6. An Introduction to Determinants and Cramer's Rule

In this section, you are introduced to a formula for solving systems of simultaneous equations, called *Cramer's Rule.* We begin with solutions to systems with two equations in two unknowns and three equations in three unknowns. The formulas extend to the general n equations in n unknowns case but not derived; the derivation will be given in a later section. The key ingredient of the formula for solving systems of equations is the *determinant* of a matrix.

Cramer's Rule for (2×2) *Matrices*

Consider a system of two simultaneous equations in two unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

which can be written as $\mathbf{Ax}=\mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \neq \mathbf{0}.$$

We have seen two (intimately related) ways to solve system of equations. One way is by elimination; another is by computing the inverse of $\mathbf{A}^{-1}$ and then finding $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$. The elimination method for computing the inverse of $\mathbf{A}$ clearly shows the connection between the two methods. In this section, we consider yet another method, centered on the concept of a determinant (which as you will see later, is also closely connected to the inverse of $\mathbf{A}$).

By solving the system in the usual way, we can show that the general solution to the system is

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}.$$

Some observations:

- The denominators of both x_1 and x_2 are the same and are made up from only the elements of the coefficient matrix $\mathbf{A}$. We call the expression in the denominator the 'determinant' of $\mathbf{A}$, denoted by

  $$|\mathbf{A}| = a_{11}a_{22} - a_{21}a_{12}$$

  Determinants are also sometimes denoted as

  $$\det(\mathbf{A}), \text{ or } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

- If the denominator is zero, the system will not have a unique solution, since we cannot divide by zero (the system will have either no solution, or an infinite number of solutions). The system will have a unique solution only if $|\mathbf{A}| \neq 0$.

- Observe that the numerator of x_1 can be expressed as the determinant of the matrix $\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$, which is just the matrix $\mathbf{A}$ with its first column replaced by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Likewise, the numerator of x_2 is the determinant of the matrix $\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$, which is the matrix $\mathbf{A}$ with its second column replaced by $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

In other words, the solutions can be written in terms of determinants as

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad \text{and} \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

This pattern extends to larger systems of equations (it is called 'Cramer's Rule'). Furthermore, by using the properties of 'Determinants', we can (i) find ways of helping us compute solutions of larger systems more easily, and (ii) we can often say a lot regarding the characteristics of solutions of systems of equations, without actually solving them. More on all this later. For now, memorize the formula for the determinant of a (2×2) matrix.

One special case should be considered separately. Consider the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned}$$

Such a system is called a homogenous system. If the determinant of the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is non-zero, then there is a unique solution $x_1 = 0$ and $x_2 = 0$. This is sometimes referred to as the trivial solution (these are just two lines that intersect at the origin). Non-trivial solutions will exist only if the determinant of the coefficient matrix is zero. However, Cramer's rule will not give you the solutions in this case, since it results in a "0/0" result).

Cramer's Rule for (3×3) *Matrices*

The general 3 equation 3 unknown system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

takes a bit more work to solve, but with a little bit of patience you can show that the solution is

$$x_1 = \frac{b_1 a_{22} a_{33} + a_{12} a_{23} b_3 + a_{13} b_2 a_{32} - a_{13} a_{22} b_3 - b_1 a_{23} a_{32} - a_{12} b_2 a_{33}}{a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}}$$

$$x_2 = \frac{a_{11} b_2 a_{33} + b_1 a_{23} a_{31} + a_{13} a_{21} b_3 - a_{13} b_2 a_{31} - a_{11} a_{23} b_3 - b_1 a_{21} a_{33}}{a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}}$$

$$x_3 = \frac{a_{11} a_{22} b_3 + a_{12} b_2 a_{31} + b_1 a_{21} a_{32} - b_1 a_{22} a_{31} - a_{11} b_2 a_{32} - a_{12} a_{21} b_3}{a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}}$$

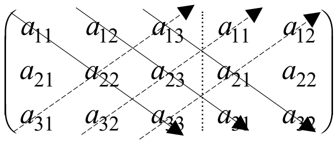
You wouldn't want to memorize this, at least not in this form. However, observe again that the denominators of all three are the same, and composed only of the coefficients (the a 's). Again, the system will have a unique solution only if the expression in the denominators does not equal zero; if it equals zero, then the system will either have infinitely many solutions, or none. The expression in the denominator again determines whether or not the system will have a unique solution, so we will collect the coefficients into the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and define the determinant of this (3×3) matrix to be

$$|\mathbf{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

This formula is also not easily memorized. There is a shortcut that is sometimes useful: extending the matrix to include the first two columns.



The formula is then easily remembered in terms of the "diagonals" where the terms under the solid arrows are added, whereas the terms under the dashed arrows are subtract.

Next, observe that the numerators of x_1, x_2, and x_3 can then be written as

$$\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

respectively, i.e. we can write

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

We will save ourselves some tedium by using $\mathbf{A}_i(\mathbf{b})$ to represent the matrix $\mathbf{A}$ with the ith column replaced by $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, thus writing

$$x_1 = \frac{|\mathbf{A}_1(\mathbf{b})|}{|\mathbf{A}|}, \; x_2 = \frac{|\mathbf{A}_2(\mathbf{b})|}{|\mathbf{A}|}, \text{ and } x_3 = \frac{|\mathbf{A}_3(\mathbf{b})|}{|\mathbf{A}|}.$$

Example Use Cramer's Rule to solve

$$\begin{aligned} 2y - z &= -7 \\ x + y + 3z &= 2 \\ -3x + 3y + 2z &= 0 \end{aligned}$$

The coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 3 \\ -3 & 3 & 2 \end{bmatrix}$$

Using the 'shortcut' rule for computing determinants, we have

$$\left(\begin{array}{ccc:cc} 0 & 2 & -1 & 0 & 2 \\ 1 & 1 & 3 & 1 & 1 \\ -3 & 3 & 2 & -3 & 3 \end{array}\right)$$

which gives $|\mathbf{A}| = 0 + (-18) + (-3) - 3 - 0 - 4 = -28$.

Also, $\mathbf{A}_1(\mathbf{b}) = \begin{bmatrix} -7 & 2 & -1 \\ 2 & 1 & 3 \\ 0 & 3 & 2 \end{bmatrix}$, and the determinant of this matrix is

$$\left(\begin{array}{ccc:cc} -7 & 2 & -1 & -7 & 2 \\ 2 & 1 & 3 & 2 & 1 \\ 0 & 3 & 2 & 0 & 3 \end{array}\right)$$

which gives $|\mathbf{A}_1(\mathbf{b})| = (-14) + 0 + (-6) - 0 - (-63) - 8 = 35$, so the solution for x is

$$x = -\frac{35}{28}$$

Similarly, the matrices $\mathbf{A}_2(\mathbf{b})$ and $\mathbf{A}_3(\mathbf{b})$ are

$$\mathbf{A}_2(\mathbf{b}) = \begin{bmatrix} 0 & -7 & -1 \\ 1 & 2 & 3 \\ -3 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_3(\mathbf{b}) = \begin{bmatrix} 0 & 2 & -7 \\ 1 & 1 & 2 \\ -3 & 3 & 0 \end{bmatrix}$$

with determinants $|\mathbf{A}_2(\mathbf{b})| = 71$ and $|\mathbf{A}_3(\mathbf{b})| = -54$, so the full solution is

$$x = -\frac{35}{28}, \quad y = -\frac{71}{28}, \quad \text{and} \quad z = \frac{54}{28} = \frac{27}{14}.$$

The visual trick for computing (3×3) matrices does not extend to larger square matrices. In the next section, we will look at a different formula for the determinant, which will apply generally.

Exercises

1. Find the determinants of the following matrices

(i) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ (ix) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (x) $\begin{bmatrix} 7 & 1 \\ 2 & 4 \end{bmatrix}$

2. Solve the following systems of equations using Cramer's Rule

(i) $\begin{matrix} 2x_1 - x_2 = 4 \\ x_1 + 2x_2 = 2 \end{matrix}$ (ii) $\begin{matrix} 3x_1 + 5x_2 = 6 \\ 6x_1 + 10x_2 = 12 \end{matrix}$ (iii) $\begin{matrix} 3x_1 + 5x_2 = 6 \\ 6x_1 + 10x_2 = 10 \end{matrix}$

(iv) $\begin{matrix} 2y - x = 4 + a \\ y + 2x = 2 \end{matrix}$ (v) $\begin{matrix} 2x_1 - 3x_2 = 0 \\ x_1 + 2x_2 = 0 \end{matrix}$ (vi) $\begin{matrix} 3x_1 + 5x_2 = 0 \\ 6x_1 + 10x_2 = 0 \end{matrix}$

For the systems that do not have a unique solution, find out whether it has zero or infinitely many solutions.

3. Solve, using Cramer's rule, the following system of equations for C and Y (take everything else as fixed)

$$C = a + bY, \; 0 < b < 1$$
$$Y = C + G.$$

4. Suppose we have the following demand and supply equation

$$Q^d = \alpha_0 + \alpha_1 P, \; \alpha_1 < 0$$
$$Q^s = \beta_0 + \beta_1 P + \beta_2 R, \; \beta_1 > 0, \; \beta_2 < 0$$

where Q^d and Q^s are the quantities demanded (by consumers) and supplied (by firms). Suppose that in equilibrium, P is the market price of the good, and R is rainfall. In equilibrium, we have

$$Q^d = Q^s \; (= Q) \; .$$

Using Cramer's Rule, solve this system of equations for the equilibrium price P and quantity Q, treating everything else as fixed. What happens to equilibrium price and quantity when rainfall R increases? How would your answer compare with the case when demand is completely inelastic (i.e. $\alpha_1 = 0$)?

5. Find the determinants of the following matrices

(i) $\begin{bmatrix} 4 & 0 & 1 \\ 19 & 1 & -3 \\ 7 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix}$

6. Use Cramer's Rule to solve

$$\begin{aligned} 4x \quad\quad\;\; + z &= 4 \\ 19x + y - 3z &= 3 \\ 7x + y \quad\quad\;\; &= 1 \end{aligned}$$

7 Let $\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. Prove that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$

This is a very useful result, and holds for the general case, not only for (2×2) *case: if* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $(n\times n)$ *matrices, then* $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$. *Note that it is essential, however, for both* $\mathbf{A}$ *and* $\mathbf{B}$ *to be square matrices of the size dimension (why?)*

8. Find the determinants of

(i) $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ (iii) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

9. Find the determinants of

(i) $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{bmatrix}$

10. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \\ 0 & -3 & -1 \end{bmatrix}$.

(a) Show that the third equation can be written as a linear combination of the first two rows (i.e., you can find c_1 and c_2 such that

$$c_1\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + c_2\begin{bmatrix} 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -3 & -1 \end{bmatrix}.$$

(b) Show that $\det \mathbf{A} = 0$.