1. Matrices: Definitions and Basic Operations

Analyzing economic models often involve working with large numbers of linear equations. Matrix algebra provides a set of tools for dealing with such objects.

Definitions

A **matrix** is a rectangular collection of numbers. The following is a matrix with m rows and n columns, i.e., "m by n" matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The "dimension" of the matrix is often written $(m \times n)$. The number that appears in the (i, j)th position is called the (i, j)th **element** or (i, j)th **entry** or (i, j)th **component** of the matrix. Note that we count rows from top to bottom, and columns from left to right.

If $m = n$, the matrix is a **square matrix**.
If $m = 1$ and $n > 1$, it is called a **row vector**.
If $m > 1$ and $n = 1$, we have a **column vector**.
The term "vector" is used in many ways in mathematics. Sometimes a vector is used to refer to an ordered list of numbers $(x_1, x_2, ..., x_n)$. *Such an object has no dimension. It merely is an ordered sequence of length n. Column and row vectors are two dimensional objects. In these notes, a "vector" will always refer to a column vector.*

If $m = n = 1$, then we have a **scalar**.
i.e., scalars are just numbers, though sometimes they are treated explicitly as matrices.

<u>Example</u> A row vector $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$, a column vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, a square matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

It is often convenient to write an $(m \times n)$ matrix as

$$\mathbf{A} = (a_{ij})_{m \times n} .$$

It is often convenient to refer to the (i, j)th element of $\mathbf{A}$ using

$$[\mathbf{A}]_{ij} .$$

Two matrices of the same dimension $m \times n$ are said to be equal if each of their corresponding elements are equal, i.e.,

$$\mathbf{A} = \mathbf{B} \iff [\mathbf{A}]_{ij} = [\mathbf{B}]_{ij} \; \forall i = 1, 2, ..., m, \; \forall j = 1, 2, ..., n.$$

Matrices of different dimensions cannot be equal.

Square matrices play an important role in matrix algebra. The present below a list of some important types of square matrices. In the definitions, the **diagonal** elements of $(n \times n)$ square matrix refer to the (i,i)th elements, i.e.,

$$[\mathbf{A}]_{ii}, i = 1,2,\ldots,n .$$

An **identity matrix** is a square matrix with diagonal elements equal to one and off-diagonal elements equal to zero, e.g.,

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

i.e., a square matrix **A** is an identity matrix if

$$[\mathbf{A}]_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \text{ for all } i, j = 1,2,\ldots,n .$$

An identity matrix is always written **I**. A subscript is added to indicate its dimension, though often it is left out. The identity matrix plays a role in matrix algebra akin to the role played by the number "1" in the number system.

A **diagonal matrix** is a square matrix with off-diagonal matrix equal to zero, i.e., the square matrix $\mathbf{A}$ is diagonal if

$$[\mathbf{A}]_{ij} = 0 \text{ for all } i \neq j, \; i, j = 1,2,\ldots,n .$$

Diagonal matrices are sometimes written $diag(a_1, a_2, \ldots, a_n)$. Identity matrices are examples of diagonal matrices.

<u>Example</u> $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = diag(1,4,3)$ is a diagonal matrix.

Note that there is nothing in the definition of a diagonal matrix that says its diagonal matrix cannot be zero.

A **lower triangular** square matrix is one where

$$[\mathbf{A}]_{ij} = 0 \text{ for all } i < j, \; i, j = 1,2,\ldots,n .$$

An **upper triangular** square matrix is one where

$$[\mathbf{A}]_{ij} = 0 \text{ for all } i > j, \; i, j = 1,2,\ldots,n .$$

<u>Example</u> $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 2 & 1 & 3 \end{bmatrix}$ is lower triangular, $\mathbf{B} = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is upper triangular, $\mathbf{C} = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 3 & 1 \\ 5 & 6 & 3 \end{bmatrix}$ is neither.

A symmetric matrix is a square matrix $\mathbf{A}$ such that

$$[\mathbf{A}]_{ij} = [\mathbf{A}]_{ji} \text{ for all } i, j = 1,2,\ldots,n$$

<u>Example</u> The matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 6 \\ 2 & 6 & 3 \end{bmatrix}$ is symmetric. The matrix $\mathbf{B} = \begin{bmatrix} 1 & 3 & 2 \\ 7 & 4 & 6 \\ 2 & 6 & 3 \end{bmatrix}$ is not.

Basic Operations (Addition, Scalar Multiplication, Subtraction, Transpose)

Addition: Let $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ be two $(m\times n)$ matrices. Then

$$\mathbf{A}+\mathbf{B} = (a_{ij}+b_{ij})_{m\times n},$$

i.e., addition of matrices is defined to be element by element addition.

Example $$\begin{bmatrix}1 & 4\\ 3 & 2\\ 6 & 5\end{bmatrix}+\begin{bmatrix}6 & 9\\ 1 & 2\\ 1 & 10\end{bmatrix} = \begin{bmatrix}1+6 & 4+9\\ 3+1 & 2+2\\ 6+1 & 5+10\end{bmatrix} = \begin{bmatrix}7 & 13\\ 4 & 4\\ 7 & 15\end{bmatrix}$$

Matrices being added together obviously must have the same dimensions. It should also be obvious that

$$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$$

$$(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$$

This means that as far as addition is concerned, we can manipulate matrices in the same way we manipulate ordinary numbers (as long as they have the same dimensions)

Scalar Multiplication Let $\mathbf{A}$ be a $(m\times n)$ matrix, and α be a scalar. Then we define

$$\alpha\mathbf{A} = (\alpha a_{ij})_{m\times n}$$

i.e., the product of a scalar and a matrix is defined to be the multiplication of each element of the matrix by the scalar.

Example $$b\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{bmatrix} = \begin{bmatrix}ba_{11} & ba_{12}\\ ba_{21} & ba_{22}\\ ba_{31} & ba_{32}\end{bmatrix}$$

We can use scalar multiplication to define matrix subtraction. Let $\mathbf{A}$ and $\mathbf{B}$ be $(m\times n)$ matrices. Then $\mathbf{A}-\mathbf{B} = \mathbf{A}+(-1)\mathbf{B}$.

Example $$\begin{bmatrix}1 & 4\\ 3 & 2\\ 6 & 5\end{bmatrix}-\begin{bmatrix}6 & 9\\ 1 & 2\\ 1 & 10\end{bmatrix} = \begin{bmatrix}1-6 & 4-9\\ 3-1 & 2-2\\ 6-1 & 5-10\end{bmatrix} = \begin{bmatrix}-5 & -5\\ 2 & 0\\ 5 & -5\end{bmatrix}$$

Transpose When we transpose a matrix, we write its rows as its column, and its columns as its rows. The transpose of a matrix $\mathbf{A}$, denoted either by $\mathbf{A}^{\mathrm{T}}$ or $\mathbf{A}'$. For example,

$$\begin{bmatrix}1 & 4\\ 3 & 2\\ 6 & 5\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}1 & 3 & 6\\ 4 & 2 & 5\end{bmatrix}$$

Put more succinctly,

$$[\mathbf{A}^{\mathrm{T}}]_{ij} = [\mathbf{A}]_{ji}.$$

We can use the transpose operator is in defining symmetric matrices. A symmetric matrix is one where $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$.

Exercises

1. Let $\mathbf{A} = \begin{bmatrix} 7 & 13 \\ 4 & 4 \\ 7 & 15 \end{bmatrix}$. What is the dimension of $\mathbf{A}$? What is $[\mathbf{A}]_{12}$? What is $[\mathbf{A}]_{31}$?

2. Suppose $\mathbf{A} = (a_{ij})_{2\times 4}$ where $a_{ij} = i + j$. Write out the matrix in full.

3. Write out in full the matrix

(i) $(a_{ij})_{4\times 4}$ where $a_{ij} = 1$ when $i = j$, 0 otherwise.

(ii) $(a_{ij})_{4\times 4}$ where $a_{ij} = 0$ if $i \neq j$. (Fill in the rest of the entries '*')

(iii) $(a_{ij})_{5\times 5}$ where $a_{ij} = 0$ when $i < j$. (Fill in the rest of the entries with '*')

(vi) $(a_{ij})_{5\times 5}$ where $a_{ij} = 0$ when $i > j$. (Fill in the rest of the entries with '*')

These are all square matrices. Matrices (i) and (ii) are called diagonal matrices. Matrix (iii) is a "lower triangular matrix", and (iv) is an "upper triangular matrix" (so we have in (iii) and (iv) matrices that are square and triangular!)

4. Give an example of a (4×4) matrix such that $[\mathbf{A}]_{ij} = [\mathbf{A}]_{ji}$.

5. If $\begin{bmatrix} u+2v & 1 & 3 \\ 9 & 0 & 4 \\ 3 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 9 & 0 & u+v \\ 3 & 4 & 7 \end{bmatrix}$, what is u and v?

6. Let v_1, v_2, v_3, v_4 represent cities, and suppose there are one-way flights from v_1 to v_2 and v_3, from v_2 to v_3 and v_4, and two-way flights between v_1 and v_4. Write out a matrix $\mathbf{A}$ such that $[\mathbf{A}]_{ij} = 1$ if there is a flight from v_i to v_j, and zero otherwise.

7. What is the dimension of the matrix $\begin{bmatrix} 1 & 8 & 3 \\ 9 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix}$?

8. Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Is $\mathbf{A} = \mathbf{B}$?

Matrices with all zero entries are called zero matrices and written $\mathbf{0}_{m,n}$, or $\mathbf{0}_n$ if square, or simply $\mathbf{0}$ if the dimensions can be easily obtained from context.

9. If $\begin{bmatrix} 1 & u \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ 1 & 1 \\ 1 & v \end{bmatrix} = \begin{bmatrix} 4 & 13 \\ 4 & 3 \\ w & 7 \end{bmatrix}$, what is u, v, and w?

10. If $2\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 8 \\ 1 & 5 \end{bmatrix}$, what is $\mathbf{A}$? If $\mathbf{B} - \frac{1}{2}\begin{bmatrix} 3 & 4 \\ 1 & 8 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 5 \\ 3 & 1 \end{bmatrix}$, what is $\mathbf{B}$?

11. Which of the following matrices are symmetric?

(a) $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 5 & 4 & b \\ 3 & 4 & 3 & 3 \\ 5 & b & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 5 & 4 & b \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 3 & 5 \\ 2 & 5 & 4 & b \\ 3 & 4 & 3 & 3 \\ 5 & b & 3 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

12. True or False?

(a) Symmetric matrices must be square.

(b) A scalar is symmetric.

(c) If $\mathbf{A}$ is symmetric, then $\alpha\mathbf{A}$ is symmetric.

(d) The sum of symmetric matrices is symmetric.

(e) If $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$, then $\mathbf{A}$ is symmetric.

13. (a) Find $\mathbf{A}$ and $\mathbf{B}$ if they satisfy

$$2\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 3 & 0 \end{bmatrix}$$
$$\mathbf{A} + 2\mathbf{B} = \begin{bmatrix} 4 & 2 & 3 \\ 5 & 1 & 1 \end{bmatrix}$$

simultaneously.

(b) If $\mathbf{A} + \mathbf{B} = \mathbf{C}$ and $3\mathbf{A} - 2\mathbf{B} = \mathbf{0}$ simultaneously, find $\mathbf{A}$ and $\mathbf{B}$ in terms of $\mathbf{C}$.