4. Systems of Equations

Solving economic models often requires solving systems of equations. Here we discuss solving systems of **linear** equations, meaning that the equations in the system take the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = c \ .$$

We look at several examples of 2- and 3-variable systems, carefully chosen to illustrate possible outcomes. The solution method is by "elimination". Follow this method strictly even if it seems unnecessarily pedantic, as we will refer to various aspects of it when discussing more advanced topics.

Two equations in two unknowns

The following is an example of a system of two equations in two unknowns:

$$\begin{array}{ll} [1] & a_{11}x + a_{12}y = b_1 \\ [2] & a_{21}x + a_{22}y = b_2 \end{array} \tag{4.1}$$

Note the way we write the system of equations: the order of the variables on the LHS is the same for all variables. The constants are placed on the right. For the moment, we exclude situations where $b_1 = b_2 = 0$ .We will deal with such situations later.

An equation such as $2x + y = 4$ represents a line in the *x*-*y* space. A system of two equations such as

$$\begin{array}{ll} [1] & 2x + \ y = 4 \\ [2] & 5x - 2y = 1 \end{array}$$

represents two lines in the *x*-*y*. With two lines, there are three possibilities: the two lines intersect, the two lines are parallel, or the two lines coincide. Solving a system of equations means finding a set of points that satisfies all the equations in the system. If the two lines intersect, then the solution is a single point, namely the point of intersection. If the two lines are parallel, then there is no solution. If the two lines coincide, then there is an infinite number of solutions – every point on the line is a solution.

We can modify a system such as (4.1) without changing its solutions. For instance, the solution to (4.1) will not change if we swapped equations [1] and [2] and write the system as

$$\begin{array}{ll} [1] & a_{21}x + a_{22}y = b_2 \\ [2] & a_{11}x + a_{12}y = b_1 \end{array}$$

The solution will not change if we multiple one equation by a non-zero constant: (4.1) and the following:

$$\begin{array}{ll} [1] & 2a_{11}x + 2a_{12}y = 2b_1 \\ [2] & a_{21}x + a_{22}y = b_2 \end{array}$$

will obviously have the same solutions. Modifying one equation by adding to it a multiple of another equation to it will also not change the solution. If a point (x_0, y_0) is a solution to a system of equations such as (4.1), then that point satisfies both equations, then it must also satisfy

$$(a_{11}x + a_{12}y) + c(a_{21}x + a_{22}y) = b_1 + cb_2 \ .$$

If a solution satisfies $(a_{11}x + a_{12}y) + c(a_{21}x + a_{22}y) = b_1 + cb_2$ and $(a_{21}x + a_{22}y) = b_2$, then the solution will also satisfy $(a_{11}x + a_{12}y) = b_1$.

We use these facts to find solutions to systems of equations.

Example 1

$$\begin{aligned} &[1] \quad 2x + y = 4 \\ &[2] \quad 5x - 2y = 1 \end{aligned} \tag{4.2}$$

To solve (4.2), do the following:

Call the coefficient on the first variable (x) in the first equation is called the first pivot. This coefficient must be non-zero (we deal with the zero case later). Use this pivot to eliminate x in the second equation by subtracting 5/2 times the first equation from the second equation, which changes the system to:

$$\begin{matrix} [1] & \boxed{2}x + y = 4 \\ [2] & 5x - 2y = 1 \end{matrix} \xrightarrow[{[2]=[2]-\frac{5}{2}[1]}]{} \begin{matrix} \boxed{2}x + & y = 4 \\ & \boxed{-4.5}y = -9 \end{matrix} \tag{4.3}$$

The coefficient -4.5 on y in the new second equation is called the second pivot (it must be non-zero to be called a pivot). Next multiply the [2] by $-1/4.5$

$$\begin{matrix} \boxed{2}x + & y = 4 \\ & \boxed{-4.5}y = -9 \end{matrix} \xrightarrow[{[2]=(-1/4.5)[2]}]{} \begin{matrix} \boxed{2}x + & y = 4 \\ & \boxed{1}y = 2 \end{matrix} \tag{4.4}$$

Eliminate y from [1] by subtracting [2] from [1]:

$$\begin{matrix} \boxed{2}x + & y = 4 \\ & \boxed{1}y = 2 \end{matrix} \xrightarrow[{[1]=[1]-[2]}]{} \begin{matrix} \boxed{2}x & = 2 \\ & \boxed{1}y = 2 \end{matrix} \tag{4.5}$$

Finally multiply [1] by 1/2:

$$\begin{matrix} \boxed{2}x & = 2 \\ & \boxed{1}y = 2 \end{matrix} \xrightarrow[{[1]=(1/2)[1]}]{} \begin{matrix} \boxed{1}x & = 1 \\ & \boxed{1}y = 2 \end{matrix} \tag{4.6}$$

The solution is $(x, y) = (1, 2)$.

The following way of writing the calculations above is tidier: the system (4.2) can be written as

$$\mathbf{Ax} = \mathbf{b} \tag{4.7}$$

where $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Write down the "augmented matrix" $[\mathbf{A} \,\vdots\, \mathbf{b}]$

$$\left[\begin{array}{cc:c} 2 & 1 & 4 \\ 5 & -2 & 1 \end{array}\right] \tag{4.8}$$

The elementary row operations reduces (4.8) to

$$\left[\begin{array}{cc:c} \boxed{2} & 1 & 4 \\ 0 & \boxed{1} & 2 \end{array}\right] \tag{4.9}$$

and finally to

$$\left[\begin{array}{cc:c} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \end{array}\right] \tag{4.10}$$

The left side of (4.9) is called the **row echelon form** (REF) of **A**, and the left side of (4.10) is called the **reduced row echelon form** (RREF) of **A**. A matrix is said to be in REF if (i) the "leading entry", i.e., the first non-zero element, or "pivot", of each row lies to the right of the pivots of the rows above it, and (ii) all zero rows are at the bottom of the matrix. A matrix is in RREF if it is in REF and (iii) all pivots are 1, and (iv) all other elements in each pivot's column are zero. The elimination method is called Gauss-Jordan elimination.

Example 2 Consider now the system

$$\begin{array}{ll} [1] & 2x+y=4 \\ [2] & 2x+y=3 \end{array} \tag{4.11}$$

It should be clear that the two equations are parallel so there will not be a solution. What happens if we follow the method above?

The first pivot is 2. We use this pivot to eliminate *x* in the second equation by subtracting 2/2 times the first equation from the second equation, which changes the system to:

$$\begin{array}{ll} [1] & 2x+\ y=4 \\ [2] & 2x+\ y=3 \end{array} \xrightarrow[{[2]=[2]-[1]}]{} \begin{array}{l} 2x\ +\ y\ =\ 4 \\ \quad\ \ 0y\ =1 \end{array} \tag{4.12}$$

There is no pivot in the second equation (the coefficient on *y* is zero and therefore does not count as a pivot). The second "equation" also produces a contradiction. This is what happens when there is no solution in a system of two equations in two unknowns. There is only one pivot, and the elimination process produces a nonsensical 'equation'.
In the 'augmented matrix notation:

$$\left[\begin{array}{cc|c} 2 & 1 & 4 \\ 2 & 1 & 3 \end{array}\right] \tag{4.13}$$

is reduced to

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} \boxed{2} & 1 \\ 0 & 0 \end{matrix}}_{\text{REF}} & \begin{matrix} 4 \\ 1 \end{matrix} \end{array}\right] \tag{4.14}$$

We can further reduce to

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} \boxed{1} & 0.5 \\ 0 & 0 \end{matrix}}_{\text{RREF}} & \begin{matrix} 2 \\ 1 \end{matrix} \end{array}\right] \tag{4.15}$$

but there is no point in doing so.

Example 3 Next consider the system

$$\begin{array}{ll} [1] & 2x+\ \ y=4 \\ [2] & 4x+2y=8 \end{array} \tag{4.16}$$

It should be obvious that the two equations represent the same line (the second equation is simply twice that of the first). We apply the elimination method here: the first pivot is 2. We use this pivot to eliminate *x* in the second equation by subtracting 4/2 times the first equation from the second equation, which changes the system to:

$$\begin{array}{ll} [1] & 2x+\ y=4 \\ [2] & 4x+2y=8 \end{array} \xrightarrow[{[2]=[2]-2[1]}]{} \begin{array}{l} 2x\ +\ y\ =\ 4 \\ \quad\ \ 0y\ =0 \end{array} \tag{4.17}$$

There is no pivot in the second equation. This is what happens when two equations coincide. There is only one pivot, and the second equation produces $0=0$. In this case there will be an infinite number of solutions. We write the (infinite number of) solutions in this way. The variable *y* is free. Let $y=s$, then from $2x+y=4$ we get $x=(4-s)/2$. The solutions are

$$(x,y)=\left(\tfrac{1}{2}(4-s),s\right) \text{ for } s\in(-\infty,\infty).$$

In the ‘augmented matrix notation:

$$\left[\begin{array}{cc:c} 2 & 1 & 4 \\ 4 & 2 & 8 \end{array}\right] \tag{4.18}$$

is reduced to

$$\left[\begin{array}{cc:c} \boxed{2} & 1 & 4 \\ \underbrace{0 \quad 0}_{\text{REF}} & & 0 \end{array}\right] \tag{4.19}$$

is reduced to

$$\left[\begin{array}{cc:c} \boxed{1} & 0.5 & 2 \\ \underbrace{0 \quad 0}_{\text{RREF}} & & 0 \end{array}\right] \tag{4.20}$$

Example 4 We consider a fourth case: suppose

$$\begin{aligned} &[1] \quad & 2y &= 4 \\ &[2] \quad 4x + & y &= 6 \end{aligned} \tag{4.21}$$

The coefficient on x in the first equation is zero, so there is no first pivot. The coefficient on y does not count. What do we do here? Swap equations to get

$$\begin{aligned} &[1] \quad 2y = 4 \\ &[2] \quad 4x + y = 6 \end{aligned} \xrightarrow[{[2] \leftrightarrow [1]}]{} \begin{aligned} 4x + y &= 6 \\ 2y &= 4 \end{aligned} \tag{4.22}$$

Then we see immediately that there are two pivots, and without further elimination we have $(x, y) = (1, 2)$.

The three operations: swapping equations, multiplying an equation by some number, and adding (or subtracting) a multiple of one equation to/from another, are called elementary row operations. These three elementary row operations are all you need to find solutions to systems of equations (whether there are many, one or none).

Three equations in two variables What if there are three equations in two variables? Here we have all three possibilities. Consider

Example 5

$$\begin{aligned} &[1] \quad & x + y &= 2 \\ &[2] \quad & x - y &= 0 \\ &[3] \quad & -x + 2y &= 2 \end{aligned} \tag{4.23}$$

Solving [1] and [2] gives $(x, y) = (1, 1)$, but this is inconsistent with [3] since $-1 + 2(1) = 1 \neq 2$. Solving [1] and [3] gives $(x, y) = \left(\frac{2}{3}, \frac{4}{3}\right)$, but this solution is inconsistent with [2] since $\frac{2}{3} - \frac{4}{3} = -\frac{2}{3} \neq 0$. Solving [2] and [3] gives $(x, y) = (2, 2)$, which is inconsistent with [1] since $2 + 2 = 4 \neq 2$. This system has no solution. This problem here is that there are three lines. For there to be a solution, all three lines must intersect at a single point. The three equations in (4.23) do not.

The elimination procedure proceeds as follows: the first pivot is 1. Using this, we subtract [1] from [2], and add [1] to [3]

$$\begin{array}{lr} [1] & x+\ y=2 \\ [2] & x-\ y=0 \\ [3] & -x+2y=2 \end{array} \xrightarrow[\substack{[2]=[2]-[1] \\ [3]=[3]+[1]}]{} \begin{array}{r} x+y=2 \\ -2y=-2 \\ 3y=4 \end{array} \tag{4.24}$$

The second pivot is -2. Using this to eliminate the y in the third equation gives

$$\begin{array}{lr} [1] & x+y=2 \\ [2] & -2y=-2 \\ [3] & 3y=4 \end{array} \xrightarrow[{[3]=[3]+\frac{3}{2}[1]}]{} \begin{array}{r} x+y=2 \\ -2y=-2 \\ \text{"}0=1\text{"} \end{array} \tag{4.25}$$

The last equation reduces to a contradiction. In the 'augmented matrix notation, where we continue the reduction to RREF:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \\ -1 & 2 & 2 \end{array}\right] \xrightarrow[\substack{[2]=[2]-[1] \\ [3]=[3]+[1]}]{} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 3 & 4 \end{array}\right] \xrightarrow[{[3]=[3]+\frac{3}{2}[1]}]{} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \\ \underbrace{0 \quad 0}_{\text{REF}} & & 1 \end{array}\right] \xrightarrow[{[2]=(-1/2)[2]}]{} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{[1]=[1]-[2]}]{} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \underbrace{0 \quad 0}_{\text{RREF}} & & 1 \end{array}\right]$$

Since there is no third variable, there was never going to be a "third pivot". The last equation would have either reduced to a contradiction, or to "$0=0$" as in the following example.

<u>Example 6</u> Suppose

$$\begin{array}{lr} [1] & x+\ y=2 \\ [2] & x-\ y=0 \\ [3] & -x+2y=1 \end{array} \tag{4.26}$$

Repeating the elimination procedure gives

$$\begin{array}{lr} [1] & x+\ y=2 \\ [2] & x-\ y=0 \\ [3] & -x+2y=1 \end{array} \xrightarrow[\substack{[2]=[2]-[1] \\ [3]=[3]+[1]}]{} \begin{array}{r} x+y=2 \\ -2y=-2 \\ 3y=3 \end{array} \xrightarrow[{[3]=[3]+\frac{3}{2}[1]}]{} \begin{array}{r} x+y=2 \\ -2y=-2 \\ \text{"}0=0\text{"} \end{array} \tag{4.27}$$

The third equation can be ignored. Then [1] and [2] gives $(x,y)=(1,1)$. This is a system where the three equations do in fact all intersect at a single point. Another way of seeing this is that although there are three equations, the three equations are dependent in that you can express one as a combination of the other two: you can verify that

$$\frac{1}{2}[1]-\frac{3}{2}[2] \;=\; [3].$$

The third equation is just a linear combination of the first two. A solution to any two of the equations must also solve the third.

Example 7 Another example: suppose the system is

$$\begin{aligned} &[1] \quad x + y = 2 \\ &[2] \quad 2x + 2y = 4 \\ &[3] \quad 3x + 3y = 6 \end{aligned} \tag{4.28}$$

Obviously all three equations coincide. Elimination yields

$$\begin{aligned} &[1] \quad x + y = 2 \\ &[2] \quad 2x + 2y = 4 \\ &[3] \quad 3x + 3y = 6 \end{aligned} \xrightarrow[{[3]=[3]-3[1]}]{{[2]=[2]-2[1]}} \begin{aligned} x + y &= 2 \\ 0 &= 0 \\ 0 &= 0 \end{aligned} \tag{4.29}$$

The solutions are $(x, y) = (2 - s, s)$ for all s. In terms of the augmented matrix notation, we have

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{array}\right] \xrightarrow[{[3]=[3]+3[1]}]{{[2]=[2]-2[1]}} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \\ \underbrace{0 \quad 0}_{\text{RREF}} & & 0 \end{array}\right]$$

Notice that there is only one pivot in the system.

Three variable systems We now turn to systems with two equations in three unknowns. A linear equation involving three variables x, y, and z represents a plane in the 3-dimensional space. This visualization helps you to understand what can happen in linear systems with three variables.

If you have two planes, then either the planes are parallel, are coincident, or intersect. If they intersect, the intersection produces a line in three-dimensional space. Therefore, two linear equations involving three variables will either have no solutions (if the two planes are parallel), or an infinite number of solutions represented by the entire plane (if the two planes coincide) or an infinite number of solutions represented by the line of intersection of the two planes.

Example 8 The following system has infinitely many solutions represented by a single line.

$$\begin{aligned} &[1] \quad 2x - 3y + z = 0 \\ &[2] \quad x + y + z = 1 \end{aligned} \tag{4.30}$$

Applying elimination: the first pivot is 2, using this to eliminate x in the second equations gives

$$\begin{aligned} &[1] \quad 2x - 3y + z = 0 \\ &[2] \quad x + y + z = 1 \end{aligned} \xrightarrow[{[2]=[2]-\frac{1}{2}[1]}]{} \begin{aligned} 2x - 3y + z &= 0 \\ \frac{5}{2}y + \frac{z}{2} &= 1 \end{aligned} \tag{4.31}$$

There is no pivot for z, which is a free variable. Setting $z = s$, we have $y = \frac{2}{5} - \frac{1}{5}s$, and $x = \frac{3}{5} - \frac{4}{5}s$. That is, the solutions can be expressed as

$$(x, y, z) = \left(\tfrac{3}{5} - \tfrac{4}{5}s, \tfrac{2}{5} - \tfrac{1}{5}s, s\right)$$

which describes a line through three-dimensional space. In the augmented matrix notation, where again we do the elimination until we reach RREF:

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[{[2]=[2]-\frac{1}{2}[1]}]{} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & \frac{5}{2} & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[{[2]=(2/5)[2]}]{} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} \end{array}\right] \xrightarrow[{[1]=(1/2)[1]}]{{[1]=[1]+3[2]}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{5} & \frac{3}{5} \\ \underbrace{0 \quad 1 \quad \tfrac{1}{5}}_{\text{RREF}} & & & \frac{2}{5} \end{array}\right]$$

It may be that the two pivots are the coefficient of the x and z equations.

Example 9 Consider

$$\begin{aligned} [1] \quad & 2x+2y+z=2 \\ [2] \quad & x+\ y+z=1 \end{aligned} \tag{4.32}$$

Applying elimination: we have

$$\begin{aligned} [1] \quad & 2x+2y+z=2 \\ [2] \quad & x+\ y+z=1 \end{aligned} \xrightarrow[{[2]=[2]-\frac{1}{2}[1]}]{} \begin{aligned} 2x+2y+z&=2 \\ \frac{z}{2}&=1 \end{aligned} \tag{4.33}$$

We have $z=2$, so the first equation gives $x=-y$. Setting $y=s$, the solutions are $(x,y,z)=(-s,s,2)$. In augmented matrix notation:

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[{[2]=[2]-\frac{1}{2}[1]}]{} \left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 0 & 0 & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[{[2]=2[2]}]{} \left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[{\substack{[1]=[1]-[2] \\ [1]=(1/2)[1]}}]{} \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\text{RREF}} & \begin{matrix} 0 \\ 2 \end{matrix} \end{array}\right]$$

Here the pivot columns are columns 1 and 3, associated with variables x and z. The variable y is a free variable, and it is associated with a non-pivot column.

If there are three equations in three unknowns, we can get a unique solution if the three equations are independent and non-conflicting. The three planes must intersect at a single point.

Example 10 Consider the system:

$$\begin{aligned} [1] \quad & \boxed{2}x+2y+4z=4 \\ [2] \quad & 3x+\ y+2z=2 \\ [3] \quad & 5x+2y+\ z=7 \end{aligned} \tag{4.34}$$

The first pivot is marked. We use this to eliminate x from [2] and [3].

$$\begin{aligned} [1] \quad & \boxed{2}x+2y+4z=4 \\ [2] \quad & 3x+\ y+2z=2 \\ [3] \quad & 5x+2y+\ z=7 \end{aligned} \xrightarrow[{\substack{[2]=[2]-\frac{3}{2}[1] \\ [3]=[3]-\frac{5}{2}[1]}}]{} \begin{aligned} \boxed{2}x+2y+4z&=4 \\ \boxed{-2}y-4z&=-4 \\ -3y-9z&=-3 \end{aligned} \xrightarrow[{[3]=[3]-\frac{3}{2}[2]}]{} \begin{aligned} \boxed{2}x+2y+4z&=4 \\ \boxed{-2}y-4z&=-4 \\ \boxed{-3}z&=3 \end{aligned} \tag{4.35}$$

There are three pivots, we can easily solve the system to get $z=-1$, $y=4$, and $x=0$. In augmented matrix terms:

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 4 \\ 3 & 1 & 2 & 2 \\ 5 & 2 & 7 & 7 \end{array}\right] \xrightarrow[{\substack{[2]=[2]-\frac{3}{2}[1] \\ [3]=[3]-\frac{5}{2}[1]}}]{} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 4 \\ 0 & -2 & -4 & -4 \\ 0 & -3 & -9 & -3 \end{array}\right] \xrightarrow[{[3]=[3]-\frac{3}{2}[2]}]{} \left[\begin{array}{c|c} \underbrace{\begin{matrix} 2 & 2 & 4 \\ 0 & -2 & -4 \\ 0 & 0 & -3 \end{matrix}}_{\text{REF}} & \begin{matrix} 4 \\ -4 \\ 3 \end{matrix} \end{array}\right]$$

$$\xrightarrow[{\substack{[1]=(1/2)[1] \\ [2]=(-1/2)[2] \\ [3]=(-1/3)[3]}}]{} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[{\substack{[1]=[1]-2[3] \\ [2]=[2]-2[3]}}]{} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[{[1]=[1]-[2]}]{} \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\text{RREF}} & \begin{matrix} 0 \\ 4 \\ -1 \end{matrix} \end{array}\right]$$

Larger systems of equations So far, we have dealt with 2- and 3-dimensions (points, lines, and planes) so we can 'picture' the solutions. Picturing solutions in systems of more than three equations or more than three unknowns is more difficult. However, we can still use elimination to obtain solutions where they exist.

<u>Example 11</u> Find the solution or solutions, if any, to the following system

$$\begin{array}{ll} [1] & w+\ x+2y+2z=1 \\ [2] & w+\ x+2y+\ z=2 \\ [3] & 2w+3x+3y+\ z=1 \\ [4] & w+3x+\ y+2z=1 \end{array} \tag{4.36}$$

Solution:

$$\left[\begin{array}{cccc:c} 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 3 & 3 & 1 & 1 \\ 1 & 3 & 1 & 2 & 1 \end{array}\right] \xrightarrow[\substack{[2]=[2]-[1] \\ [3]=[3]-2[1] \\ [4]=[4]-[1]}]{} \left[\begin{array}{cccc:c} 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & -3 & -1 \\ 0 & 2 & -1 & 0 & 0 \end{array}\right] \xrightarrow[{[2]\leftrightarrow[3]}]{} \left[\begin{array}{cccc:c} 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 2 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[{[4]=[4]-2[2]}]{} \left[\begin{array}{cccc:c} 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 6 & 2 \end{array}\right] \xrightarrow[{[3]\leftrightarrow[4]}]{} \left[\begin{array}{cccc:c} 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 1 & 6 & 2 \\ \underbrace{0 \quad 0 \quad 0 \quad -1}_{\text{REF}} & & & & 1 \end{array}\right]$$

$$\xrightarrow[\substack{[3]=[3]+6[4] \\ [2]=[2]-3[4] \\ [1]=[1]+2[4]}]{} \left[\begin{array}{cccc:c} 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & -4 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow[\substack{[2]=[2]+[3] \\ [1]=[1]-2[3]}]{} \left[\begin{array}{cccc:c} 1 & 1 & 0 & 0 & -13 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[{[1]=[1]-[2]}]{} \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & -17 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow[{[4]=-[4]}]{} \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & -17 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 8 \\ \underbrace{0 \quad 0 \quad 0 \quad 1}_{\text{RREF}} & & & & -1 \end{array}\right]$$

The solution is $(w,x,y,z)=(-17,4,8,-1)$.

There are four variables, and four pivots, and a unique solution. The RREF is the identity matrix.

<u>Example 12</u> Find the solution or solutions, if any, to the following system

$$\begin{array}{ll} [1] & w+\ x+2y+2z=1 \\ [2] & w+\ x+2y+\ z=2 \\ [3] & 2w+2x+4y+3z=3 \\ [4] & w+3x+\ y+2z=1 \end{array} \tag{4.37}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 2 & 4 & 3 & 3 \\ 1 & 3 & 1 & 2 & 1 \end{array}\right] \xrightarrow[\substack{[2]=[2]-[1] \\ [3]=[3]-2[1] \\ [4]=[4]-[1]}]{} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 2 & -1 & 0 & 0 \end{array}\right] \xrightarrow[{[2]\leftrightarrow[4]}]{} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 1 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[{[4]=[4]-[3]}]{} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 1 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[{[1]\leftrightarrow[1]+2[3]}]{} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 3 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[{[1]=[1]-\frac{1}{2}[2]}]{} \left[\begin{array}{cccc|c} 1 & 0 & \frac{5}{2} & 0 & 3 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{[2]=\frac{1}{2}[2] \\ [4]=-[4]}]{} \left[\begin{array}{c|c} \underbrace{\begin{array}{cccc} \boxed{1} & 0 & \frac{5}{2} & 0 \\ 0 & \boxed{1} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{array}}_{\text{RREF}} & \begin{array}{c} 3 \\ 0 \\ -1 \\ 0 \end{array} \end{array}\right]$$

There are three pivots. The pivot columns are columns 1, 2, and 4 corresponding to w, x, and z. The non-pivot column corresponds to the free variable y.

The solution is $(w,x,y,z)=\left(3-\frac{5}{2}s,\frac{s}{2},s,-1\right)$ for all s.

Homogenous Systems of Linear equations Suppose

$$\begin{aligned} a_{11}x + a_{12}y &= 0 \\ a_{21}x + a_{22}y &= 0 \end{aligned} \tag{4.38}$$

A system where the constants on the right-hand side are zero is called a homogenous system. For the system in (4.38), the two equations represent two lines that intersect the origin. That is, $(x,y)=(0,0)$ is a trivial solution. If there are two pivots, then this must be the only solution. For there to be a non-trivial solution, it must be that the two lines coincide, and every point on the line is a solution.

Example 13

$$\begin{aligned} 2x + 3y &= 0 \\ 1x + 2y &= 0 \end{aligned} \tag{4.39}$$

Solution:

$$\begin{aligned} \boxed{2}x + 3y &= 0 \\ 1x + 2y &= 0 \end{aligned} \xrightarrow[{[2]=[2]-\frac{1}{2}[1]}]{} \begin{aligned} \boxed{2}x + 3y &= 0 \\ 0x + \boxed{\tfrac{1}{2}}y &= 0 \end{aligned} \xrightarrow[{[1]=[1]-6[2]}]{} \begin{aligned} \boxed{2}x + 0y &= 0 \\ 0x + \boxed{\tfrac{1}{2}}y &= 0 \end{aligned}$$

$(x,y)=(0,0)$ is the only solution.

Using augmented matrix notation:

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow[{[2]=[2]-\frac{1}{2}[1]}]{} \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & \frac{1}{2} & 0 \end{array}\right] \xrightarrow[{[1]=[1]-6[2]}]{} \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{array}\right] \xrightarrow[\substack{[1]=(1/2)[1] \\ [2]=(2)[2]}]{} \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \end{array}\right]$$

Example 14

$$\begin{aligned}&[1]\quad 2x+4y=0\\&[2]\quad 1x+2y=0\end{aligned}\tag{4.40}$$

Solution:

$$\begin{matrix}\boxed{2}x+4y=0\\1x+2y=0\end{matrix}\xrightarrow[{[2]=[2]-\frac{1}{2}[1]}]{}\begin{matrix}\boxed{2}x+4y=0\\0x+0y=0\end{matrix}\xrightarrow[{[1]=(1/2)[1]}]{}\begin{matrix}\boxed{1}x+2y=0\\0x+0y=0\end{matrix}$$

Solutions are $(x, y) = (-2s, s)$ for all s, a line in 2-dimensional space passing through the origin.

$$\left[\begin{array}{cc:c}2&4&0\\1&2&0\end{array}\right]\xrightarrow[{[2]=[2]-\frac{1}{2}[1]}]{}\left[\begin{array}{cc:c}2&4&0\\0&0&0\end{array}\right]\xrightarrow[{[1]=(1/2)[1]}]{}\left[\begin{array}{cc:c}1&2&0\\0&0&0\end{array}\right]$$

Example 15

$$\begin{aligned}&[1]\quad 2x+2y+4z=0\\&[2]\quad 3x+\ \ y+2z=0\\&[3]\quad 5x+2y+\ \ z=0\end{aligned}\tag{4.41}$$

Solution:

$$\begin{matrix}[1]\quad \boxed{2}x+2y+4z=0\\ [2]\quad 3x+\ \ y+2z=0\\ [3]\quad 5x+2y+\ \ z=0\end{matrix}\xrightarrow[{\substack{[2]=[2]-\frac{3}{2}[1]\\ [3]=[3]-\frac{5}{2}[1]}}]{}\begin{matrix}\boxed{2}x+2y+4z=0\\ \boxed{-2}y-4z=0\\ -3y-9z=0\end{matrix}\xrightarrow[{[3]=[3]-\frac{3}{2}[2]}]{}\begin{matrix}\boxed{2}x+2y+4z=0\\ \boxed{-2}y-4z=0\\ \boxed{-3}z=0\end{matrix}$$

You can complete the upward elimination steps, though it is obvious that $(x, y, z) = (0,0,0)$ is the only solution.

Example 16

$$\begin{aligned}&[1]\quad 2x+2y+4z=0\\&[2]\quad 3x+\ \ y+2z=0\\&[3]\quad 5x+3y+6z=0\end{aligned}\tag{4.42}$$

Solution:

$$\begin{matrix}[1]\quad \boxed{2}x+2y+4z=0\\ [2]\quad 3x+\ \ y+2z=0\\ [3]\quad 5x+3y+6z=0\end{matrix}\xrightarrow[{\substack{[2]=[2]-\frac{3}{2}[1]\\ [3]=[3]-\frac{5}{2}[1]}}]{}\begin{matrix}\boxed{2}x+2y+4z=0\\ \boxed{-2}y-4z=0\\ -2y-4z=0\end{matrix}\xrightarrow[{[3]=[3]-[2]}]{}\begin{matrix}\boxed{2}x+2y+4z=0\\ \boxed{-2}y-4z=0\\ 0z=0\end{matrix}$$

$$\xrightarrow[{[1]=[1]+[2]}]{}\begin{matrix}\boxed{2}x+0y+0z=0\\ \boxed{-2}y-4z=0\\ 0z=0\end{matrix}\xrightarrow[{\substack{[1]=(1/2)[1]\\ [2]=-(1/2)[2]}}]{}\begin{matrix}\boxed{1}x+0y+0z=0\\ \boxed{1}y+2z=0\\ 0z=0\end{matrix}$$

Solutions are $(x, y, z) = (0, -2s, s)$, a line (one-dimensional object) in 3-dimensional space.

$$\left[\begin{array}{ccc:c}2&2&4&0\\3&1&2&0\\5&3&6&0\end{array}\right]\xrightarrow[{\substack{[2]=[2]-\frac{3}{2}[1]\\ [3]=[3]-\frac{5}{2}[1]}}]{}\left[\begin{array}{ccc:c}2&2&4&0\\0&-2&-4&0\\0&-2&-4&0\end{array}\right]\xrightarrow[{[3]=[3]-[2]}]{}\left[\begin{array}{ccc:c}2&2&4&0\\0&-2&-4&0\\0&0&0&0\end{array}\right]\xrightarrow[{[1]=[1]+\frac{1}{2}[2]}]{}\left[\begin{array}{ccc:c}2&0&0&0\\0&-2&-4&0\\0&0&0&0\end{array}\right]$$

$$\xrightarrow[{\substack{[1]=(1/2)[1]\\ [2]=-(1/2)[2]}}]{}\left[\begin{array}{ccc:c}\boxed{1}&0&0&0\\0&\boxed{1}&2&0\\ \underbrace{0\quad 0\quad 0}_{\text{RREF}}&&&0\end{array}\right]$$

Example 17

$$\begin{aligned} &[1] \quad 2x+2y+4z=0 \\ &[2] \quad 4x+4y+8z=0 \\ &[3] \quad 3x+3y+6z=0 \end{aligned} \tag{4.43}$$

Solution:

$$\begin{aligned} &[1] \quad \boxed{2}x+2y+4z=0 \\ &[2] \quad 4x+4y+8z=0 \\ &[3] \quad 3x+3y+6z=0 \end{aligned} \xrightarrow[{[3]=[3]-3[1]}]{{[2]=[2]-2[1]}} \begin{aligned} \boxed{2}x+2y+4z=0 \\ 0y+0z=0 \\ 0y+0z=0 \end{aligned} \xrightarrow{[1]=\frac{1}{2}[1]} \begin{aligned} \boxed{1}x+1y+2z=0 \\ 0y+0z=0 \\ 0y+0z=0 \end{aligned}$$

Solutions are $(x, y, z) = (-s-2t, s, t)$, a plane in 3-dimensional space.

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 4 & 4 & 8 & 0 \\ 3 & 3 & 6 & 0 \end{array}\right] \xrightarrow[{[3]=[3]-3[1]}]{{[2]=[2]-2[1]}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{[1]=\frac{1}{2}[1]} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]_{\text{RREF}}$$

Example 18 Find the solution or solutions, if any, to the following system

$$\begin{aligned} &[1] \quad w+x+2y+2z=0 \\ &[2] \quad w+x+2y+z=0 \\ &[3] \quad 2w+2x+4y+3z=0 \end{aligned} \tag{4.44}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 2 & 2 & 4 & 3 & 0 \end{array}\right] \xrightarrow[{[3]=[3]-2[1]}]{{[2]=[2]-[1]}} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{[3]=[3]-[2]} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{[2]=-[2]} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{[1]=[1]-2[2]} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]_{\text{RREF}}$$

There are two pivots. The pivot columns are columns 1 and 4 corresponding to *w* and *z*. The non-pivot column corresponds to the free variables *x* and *y*. The solution is $(w, x, y, z) = (-s-2t, s, t, 0)$ for all *s*.

Observe that the solutions to systems of equations 'live' in the *n*-dimensional space $\mathbb{R}^n$, where *n* is the number of variables in the system. This is true whether we have a homogenous or non-homogenous system, and the number of equations. The dimension of the solution (if there are solutions) is $n - \text{number of pivots}$. The number of pivots of course will never exceed $\min(m, n)$ where *m* is the number of equations, and *n* is the number of variables.

If the system is homogenous, solutions always exists, and will always include the zero vectors, i.e., solutions to homogenous systems of equations are either the zero vector, or is a line, plane, hyperplane, … that passes through the origin of the space. We call these "sub-spaces" of $\mathbb{R}^n$. A non-homogenous system might not have any solutions, and if they do, these solutions will not pass through the origin.

Systems of Nonlinear Equations We sometimes need to solve systems of non-linear equations, such as

$$\begin{aligned} xy &= 4 \\ x^2 + y^2 &= 8 \end{aligned} \tag{4.45}$$

The 'method' for solving systems of non-linear equations is effectively also to eliminate variables, although how best to do this depends on the system; it is case-by-case. For the system above, we might approach it in the following way: the first equation gives $x = 4/y$. Thus

$$16/y^2 + y^2 = 8 \quad \Rightarrow \quad 16 + y^4 = 8y^2$$

which gives $(y^2 - 4)^2 = 0$.

The solutions are, therefore, $(x, y) = (2,2)$ and $(x, y) = (-2,-2)$. It is easy to see that larger systems of non-linear equations can be very difficult to solve. Even small systems can be tricky, and care must be taken to avoid extraneous solutions or missing solutions. For example, take

$$\begin{aligned} y &= \sqrt{x} \\ y &= 2 - x \end{aligned} \tag{4.46}$$

Perhaps the obvious thing to do here is

$$\sqrt{x} = 2 - x \quad \Rightarrow \quad x = (2-x)^2 = 4 - 4x + x^2 \quad \Rightarrow \quad 0 = (x-4)(x-1)$$

so you might conclude that $x = 1$ or $x = 4$, with $y = 1$ and $y = 2$ respectively. However, $(x, y) = (4,2)$ is not a solution since this does not lie on the second equation. [*Qn: why is* $(x, y) = (4,-2)$ *not a solution?*]