Mathematics for Economics: Linear Algebra **Anthony Tay**

7. The Laplace Expansion

In the last section, we defined the determinant of a (3×3) matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

to be

$$|\mathbf{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

In this section, we introduce a general formula for computing determinants. Rewriting

$$\begin{aligned} |\mathbf{A}| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

note that the terms outside the brackets are the terms along the first row of the matrix

$$\begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad a_{13}} \\ a_{21} \quad a_{22} \quad a_{23} \\ a_{31} \quad a_{32} \quad a_{33} \end{bmatrix}$$

The term in brackets associated with a_{11} is the determinant of the (2×2) matrix after deleting the 1st row and 1st column of $\mathbf{A}$:

$$(a_{22}a_{33} - a_{23}a_{32}) = \begin{vmatrix} \boxed{\cancel{a_{11}}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

The term in brackets associated with a_{12} is the determinant of the (2×2) matrix after deleting the 1st row and 2nd column of $\mathbf{A}$:

$$(a_{21}a_{33} - a_{23}a_{31}) = \begin{vmatrix} \cancel{a_{11}} & \boxed{\cancel{a_{12}}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}$$

The term in brackets associated with a_{13} is the determinant of the (2×2) matrix after deleting the 1st row and 3nd column of $\mathbf{A}$:

$$(a_{21}a_{32} - a_{22}a_{31}) = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \boxed{\cancel{a_{13}}} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{vmatrix}.$$

There is the matter of the minus sign in front of a_{12}. This can be achieved by multiplying into each term $(-1)^{i+j}$. Therefore, we can write

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{11}\underbrace{(-1)^{1+1}\underbrace{\begin{vmatrix} \boxed{\cancel{a_{11}}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix}}_{(1,1)\text{th Minor of } \mathbf{A}}}_{(1,1)\text{th Cofactor of } \mathbf{A}} + a_{12}\underbrace{(-1)^{1+2}\underbrace{\begin{vmatrix} \cancel{a_{11}} & \boxed{\cancel{a_{12}}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}}_{(1,2)\text{th Minor of } \mathbf{A}}}_{(1,2)\text{th Cofactor of } \mathbf{A}} + a_{13}\underbrace{(-1)^{1+3}\underbrace{\begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \boxed{\cancel{a_{13}}} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{vmatrix}}_{(1,3)\text{th Minor of } \mathbf{A}}}_{(1,3)\text{th Cofactor of } \mathbf{A}}$$

Some names have been introduced in the formula above: the determinant of the matrix after removing the i th row and j th column is called the (i,j)th "minor" of $\mathbf{A}$; including the sign $(-1)^{i+j}$ gives us the (i,j)th "cofactor" of $\mathbf{A}$.

Example The determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 3 \\ -3 & 3 & 2 \end{bmatrix}$$

is

$$|\mathbf{A}| = (0)(-1)^{1+1}\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} + (2)(-1)^{1+2}\begin{vmatrix} 1 & 3 \\ -3 & 2 \end{vmatrix} + (-1)(-1)^{1+3}\begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} = -28;$$

The cofactor expansion for computing determinants is not unique. For instance, we could have written the original formula as

$$\begin{aligned} |\mathbf{A}| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &= -a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ &= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{23}(a_{11}a_{32} - a_{12}a_{31}) \end{aligned}$$

which can be written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{21}\underbrace{(-1)^{2+1}\underbrace{\begin{vmatrix} \cancel{a_{11}} & a_{12} & a_{13} \\ \boxed{\cancel{a_{21}}} & \cancel{a_{22}} & \cancel{a_{23}} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix}}_{(2,1)\text{th Minor of } \mathbf{A}}}_{(2,1)\text{th Cofactor of } \mathbf{A}} + a_{22}\underbrace{(-1)^{2+2}\underbrace{\begin{vmatrix} a_{11} & \cancel{a_{12}} & a_{13} \\ \cancel{a_{21}} & \boxed{\cancel{a_{22}}} & \cancel{a_{23}} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}}_{(2,2)\text{th Minor of } \mathbf{A}}}_{(2,2)\text{th Cofactor of } \mathbf{A}} + a_{23}\underbrace{(-1)^{2+3}\underbrace{\begin{vmatrix} a_{11} & a_{12} & \cancel{a_{13}} \\ \cancel{a_{21}} & \cancel{a_{22}} & \boxed{\cancel{a_{23}}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{vmatrix}}_{(2,3)\text{th Minor of } \mathbf{A}}}_{(2,3)\text{th Cofactor of } \mathbf{A}}$$

Alternatively, we could have expanded along a column:

$$\begin{aligned} |\mathbf{A}| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &= a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} \\ &= -a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{32}(a_{11}a_{23} - a_{13}a_{21}) \end{aligned}$$

in which case, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$\underbrace{a_{12}(-1)^{1+2} \underbrace{\begin{vmatrix} \cancel{a_{11}} & \boxed{\cancel{a_{12}}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}}_{(1,2)\text{th Minor of } \mathbf{A}}}_{(1,2)\text{th Cofactor of } \mathbf{A}} + \underbrace{a_{22}(-1)^{2+2} \underbrace{\begin{vmatrix} a_{11} & \cancel{a_{12}} & a_{13} \\ \cancel{a_{21}} & \boxed{\cancel{a_{22}}} & \cancel{a_{23}} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}}_{(2,2)\text{th Minor of } \mathbf{A}}}_{(2,2)\text{th Cofactor of } \mathbf{A}} + \underbrace{a_{32}(-1)^{3+2} \underbrace{\begin{vmatrix} a_{11} & \cancel{a_{12}} & a_{13} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ \cancel{a_{31}} & \boxed{\cancel{a_{32}}} & \cancel{a_{33}} \end{vmatrix}}_{(3,2)\text{th Minor of } \mathbf{A}}}_{(3,2)\text{th Cofactor of } \mathbf{A}}$$

Note that we achieved the correct signs by multiplying the (i, j)th minor with $(-1)^{i+j}$.

Expanding along any row or column would in fact give us the same expression; we can write

$$|\mathbf{A}| = \sum_{j=1}^{3} a_{ij}(-1)^{i+j} M_{ij} \quad \text{for any row } i$$

$$\text{or} \quad |\mathbf{A}| = \sum_{i=1}^{3} a_{ij}(-1)^{i+j} M_{ij} \quad \text{for any column } j.$$

where M_{ij} is the (i, j)th minor of A. This formula is known as the Laplace Expansion.

The fact that you can expand along any row or column can simplify computations substantially if there is a row or column with many zeros.

Example Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 11 \\ 2 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

using the Laplace Expansion (i) expanding along the first row, (ii) expanding down the second column.

$$\text{(i)} \quad \begin{vmatrix} 1 & 0 & 11 \\ 2 & 0 & 3 \\ 5 & 4 & 6 \end{vmatrix} = (1)(-1)^{1+1}\begin{vmatrix} 0 & 3 \\ 4 & 6 \end{vmatrix} + (0)(-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (11)(-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 5 & 4 \end{vmatrix} = -12 + 0 + 88 = 76$$

$$\text{(ii)} \quad \begin{vmatrix} 1 & 0 & 11 \\ 2 & 0 & 3 \\ 5 & 4 & 6 \end{vmatrix} = \cancel{(0)(-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}} + \cancel{(0)(-1)^{2+2}\begin{vmatrix} 1 & 11 \\ 5 & 6 \end{vmatrix}} + (4)(-1)^{3+2}\begin{vmatrix} 1 & 11 \\ 2 & 3 \end{vmatrix} = 76$$

(n×n) Determinants

Cramer's Rule and the Laplace Expansion extend to larger systems. The determinant for a general $(n \times n)$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} M_{ij} \quad \text{for any row } i$$

$$\text{or} \qquad |\mathbf{A}| = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} M_{ij} \quad \text{for any column } j$$

where the (i, j)th minor M_{ij} of an $n \times n$ matrix **A** is the determinant of the $(n-1)\times(n-1)$ matrix that remains after removing the ith row and jth column of **A**.

Example

The determinant of $\mathbf{D} = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 1 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{bmatrix}$ is

$$|\mathbf{D}| = \underbrace{3(-1)^{1+3}\begin{vmatrix} 0 & -1 & 11 \\ 1 & -1 & 3 \\ -2 & 0 & 3 \end{vmatrix}}_{=\,-39} + \underbrace{0(-1)^{2+3}\begin{vmatrix} 2 & 2 & 4 \\ 1 & -1 & 3 \\ -2 & 0 & 3 \end{vmatrix}}_{=\,0} \cdots$$

$$+ \underbrace{0(-1)^{3+3}\begin{vmatrix} 2 & 2 & 4 \\ 0 & -1 & 11 \\ -2 & 0 & 3 \end{vmatrix}}_{=\,0} + \underbrace{(-1)(-1)^{4+3}\begin{vmatrix} 2 & 2 & 4 \\ 0 & -1 & 11 \\ 1 & -1 & 3 \end{vmatrix}}_{=\,42} = 3$$

where we have expanded down the third column.

The Laplace expansion includes the (2×2) case. Define the determinant of a single number as $|a| = a$. Note that in this context the symbol $|\bullet|$ does not refer to absolute values, e.g. $|-2| = -2$, not 2. Then taking, say, a first row expansion, we have

$$|\mathbf{A}| = \sum_{j=1}^{2} a_{1j}(-1)^{1+j} M_{1j} = a_{11}(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12} = a_{11}a_{22} - a_{12}a_{21}.$$

Cramer's Rule also extends to general $(n \times n)$ systems of equations. The solution to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

is $$x_i = \frac{|\mathbf{A}_i(\mathbf{b})|}{|\mathbf{A}|}, \quad i = 1, \ldots, n,$$

where $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and $\mathbf{A}_i(\mathbf{b})$ is the matrix $\mathbf{A}$ with its ith column replaced by $\mathbf{b}$.

Exercises

1. Find the determinants of the following matrices using the Laplace expansion.

(i) $\begin{bmatrix} 4 & 0 & 1 \\ 19 & 1 & -3 \\ 7 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$

(vi) $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (v) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ (vi) $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

(vii) $\begin{bmatrix} 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \\ a_{41} & 0 & 0 \end{bmatrix}$ (viii) $\begin{bmatrix} 4 & 3 & 2 \\ 6 & 4.5 & 3 \\ 7 & 1 & 0 \end{bmatrix}$ (ix) $\begin{bmatrix} 4 & 3 & 0 \\ 9 & 1 & 3 \\ 7 & 1 & 0 \end{bmatrix}$

2. Use Cramer's Rule to solve

$$\begin{aligned} 4x \quad\quad + z &= 4 \\ 19x + y - 3z &= 3 \\ 7x + y \quad\quad &= 1 \end{aligned}$$

3. Solve the following system of equations

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 + 4x_4 &= 2 \\ - x_2 \quad + 11x_4 &= 3 \\ x_1 - x_2 \quad + 3x_4 &= 1 \\ -2x_1 \quad - x_3 + 3x_4 &= 2 \end{aligned}$$

4. Find the determinants using the Laplace expansion

(i) $\begin{bmatrix} a_{11} & 0 & 0 & a_{41} \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ (ii) $\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ (iii) $\begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ 0 & a_{22} & a_{32} & a_{42} \\ 0 & 0 & a_{33} & a_{43} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$

(iv) $\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

5. Let $\mathbf{A}$ be an arbitrary $(n \times n)$ matrix. Show that if we multiply every element of a single row or column by c, then the determinant of the new matrix is $c\,|\mathbf{A}|$. What is the determinant of $c\mathbf{A}$?

6. Let $\mathbf{E}$ be the matrix obtained by switching the 1st and last rows of an $(n \times n)$ identity matrix, i.e.

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

(a) Show using the Laplace expansion that $\det \mathbf{E} = -1$;

(b) Does your answer depend on whether n is even or odd?

(c) Use your result to prove that the determinant of the matrix formed by switching *any* two rows of the identity matrix is -1.

7. (a) Let

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}$$

and suppose $\mathbf{A}$ is some $(3 \times n)$ matrix. Describe the rows of the product $\mathbf{EA}$ in terms of the rows of $\mathbf{A}$; What is the determinant of the matrix $\mathbf{E}$?

(b) Let $\mathbf{E}$ be a matrix that carries out the elementary row operation of subtracting a multiple of one row from another row. What is the determinant of $\mathbf{E}$?

8. Let $\mathbf{A} = \begin{bmatrix} 2 & 4 & -1 & 2 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Find $|\mathbf{A}^{-1}|$.