8. Properties of Determinants

A very important result, stated here without proof, is that

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$$

if **A** and **B** are square matrices that can be multiplied together.

We can use this result to relate elementary row operations to the determinants of square matrices, generating several important properties of determinants. This in turn provides a way to simplify the computation of determinants (including determinants of larger matrices).

(1) If two rows of **A** are interchanged, the sign of the determinant changes.

(2) If all the elements of a single row or column of **A** are multiplied by α, then the determinant of the new matrix is $\alpha|\mathbf{A}|$.

(3) If a multiple of a row (column) is added to another row (column), the determinant remains unchanged.

We illustrate these using examples. It should be easy for you to generalize from these examples. Throughout, we use

$$\mathbf{A} = \begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix}$$

as an example. You can verify that the determinant of this matrix is -24.

Recall that switching two rows of the matrix:

e.g. switching rows 1 and 2: $\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 3 & 3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$

is equivalent to premultiplying $\mathbf{A}$ with that matrix

$$\mathbf{E}_{(1)\leftrightarrow(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

You can verify this:

$$\mathbf{E}_{(1)\leftrightarrow(2)}\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$

The matrix $\mathbf{E}_{(1)\leftrightarrow(2)}$ is obtained by switching the 1st and 2nd rows of the (3×3) identity matrix. Using the Laplace expansion, you can easily show that

$$\left|\mathbf{E}_{(1)\leftrightarrow(2)}\right| = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

This is true in general: the determinant of the identity matrix is one; when two rows of the identity matrix are switched, the determinant of the new matrix becomes -1.

This means that switching two rows of a matrix results in a switch in the sign of its determinant.

$$|\mathbf{E}_{(1)\leftrightarrow(2)}\mathbf{A}| = |\mathbf{E}_{(1)\leftrightarrow(2)}||\mathbf{A}| = -|\mathbf{A}|$$

In our example:

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 3 & 2 \\ 1 & 5 & 2 \end{vmatrix} = \left|\mathbf{E}_{(1)\leftrightarrow(2)}\mathbf{A}\right| = \left|\mathbf{E}_{(1)\leftrightarrow(2)}\right|\left|\mathbf{A}\right| = (-1)(-24) = 24$$

In general, switching two rows of a matrix switches the sign of its determinant.

The second type of row operator is multiplying a row of a matrix by some number:

e.g. multiply row 1 of $\mathbf{A}$ by 1/3: $\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & \frac{2}{3} \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix}$

This is equivalent to premultiplying the matrix with

$$\mathbf{E}_{\frac{1}{3}\times(1)} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

You can easily verify that the determinant of this matrix is $\frac{1}{3}$. Therefore

$$\begin{vmatrix} 1 & 1 & \frac{2}{3} \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{vmatrix} = \left|\mathbf{E}_{\frac{1}{3}\times(1)}\mathbf{A}\right| = \left|\mathbf{E}_{\frac{1}{3}\times(1)}\right|\left|\mathbf{A}\right| = \left(\tfrac{1}{3}\right)(-24) = -8$$

Multiplying one row of a matrix by α multiplies the determinant by α.

Finally, the third type of elementary row operator is to add/subtract a constant times one row to another row:

e.g. subtract half of row 2 of $\mathbf{A}$ from row 3: $\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 4.5 & 0.5 \end{bmatrix}$

This is equivalent to premultiplying the matrix with

$$\mathbf{E}_{(3)-\frac{1}{2}(2)\rightarrow(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

obtained by subtracting half of the second row of the identity matrix from row 3 of the identity matrix. It is obvious that the determinant of this matrix is one. Therefore

$$\begin{vmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 4.5 & 0.5 \end{vmatrix} = \left|\mathbf{E}_{(3)-\frac{1}{2}(2)\rightarrow(3)}\mathbf{A}\right| = \left|\mathbf{E}_{(3)-\frac{1}{2}(2)\rightarrow(3)}\right|\left|\mathbf{A}\right| = (1)\left|\mathbf{A}\right| = -24$$

Adding/subtracting a constant times one row of a matrix to another row of the matrix does not change its determinant.

Since the determinant of a triangular matrix is simply the product of the elements in the diagonal, these results imply that to compute a determinant, we can reduce a matrix to a triangular matrix, compute the determinant of that, and reverse the effects of the row operations used.

	Row Operation	gives	Effect on det.
$\begin{matrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{matrix}$	switch rows 1 and 3, ↵	$\begin{matrix} 1 & 5 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 2 \end{matrix}$	$[\times(-1)]$
	row 2 minus 2 times row 1, ↵	$\begin{matrix} 1 & 5 & 2 \\ 0 & -9 & -1 \\ 3 & 3 & 2 \end{matrix}$	[*no change*]
	row 3 minus 3 times row 1, ↵	$\begin{matrix} 1 & 5 & 2 \\ 0 & -9 & -1 \\ 0 & -12 & -4 \end{matrix}$	[*no change*]

Row Operation	gives	Effect on det.
row 2 minus row 3, ↵	$\begin{matrix} 1 & 5 & 2 \\ 0 & 3 & 3 \\ 0 & -12 & -4 \end{matrix}$	[*no change*]
row 3 plus 4 times row 2, ↵	$\begin{matrix} 1 & 5 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 8 \end{matrix}$	[*no change*]

Determinant of this last triangular matrix is $(1\times3\times8) = 24$.

Looking back at the row operations used, we find we switch the sign once. Therefore, the determinant of the original matrix is

$$\begin{vmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{vmatrix} = -24.$$

Exercises

1. The following matrices are derived from the matrix

$$\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix}$$

using elementary row operations. Find their determinants.

(i) $\begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 3 & 2 \\ 4 & 2 & 6 \\ 1 & 5 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 5 & 13 & 6 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix}$

2. It is true generally that $|\mathbf{A}| = |\mathbf{A}'|$. Verify this for (2×2) and (3×3) matrices.

3. Find the determinant of the matrix

$$\mathbf{D} = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & -1 & 1 & 11 \\ 1 & -1 & 7 & 3 \\ -2 & 0 & -1 & 3 \end{bmatrix}$$

by first reducing to a triangular matrix.