45. Complex Numbers

What values of x result in $ax^2 + bx + c = 0$? Some simple calculations give the answer:

$$ax^2 + bx + c = 0$$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$
$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The "roots" of the quadratic equation $ax^2 + bx + c = 0$ are the two values given in the last line. But what if $b^2 - 4ac > 0$? E.g., applying the formula to $x^2 + 1 = 0$, we have $x = \pm\sqrt{-1}$. If we accept the existence of such a thing, and accept that $(\sqrt{-1})(\sqrt{-1}) = -1$, then we find that the solution works, since

$$(-\sqrt{-1})(-\sqrt{-1}) + 1 = -1 + 1 = 0 \text{ and } \sqrt{(-1)}(\sqrt{-1}) + 1 = -1 + 1 = 0 .$$

Another example: using the formula, the roots of the quadratic equation

$$x^2 + 2x + 5 = 0$$

are

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm \sqrt{16(-1)}}{2} = -1 \pm 2\sqrt{-1} .$$

If we treat $\sqrt{-1}$ just as we would any 'normal' number, then the solution again works, since

$$(-1 + 2\sqrt{-1})^2 + 2(-1 + 2\sqrt{-1}) + 5 = (1 - 4\sqrt{-1} - 4) + (-2 + 4\sqrt{-1}) + 5 = 0$$

and $$(-1 - 2\sqrt{-1})^2 + 2(-1 - 2\sqrt{-1}) + 5 = (1 + 4\sqrt{-1} - 4) + (-2 - 4\sqrt{-1}) + 5 = 0 .$$

Then we can say that the equation $ax^2 + bx + c = 0$ will always have two roots. (This is also true even if a, b, c themselves involve $\sqrt{-1}$!)

The object $\sqrt{-1}$ is called the "unit imaginary number" and is often denoted i or j (we well use i). Complex numbers are defined as number of the form $z = a + bi$ where a and b are real numbers. The number a is called the 'real part of z' and b is called the imaginary part of z. Sometimes the complex number is written as

$$z = \text{Re}(z) + \text{Im}(z)i .$$

Addition and multiplication are *defined* for complex numbers as

$$\begin{aligned}(a_1 + b_1 i) + (a_2 + b_2 i) &= (a_1 + a_2) + (b_1 + b_2)i \\ (a_1 + b_1 i)(a_2 + b_2 i) &= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_2 b_1 + a_1 b_2)i\end{aligned}$$

These natural definitions for addition and multiplication (basically treating i no differently from real numbers) follows all the essential properties for addition and multiplication that are satisfied by real numbers. Furthermore, real numbers are a special case of complex numbers with $b = 0$, and the addition/multiplication definitions reduce to addition/multiplication of real numbers. Complex numbers can be viewed as a generalization of real numbers.

Given a complex number $z = a + bi$, its conjugate is defined as $\bar{z} = a - bi$. The product of a complex number with its conjugate always results in a real number.

The numbers '0' and '1' play an important role in the real number system, since $0 + x = x$ and $0x = 0$, and $1x = x$. In the complex number system these roles are played by the complex numbers $0 + 0i$ and $1 + 0i$ respectively. The reciprocal of a real number x is that number x^{-1} such that $x(x^{-1}) = 1$. It is straightforward to show that for a complex number $z = a + bi$,

$$z^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right)$$

which exists for all z except $z = 0 + 0i$. Division is defined by

$$\frac{z_1}{z_2} = z_1 z_2^{-1} .$$

Polar form Complex numbers are often viewed as a pair of numbers (a,b) or $(\text{Re}(z), \text{Im}(z))$ and plotted as a point on a 2-dimensional Cartesian plane (your usual x-y axes) with the imaginary coordinate on the y-axis, and the real coordinate on the x-axis (which serves as your usual 'real number line'. The definitions of addition and multiplication can be written as

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

$$(a_1 + b_1)(a_2 + b_2) = (a_1 a_2 - b_1 b_2, b_1 a_2 + a_1 b_2)$$

The complex 'zero' is (0,0) and the multiplicative identity '1' is $(1,0)$. The imaginary number i is $(0,1)$.

The modulus of a real number $|x|$ is defined as the length of a real number, which amounts to its 'absolute value' $\sqrt{x^2}$. For complex numbers, the length is

$$|z| = \sqrt{a^2 + b^2}\ .$$

Viewing a complex number as a point in the 2-dimensional plan, we can write a complex number in *polar form*. Using the definition of the sine and cosine of an angle θ, we have:

$$z = r[\cos\theta + i\sin\theta]$$

where r is $|z|$, and θ is the anti-clockwise angle made by the line connecting z to the origin, and the positive x-axis. Note that r cannot be negative. The value θ, in radians, is called the argument of z, and written $Arg(z) = \theta$.

Complex exponentials We can define and work with functions of complex numbers, i.e., we can talk about $f(z) = az^2 + bz + c$, etc. (which is what we really are doing anyway, if we allow for complex roots!). One function is particularly useful, and that is

$$f(z) = e^{\text{Re}(z)}(\cos(\text{Im}(z)) + i\sin(\text{Im}(z)))\ .$$

When z is real, i.e. $z=(x,0)$, this function reduces to $f(z)=e^x$. Furthermore, while we have not discussed (and will not discuss) differentiation of complex function – such a thing exists! – it turns out that this is the only complex function such that $f'(z)=f(z)$, which is a property shared by the real function $f(x)=e^x$. For this reason, we <u>define</u> the complex exponential function to be the complex function

$$e^z = e^{\text{Re}(z)}(\cos(\text{Im}(z)) + i\sin(\text{Im}(z))) .$$

In particular, we have the useful special case

$$e^{(0,\theta)} = e^{i\theta} = e^0(\cos\theta + i\sin\theta) = \cos\theta + i\sin\theta .$$

(Since $\sin\pi = 0$ and $\cos\pi = -1$, we also get the famous identity $e^{i\pi} = -1$.)

Note that

$$\begin{aligned} e^{i\theta_1+i\theta_2} &= e^{i(\theta_1+\theta_2)} \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 \\ &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= e^{i\theta_1}e^{i\theta_2} \end{aligned}$$

As we have pointed out, $e^{(x,0)} = e^x$, so we can also write

$$e^z = e^x e^{iy} .$$

From here, one can see that

$$e^{z_1}e^{z_2} = e^{a_1}e^{b_1 i}e^{a_2}e^{b_2 i} = e^{a_1+a_2}e^{(b_1+b_2)i} = e^{z_1+z_2}$$

as expected of the exponential function.

Referring back to the polar form of complex numbers, we see that we can also write

$$z = re^{i\theta} \text{ where } r = |z| .$$

This help us in computing roots of complex numbers. The n-th root of a complex number $z_0 = r_0 e^{\theta_0}$ is that complex number $z = re^{i\theta}$ such that $z^n = r^n e^{in\theta} = z_0$. We can immediately see that

$$r = r_0^{1/n} \text{ and } \theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}, k = 0, \pm 1, \pm 2, \ldots .$$

Complex numbers are useful because we do not have to worry about situations where we run into the root of a negative number. Perhaps the easiest example where we see this is when finding solutions to quadratic equations. Instead of having to say that the quadratic equation will have real roots only if $b^2 - 4ac \geq 0$, when we extend to the complex numbers, there are always two roots. It turns out that many calculations actually become *easier* when you extend the problem to complex numbers. As an illustration, take the classic example $\int e^x \cos x \, dx$, which can be solved quite easily by taking integration by parts twice. Instead of integration by parts, suppose we work on the integral

$$\int e^x (\cos x + i \sin x) dx .$$

Since

$$\int e^x (\cos x + i \sin x) dx = \int e^x \cos x \, dx + i \int e^x \sin x \, dx ,$$

the solution of the original problem is just the real part of the complex integral. Since $e^{ix} = \cos x + i \sin x$, the complex integral is just

$$\int e^x (\cos x + i \sin x) dx = \int e^x e^{ix} dx = \int e^{x(1+i)} dx = \frac{1}{1+i} e^{x+xi}$$

and real part of which is easily calculated to be

$$\frac{e^x}{2} (\cos x + \sin x) .$$

Exercises

1. Verify that for complex numbers z_1, z_2, z_3 we have

   a. $z_1 + z_2 = z_2 + z_1$

   b. $z_1 z_2 = z_2 z_1$

   c. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

   d. $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

   e. $z_3 (z_1 + z_2) = z_3 z_1 + z_3 z_2$.

2. Suppose that $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Find the expression for $\frac{z_1}{z_2}$.

3. Show that if $z_1 z_2 = 0$, then at least one of the two complex numbers is not zero.

4. Show that $z\bar{z} = |z|^2$. Use this to show $|z_1 z_2| = |z_1||z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

5. An easy way to compute a reciprocal is to use the fact that

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}}.$$

   Use this to show that if $z = 1 + bi$, then

$$z^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right).$$

5. Show that the n-th roots of unity (i.e., the number $(1,0)$) are

$$w_n = \exp\left(i\frac{2k\pi}{n}\right), k = 0, 1, ..., n-1.$$

   What are the n-th of unity when $n = 2$? $n = 3$? $n = 4$?

6. Show that

$$\frac{1}{1+i} = \frac{1}{2} - \frac{i}{2}$$

   Use this, and the fact that $e^{x+ix} = e^x(\cos x + i \sin x)$ (why?) to show that $\int e^x \cos x dx = \frac{e^x}{2}(\cos x + \sin x)$.