilde{\mathbf{A}}| = a_{11}\widetilde{C}_{12} + a_{21}\widetilde{C}_{22} + a_{31}\widetilde{C}_{32} + a_{41}\widetilde{C}_{42},$$

where we have expanded along the second column;

(ii) the cofactors associated with the second column of $\widetilde{\mathbf{A}}$ are identical to the cofactors associated with the second column of $\mathbf{A}$: $C_{12} = \widetilde{C}_{12}$, $C_{22} = \widetilde{C}_{22}$, $C_{32} = \widetilde{C}_{32}$, $C_{42} = \widetilde{C}_{42}$, since the second column is removed when computing these cofactors. Therefore $|\widetilde{\mathbf{A}}| = a_{11}C_{12} + a_{21}C_{22} + a_{31}C_{32} + a_{41}C_{42}$.

(iii) The determinant of $\widetilde{\mathbf{A}}$ is zero, because it has two identical rows.

Therefore

$$a_{11}C_{12} + a_{21}C_{22} + a_{31}C_{32} + a_{41}C_{42} = 0.$$

In general

The sum of the product of the cofactors of one column and the elements of another column is zero.

Exercises

1. (a) Write down the three cofactors C_{13}, C_{23}, and C_{33} corresponding to the three elements in the third column of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 2 & 0 \\ 4 & 6 & 9 \end{bmatrix}.$$

and compute the determinant of $\mathbf{A}$ by expanding down the third column.

(b) Now find the sum of the products of the same three cofactors and the corresponding elements of a different column, e.g., compute

$$3C_{13} + 5C_{23} + 4C_{33}$$

and

$$1C_{13} + 2C_{23} + 6C_{33}$$

2. For the matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

write down the cofactors C_{21} and C_{22} corresponding to the elements of the second row. Find $cC_{21} + dC_{22}$ and $aC_{21} + bC_{22}$.

A Formula for the Inverse, and Cramer's Rule

The results from the preceding section can be used to develop a formula for the inverse. Suppose we premultiply $\mathbf{A}$ by the transpose of the cofactor matrix of $\mathbf{A}$. What we get is

$$C^{\mathrm{T}}(\mathbf{A})\,\mathbf{A} = \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ \boxed{C_{12} \quad C_{22} \quad C_{32} \quad C_{42}} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} |\mathbf{A}| & 0 & 0 & 0 \\ 0 & \boxed{|\mathbf{A}|} & 0 & 0 \\ 0 & 0 & |\mathbf{A}| & 0 \\ 0 & 0 & 0 & |\mathbf{A}| \end{bmatrix}$$

Each diagonal element in the right-most matrix is the sum of the product of the cofactors of a column of $\mathbf{A}$ and the elements of that column, and is therefore equal to the determinant. One example is marked out. (To reiterate: the cofactor matrix here has been transposed.) The other elements are the sum of the product of the cofactors of a column of $\mathbf{A}$ and the elements of a different column of $\mathbf{A}$, therefore is zero. Multiplying both sides by the reciprocal of the determinant, we get

$$\frac{1}{|\mathbf{A}|} C^{\mathrm{T}}(\mathbf{A})\mathbf{A} = \mathbf{I}$$

and therefore:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} C^{\mathrm{T}}(\mathbf{A}) = \frac{1}{|\mathbf{A}|} \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix}.$$

The transpose of the cofactor matrix is called the adjoint of $\mathbf{A}$, $adj(\mathbf{A})$. so the formula for the inverse is often written as

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj.(\mathbf{A}),$$

which is called the adjoint formula for the inverse.

Now we prove Cramer's Rule. Take the general n equations in n unknowns system of simultaneous equations.

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ &... \\ &a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{aligned}$$

and write it as $\mathbf{Ax} = \mathbf{b}$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

The solution is, using the inverse $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Writing this using the adjoint formula:

$$\begin{bmatrix} x_1 \\ \boxed{x_2} \\ \vdots \\ x_n \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Take as a specific example the solution for x_2. We have

$$x_2 = \frac{b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2}}{|\mathbf{A}|}$$

Finally, observe that the numerator of this expression is the determinant of the matrix

$$\mathbf{A}_2(\mathbf{b}) = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}$$

(recall again that in computing the cofactors of the column, that column is deleted.)

This same argument holds for any x_i, therefore we get Cramer's Rule:

$$x_i = \frac{|\mathbf{A}_i(\mathbf{b})|}{|\mathbf{A}|}, \; i = 1,\ldots,n.$$

In other words, solving a system by Cramer's Rule is exactly equivalent to solving the system by using the inverse. Cramer's Rule is simply a shortcut for implementing the inverse matrix approach.

Exercises

1. Under what condition will a system of n equations in n unknowns have a unique solution?

2. Earlier you computed the cofactor matrix and determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 2 & 0 \\ 4 & 6 & 9 \end{bmatrix}.$$

   Use the adjoint formula to compute the inverse of $\mathbf{A}$. Verify your answer by computing the product $\mathbf{A}^{-1}\mathbf{A}$.

3. Use the adjoint formula to compute the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

4. Find the inverse of matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

   Can you generalize to arbitrary diagonal matrices of dimension $(n \times n)$?