**3. Introduction to the Inverse Matrix**

The inverse of a square matrix $\mathbf{A}$ is the matrix, denoted by $\mathbf{A}^{-1}$, such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

*Example* The inverse of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ is

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

since

$$\mathbf{A}^{-1}\mathbf{A} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

One application of matrix inverses is in solving simultaneous equations. Take for example

$$\begin{aligned} 2x_1 - x_2 &= 4 \\ x_1 + 2x_2 &= 2 \end{aligned}$$

which can be written in matrix form as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{and } \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Since we know $\mathbf{A}^{-1}$, we can simply (pre)multiply $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A}^{-1}$ to get

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}.$$

Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, and $\mathbf{Ix} = \mathbf{x}$, we have $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. This is the solution to the system.

$$\mathbf{A}^{-1}\mathbf{b} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

You can verify on your own that $x_1 = 2$ and $x_2 = 0$ solves the equations.

The formula for the inverse of an arbitrary $(2 \times 2)$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \text{ where } |\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}.$$

To show this, multiply the two together:

$$\frac{1}{|\mathbf{A}|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{12}a_{22} - a_{12}a_{22} \\ a_{12}a_{22} - a_{12}a_{22} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} |\mathbf{A}| & 0 \\ 0 & |\mathbf{A}| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is worthwhile committing the formula for the inverse of a $(2 \times 2)$ matrix to memory.

The expression $|\mathbf{A}|$ is called the 'determinant' of $\mathbf{A}$, something we will discuss in detail in later sections. Note that if $|\mathbf{A}|=0$, then the inverse will not exist (we say that $\mathbf{A}$ is 'singular'). If the determinant is not zero, the inverse exists, and $\mathbf{A}$ is said to be 'non-singular'. If $\mathbf{A}$ is non-singular, the system

$$\mathbf{Ax}=\mathbf{b}$$

will have a unique solution, otherwise it may have many, or none. If $\mathbf{b}=\mathbf{0}$ and $\mathbf{A}$ is non-singular, the unique solution is the trivial solution $\mathbf{x}=\mathbf{0}$ since $\mathbf{Ax}=\mathbf{0} \Rightarrow \mathbf{A}^{-1}\mathbf{Ax}=\mathbf{A}^{-1}\mathbf{0} \Rightarrow \mathbf{x}=\mathbf{0}$. For the system $\mathbf{Ax}=\mathbf{0}$ to have non-trivial solutions, it must be that $\mathbf{A}$ is singular.

**Exercises**

1. Find the inverse of the following matrices

(i) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

(ix) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (x) $\begin{bmatrix} 7 & 1 \\ 2 & 4 \end{bmatrix}$ (xi) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (xii) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

2. Find the inverse of the matrix

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Verify by direct multiplication.

3. Make a guess as to the inverse of the $(n \times n)$ matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Verify your conjecture by direct multiplication.

4. Find the inverse of the matrix $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$.

5. Solve the following systems of equations by computing the inverse of the coefficient matrix:

(i) $\begin{aligned} 2x_1 - x_2 &= 4 \\ x_1 + 2x_2 &= 2 \end{aligned}$ (ii) $\begin{aligned} 3x_1 + 5x_2 &= 6 \\ 6x_1 + 10x_2 &= 12 \end{aligned}$ (iii) $\begin{aligned} 3x_1 + 5x_2 &= 6 \\ 6x_1 + 10x_2 &= 10 \end{aligned}$

(iv) $\begin{aligned} 2y - x &= 4 + a \\ y + 2x &= 2 \end{aligned}$ (v) $\begin{aligned} 2x_1 - 3x_2 &= 0 \\ x_1 + 2x_2 &= 0 \end{aligned}$ (vi) $\begin{aligned} 3x_1 + 5x_2 &= 0 \\ 6x_1 + 10x_2 &= 0 \end{aligned}$

6. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 1.5 & 2 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

   (i) Show that $\mathbf{AB} = \mathbf{AC}$.

   (ii) Show that $\mathbf{A}^{-1}$ does not exist.

   *Remark* *The example in (i) shows that* $\mathbf{AB} = \mathbf{AC}$ *does not imply that* $\mathbf{B} = \mathbf{C}$ *in general. However, if* $\mathbf{A}^{-1}$ *exists, then it must be that* $\mathbf{B} = \mathbf{C}$*, since*

$$\mathbf{AB} = \mathbf{AC} \Rightarrow \mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC} \Rightarrow \mathbf{B} = \mathbf{C}.$$

7. We defined the inverse of $\mathbf{A}$ as the matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Show that this implies that $\mathbf{AA}^{-1} = \mathbf{I}$. That is, it doesn't matter whether you premultiplying or postmultiplying $\mathbf{A}$ with $\mathbf{A}^{-1}$, you still get $\mathbf{I}$ as a result.

8. Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find the inverse of $\mathbf{A}^{\mathrm{T}}$ (assume $a, b, c, d$ are such that the inverse exists), and show that $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$.

9. Let $\mathbf{A}$ be an $(n \times n)$ matrix whose inverse exists. Show that $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$. (You don't need to know how to compute an $(n \times n)$ for this. Start with the fact that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and take transposes.

10. Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, and assume that their inverses exist. Show that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

11. Let $\mathbf{A}$ and $\mathbf{B}$ be $(n \times n)$ matrices whose inverses exist. Show that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

12. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$. Find the inverse of $\mathbf{AB}$. Does the relationship $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ hold for these two matrices? Why?

13. Is it true that $(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1} + \mathbf{B}^{-1}$? Give a counterexample.