2. Matrix Algebra: Multiplication

Let **A** be a $(m \times n)$ matrix and **B** be a $(n \times p)$ matrix, i.e., the number of columns of **A** and the number of rows of **B** must be the same. Then product **AB** is defined as the $(m \times p)$ matrix whose $(i,j)^{th}$ element is defined by

$$[\mathbf{AB}]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

That is, the $(i,j)^{th}$ element of the product **AB** is defined as the sum of the product of the elements of the i^{th} row of **A** with the corresponding elements in the j^{th} column of **B**. For example, the $(1,1)^{th}$ element of **AB** is

$$[\mathbf{AB}]_{1,1} = \sum_{k=1}^{n} a_{1k} b_{k1} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + ... + a_{1n}b_{n1}$$

The $(2,3)$th element is

$$[\mathbf{AB}]_{2,3} = \sum_{k=1}^{n} a_{2k} b_{k3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} + ... + a_{2n}b_{n3}$$

and so on. Visually, for a product of a (3×3) matrix into a (3×2) matrix, we have

$$\begin{bmatrix} \boxed{a_{11} \; a_{12} \; a_{13}} \\ a_{21} \; a_{22} \; a_{23} \\ a_{31} \; a_{32} \; a_{33} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} b_{11} \\ b_{21} \\ b_{31} \end{matrix}} & \begin{matrix} b_{12} \\ b_{22} \\ b_{32} \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}} & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} \boxed{a_{11} \; a_{12} \; a_{13}} \\ a_{21} \; a_{22} \; a_{23} \\ a_{31} \; a_{32} \; a_{33} \end{bmatrix} \begin{bmatrix} \begin{matrix} b_{11} \\ b_{21} \\ b_{31} \end{matrix} & \boxed{\begin{matrix} b_{12} \\ b_{22} \\ b_{32} \end{matrix}} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & \boxed{a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}} \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \; a_{12} \; a_{13} \\ \boxed{a_{21} \; a_{22} \; a_{23}} \\ a_{31} \; a_{32} \; a_{33} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} b_{11} \\ b_{21} \\ b_{31} \end{matrix}} & \begin{matrix} b_{12} \\ b_{22} \\ b_{32} \end{matrix} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ \boxed{a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}} & \bullet \\ \bullet & \bullet \end{bmatrix}$$ and so on...

<u>Example</u> Let $\mathbf{A} = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 7 \\ 6 & 9 \end{bmatrix}$. Then

$$\mathbf{AB} = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} (2)(4)+(8)(6) & (2)(7)+(8)(9) \\ (3)(4)+(0)(6) & (3)(7)+(0)(9) \\ (5)(4)+(1)(6) & (5)(7)+(1)(9) \end{bmatrix} = \begin{bmatrix} 56 & 86 \\ 12 & 21 \\ 26 & 44 \end{bmatrix}$$

<u>Example</u> Let $\mathbf{A} = \begin{bmatrix} 6 & 5 & -1 \\ 1 & 0 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 0 & 1 \end{bmatrix}$. Then

$$\mathbf{AB} = \begin{bmatrix} 6 & 5 & -1 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (6)(4)+(5)(5)+(-1)(0) & (6)(-1)+(5)(2)+(-1)(1) \\ (1)(4)+(0)(5)+(4)(0) & (1)(-1)+(0)(2)+(4)(1) \end{bmatrix} = \begin{bmatrix} 49 & 3 \\ 4 & 3 \end{bmatrix}$$

<u>Example</u> The simultaneous equations

$$2x_1 - x_2 = 4$$
$$x_1 + 2x_2 = 2$$

can be written in matrix form as

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \text{ or } \mathbf{Ax} = \mathbf{b} \text{ where } \mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \; \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Partitioned Matrices

We can extend the matrix multiplication (and addition, subtraction, etc) to partitioned matrices. Given an $(n \times n)$ matrix, we can break it up into blocks of 'submatrices'. For instance, we can write

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 & 6 \\ 2 & 8 & 2 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 1 & 7 \end{bmatrix} = \left[\begin{array}{c:ccc} 1 & 3 & 2 & 6 \\ 2 & 8 & 2 & 1 \\ \hdashline 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 1 & 7 \end{array}\right] = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{A}_{12} \\ \mathbf{a}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

where $\mathbf{a}_{11} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{A}_{12} = \begin{bmatrix} 3 & 2 & 6 \\ 8 & 2 & 1 \end{bmatrix}$, $\mathbf{a}_{21} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{A}_{22} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 1 & 7 \end{bmatrix}$.

There are of course many ways of partitioning the same matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 & 6 \\ 2 & 8 & 2 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 1 & 7 \end{bmatrix} = \left[\begin{array}{c:ccc} 1 & 3 & 2 & 6 \\ 2 & 8 & 2 & 1 \\ \hdashline 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 1 & 7 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 3 & 2 & 6 \\ 2 & 8 & 2 & 1 \\ 3 & 1 & 2 & 4 \\ \hdashline 4 & 2 & 1 & 3 \\ 3 & 1 & 1 & 7 \end{array}\right].$$

Addition of Partitioned Matrices Consider two $(n \times m)$ matrices $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} \underbrace{\mathbf{A}_{11}}_{n_1 \times m_1} & \underbrace{\mathbf{A}_{12}}_{n_1 \times m_2} \\ \underbrace{\mathbf{A}_{21}}_{n_2 \times m_1} & \underbrace{\mathbf{A}_{22}}_{n_2 \times m_2} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \underbrace{\mathbf{B}_{11}}_{n_1 \times m_1} & \underbrace{\mathbf{B}_{12}}_{n_1 \times m_2} \\ \underbrace{\mathbf{B}_{21}}_{n_2 \times m_1} & \underbrace{\mathbf{B}_{22}}_{n_2 \times m_2} \end{bmatrix}$$

where emphasize that **A** and **B** are of the same size and partitioned in a similar fashion. Then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} \underbrace{\mathbf{A}_{11} + \mathbf{B}_{11}}_{n_1 \times m_1} & \underbrace{\mathbf{A}_{12} + \mathbf{B}_{12}}_{n_1 \times m_2} \\ \underbrace{\mathbf{A}_{21} + \mathbf{B}_{21}}_{n_2 \times m_1} & \underbrace{\mathbf{A}_{22} + \mathbf{B}_{22}}_{n_2 \times m_2} \end{bmatrix}.$$

Multiplication of Partitioned Matrices Consider two matrices $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} \underbrace{\mathbf{A}_{11}}_{n_1 \times m_1} & \underbrace{\mathbf{A}_{12}}_{n_1 \times m_2} \\ \underbrace{\mathbf{A}_{21}}_{n_2 \times m_1} & \underbrace{\mathbf{A}_{22}}_{n_2 \times m_2} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \underbrace{\mathbf{B}_{11}}_{n_1 \times m_1} & \underbrace{\mathbf{B}_{12}}_{n_1 \times m_2} \\ \underbrace{\mathbf{B}_{21}}_{n_2 \times m_1} & \underbrace{\mathbf{B}_{22}}_{n_2 \times m_2} \end{bmatrix}$$

Exercises

1. Find $\mathbf{AB}$ when

(a) $\mathbf{A}=\begin{bmatrix}0 & 2 & 0\\ 3 & 0 & 4\\ 2 & 3 & 0\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}8 & 0\\ 0 & 1\\ 3 & 5\end{bmatrix}$ (b) $\mathbf{A}=\begin{bmatrix}2 & 5 & -1\\ 1 & 0 & 4\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}4\\ 5\\ 1\end{bmatrix}$

2. Let $\mathbf{A}=\begin{bmatrix}2 & 8\\ 3 & 0\\ 5 & 1\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}2 & 0\\ 3 & 8\end{bmatrix}$, and $\mathbf{C}=\begin{bmatrix}7 & 2\\ 6 & 3\end{bmatrix}$

(i) Compute $\mathbf{BC}$; (ii) Compute $\mathbf{CB}$; (iii) Can $\mathbf{BA}$ be computed?

Remark: This exercise shows that for any two matrices $\mathbf{A}$ *and* $\mathbf{B}$, $\mathbf{AB} \neq \mathbf{BA}$. *We distinguish between* <u>*pre-multiplication*</u> *and* <u>*post-multiplication*</u>. *In the product* $\mathbf{AB}$, *we say that* $\mathbf{B}$ *is pre-multiplied by* $\mathbf{A}$, *or* $\mathbf{A}$ *is post-multiplied by* $\mathbf{B}$.

3. Let $\mathbf{d}=\begin{bmatrix}1\\ 3\\ 2\\ 2\end{bmatrix}$ and $\mathbf{f}=\begin{bmatrix}4 & 2 & 1 & 6\end{bmatrix}$. Compute (i) $\mathbf{fd}$ (ii) $\mathbf{df}$ (iii) $\mathbf{d}^{\mathrm{T}}\mathbf{d}$.

4. Show that for any vector $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$, the product $\mathbf{x}^{\mathrm{T}}\mathbf{x} \geq 0$. When will $\mathbf{x}^{\mathrm{T}}\mathbf{x} = 0$?

5. (i) Compute $\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}\begin{bmatrix}-2 & 4\\ 1 & -2\end{bmatrix}$

(ii) Let $\mathbf{A} = \begin{bmatrix}1 & b\\ -\frac{1}{b} & -1\end{bmatrix}$. Compute $\mathbf{A}^2 = \mathbf{AA} = \begin{bmatrix}1 & b\\ -\frac{1}{b} & -1\end{bmatrix}\begin{bmatrix}1 & b\\ -\frac{1}{b} & -1\end{bmatrix}$

Remark A matrix with all elements equal zero is called the zero matrix $\mathbf{0}$. *Obviously,* $\mathbf{A}+\mathbf{0}=\mathbf{A}$ *and* $\mathbf{A0}=\mathbf{0A}=\mathbf{0}$. *Note however that* $\mathbf{AB}=\mathbf{0}$ *does* ***not*** *imply* $\mathbf{A}=\mathbf{0}$ *or* $\mathbf{B}=\mathbf{0}$. *In fact, the square of a non-zero matrix can be a zero matrix!*

Matrix multiplication therefore does not behave like the usual multiplication of numbers: the order of multiplication is important, and $\mathbf{AB}=\mathbf{0}$ *does not imply that either* $\mathbf{A}=\mathbf{0}$ *or* $\mathbf{B}=\mathbf{0}$. *Matrix multiplication however* <u>*does*</u> *follow associative and distributive laws:*

$$\begin{aligned}(\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC})\\ \mathbf{A}(\mathbf{B}+\mathbf{C}) &= \mathbf{AB}+\mathbf{AC}\\ (\mathbf{A}+\mathbf{B})\mathbf{C} &= \mathbf{AC}+\mathbf{BC}\end{aligned}$$

These are easy to prove. For instance, let the dimensions of **A, B**, *and* **C** be $(m\times n)$, $(n\times p)$, and $(p\times q)$ respectively. Then $\mathbf{AB}$ is $(m\times p)$. We have

$$[(\mathbf{AB})\mathbf{C}]_{ij} = \sum_{k=1}^{p}[\mathbf{AB}]_{ik}[\mathbf{C}]_{kj} = \sum_{k=1}^{p}\left(\sum_{l=1}^{n}[\mathbf{A}]_{il}[\mathbf{B}]_{lk}\right)[\mathbf{C}]_{kj} = \sum_{l=1}^{n}[\mathbf{A}]_{il}\left(\sum_{k=1}^{p}[\mathbf{B}]_{lk}[\mathbf{C}]_{kj}\right) = \sum_{l=1}^{n}[\mathbf{A}]_{il}[\mathbf{BC}]_{lj} = [\mathbf{A}(\mathbf{BC})]_{ij}$$

6. Compute

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ (ii) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Remark *The square matrix*

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

is called the identity matrix. It behaves like the number '1' in regular multiplication: for any matrix $\mathbf{A}$, $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$. *To emphasize the dimension of an identity matrix, we sometimes write* $\mathbf{I}_n$ *if it is* $(n \times n)$.

7. Compute

(i) $\begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ (ii) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$

(iii) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ (iv) $\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$

8. Show that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + b_2\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + b_3\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}$$

In other words, $\mathbf{Ab}$ is a "linear combination" of the columns of $\mathbf{A}$, with weights given in $\mathbf{b}$.

9. Show that

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$
$$= b_1\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} + b_2\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} + b_3\begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix} + b_4\begin{bmatrix} a_{41} & a_{42} & a_{43} \end{bmatrix}$$

10. Write the simultaneous equations

$$\begin{aligned} 4x \quad\quad + z &= 4 \\ 19x + y - 3z &= 3 \\ 7x + y \quad\quad &= 1 \end{aligned}$$

in matrix notation.

11. For $\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix}$, prove that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ by multiplying out the matrices.

Remark: This result holds generally. For any $(m \times n)$ *matrix* $\mathbf{A}$ *and any* $(n \times p)$ *matrix* $\mathbf{B}$, *we have* $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$. *The proof is a very good exercise in using the notation we developed earlier. We want to show that the* (i, j)th *element of* $(\mathbf{AB})^{\mathrm{T}}$ *is equal to the* (i, j)th *element of* $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$. *By the definition of the transpose,* (i, j)th *element of* $(\mathbf{AB})^{\mathrm{T}}$ *is the* (j, i)th *element of* $\mathbf{AB}$, *therefore*

$$\begin{aligned}
[(\mathbf{AB})^{\mathrm{T}}]_{ij} &= [\mathbf{AB}]_{ji} \\
&= \sum_{k=1}^{n} a_{jk} b_{ki} \\
&= \sum_{k=1}^{n} b_{ki} a_{jk} \\
&= \sum_{k=1}^{n} \left[\mathbf{B}^{\mathrm{T}}\right]_{ik} \left[\mathbf{A}^{\mathrm{T}}\right]_{kj} \\
&= \left[\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\right]_{ij} \ .
\end{aligned}$$

12. Prove that $(\mathbf{ABC})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

13. Let $\mathbf{X}$ be a general $(n \times k)$ matrix. Explain why $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is square. Explain why $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is symmetric.

14. Define the trace of a $(n \times n)$ square matrix $\mathbf{A}$ to be

$$trace(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

That is, the trace of a square matrix is simply the sum of its diagonal elements. Show that

(i) $trace(\mathbf{A} + \mathbf{B}) = trace(\mathbf{A}) + trace(\mathbf{B})$ (*of course, both matrices must be the same size*)

(ii) $trace(\mathbf{A}) = trace(\mathbf{A}^{\mathrm{T}})$

(iii) $trace(\mathbf{AB}) = \mathrm{trace}(\mathbf{BA})$ (*here* $\mathbf{A}$ *and* $\mathbf{B}$ *need not be of the same size*).

Hint: for the last one, look at the proof given in question 11 and adapt it.