## 22. Constrained Optimization **Anthony Tay**

*22.1 Introduction*

Most optimization problems in economics have to account for constraints, e.g.

(1) $\max u(x,y)$ such that $px + y = m$

where $x$ and $y$ are choice variables, representing quantities consumed of two goods, and $p$ and $m$ are constants, representing price of $x$ and the available budget $m$ respectively, both in terms of the price of good $y$. The constraint in this example is an equality constraint. Constraints such as

$$px + y \le m, \text{ and } x \ge 0$$

are called inequality constraints.

Simple optimization problems with equality constraints can often be recast as unconstrained optimization problems. For instance, (1) can be rewritten as

(1′) $\max u(x, m - px)$

In this session, we introduce an alternative method called the Lagrange Multiplier (LM) method for constrained optimization. This alternative method is useful because

(i) in more complicated situations, the substitution method may be inconvenient or impossible,

(ii) the LM method can be extended to handle inequality constraints,

(iii) for optimization problems in economics, the LM method introduces into each problem new quantities (called Lagrange Multipliers) with important economic interpretations; and

The LM method is also used in statistical hypothesis testing (we shall omit this application).

We will

-- see what the Lagrange Multiplier method is,

-- discuss economic interpretations of the Lagrange Multipliers,

-- explain why the method works

*22.2 The Lagrange Multiplier Method (two variable, one equality constraint)*

The Problem:

(2) max(min) $f(x, y)$ subject to $g(x, y) = c$.

where the functions $f$ and $g$ may include parameters. To keep the expressions uncluttered, we do not write down these parameters.

The Lagrange Multiplier Method:

Step 1. Write down the Lagrangian function, introducing a new variable $\lambda$ (the Lagrange Multiplier)

(3) $$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c)$$

Step 2. *First Order Conditions*: Differentiate $\mathcal{L}(x, y, \lambda)$ with respect to $x$, $y$, and $\lambda$, equate the partial derivatives to 0, and solve the equations simultaneously to obtain the stationary points $x^*$, $y^*$, and $\lambda^*$ :

(3a) $$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} = f_1'(x,y) - \lambda g_1'(x,y) = 0$$

(3b) $$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} = f_2'(x,y) - \lambda g_2'(x,y) = 0$$

(3c) $$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} = -g(x,y) + c = 0$$

These will be the possible solutions to the optimization problem.

Note that the conditions (3a), (3b), and (3c) only give necessary conditions. Further arguments will be needed to check if the solutions to these three equations are indeed solutions to the optimization problem.

Example 22.2.1 Maximize $xy$ subject to $x + 3y = 24$

Setting up the Lagrangian: $\mathcal{L}(x,y,\lambda) = xy - \lambda(x + 3y - 24)$

First-Order Conditions:

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} = y - \lambda = 0, \qquad \text{or } y = \lambda$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} = x - 3\lambda = 0, \qquad \text{or } x = 3\lambda$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} = -x - 3y + 24 = 0, \qquad \text{or } x + 3y = 24$$

Combining these equations we have $3\lambda + 3\lambda = 24$. Thererfore $\lambda^* = 4$, $y^* = 4$, and $x^* = 12$ is a possible solution to our problem.

Example 22.2.2 Maximize/Minimize $3xy$ subject to $x^2 + y^2 = 8$

Setting up the Lagrangian: $\mathcal{L}(x,y,\lambda) = 3xy - \lambda(x^2 + y^2 - 8)$

First-Order Conditions:

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} = 3y - 2\lambda x = 0 \qquad \text{or } 3y = 2\lambda x$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} = 3x - 2\lambda y = 0 \qquad \text{or } 3x = 2\lambda y$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} = -x^2 - y^2 + 8 = 0 \qquad \text{or } x^2 + y^2 = 8$$

Therefore we have $3x = 2\lambda y = 2\lambda\, 2\lambda x/3 = 4\lambda^2 x/3 \Rightarrow \lambda^2 = 9/4 \Rightarrow \lambda = \pm 3/2$

When $\lambda = 3/2$: $3y = 2\lambda x \quad \Rightarrow \quad y = x$ so $x^2 + y^2 = 8$

$\Rightarrow \quad 2x^2 = 8 \quad \Rightarrow \quad x = \pm 2,\ y = \pm 2;$

When $\lambda = -3/2$: $3y = 2\lambda x \quad \Rightarrow \quad y = -x$ so $x^2 + y^2 = 8$

$\Rightarrow \quad 2x^2 = 8 \quad \Rightarrow \quad x = \pm 2,\ y = \mp 2.$

Possible solutions for $(x^*, y^*)$ are (2,2), (2, –2), (–2,2) and (–2,–2), and the values of our objective function at these points are 12, –12, –12 and 12 respectively.

## *22.3 Economic Interpretation of the Lagrangian Multiplier*

Given the problem

$$\max(\min) f(x, y) \text{ subject to } g(x, y) = c,$$

the Lagrange Multiplier method tells us how to get the solutions $x^*$ and $y^*$ (and $\lambda^*$). Substituting this into the objective function gives us the value function $f^* = f(x^*, y^*)$. It turns out that

$$df^* = \lambda^* dc,$$

so that the optimal value of the Lagrange multiplier tells us the rate at which the value function changes with respect to changes in the constraint $c$.

Proof: Taking differentials of $f^* = f(x^*, y^*)$ gives us

$$df^* = \frac{\partial f(x^*, y^*)}{\partial x^*} dx^* + \frac{\partial f(x^*, y^*)}{\partial y^*} dy^*$$

From the F.O.C:

$$\frac{\partial f(x^*, y^*)}{\partial x^*} = \lambda^* \frac{\partial g(x^*, y^*)}{\partial x^*} \text{ and } \frac{\partial f(x^*, y^*)}{\partial x^*} = \lambda^* \frac{\partial g(x^*, y^*)}{\partial x^*}$$

so we have

$$\begin{aligned} df^* &= \lambda^* \frac{\partial g(x^*, y^*)}{\partial x^*} dx^* + \lambda^* \frac{\partial g(x^*, y^*)}{\partial y^*} dy^* \\ &= \lambda^* \left( \frac{\partial g(x^*, y^*)}{\partial x^*} dx^* + \frac{\partial g(x^*, y^*)}{\partial y^*} dy^* \right) \end{aligned}$$

The differential of the constraint $g(x^*, y^*) = c$ is

$$dc = \frac{\partial g(x^*, y^*)}{\partial x^*} dx^* + \frac{\partial g(x^*, y^*)}{\partial y^*} dy^*$$

Therefore $df^* = \lambda^* dc$ .

Typically, $c$ represents some sort of resource. Because a small change in $c$ leads to an approximate change in the value function by $df^*$, $\lambda^* dc$ represents the value of the additional resouce $dc$. Thus $\lambda^*$ acts as a price. We call it the **shadow price** of $c$. This price is in terms of the units that the objective function is measured in.

For instance, if $f$ is a profit function measured in dollars, and $c$ is a resource constraint, then an additional amount $dc$ of the resouces leads to an increase of approximately $df^*$, and the firm would be willing to pay up to (approx.) $\lambda^* dc$ for this additional resource. The additional resource is worth $\lambda^*$ per unit.

<u>Example 22.3.1</u> The problem

$$\max\ xy \text{ such that } 2x + y = 100$$

gives $x^* = 25$, $y^* = 50$, so that $x^* y^* = 1250$, and $\lambda^* = 25$

whereas the problem

$$\max\ xy \text{ such that } 2x + y = 101$$

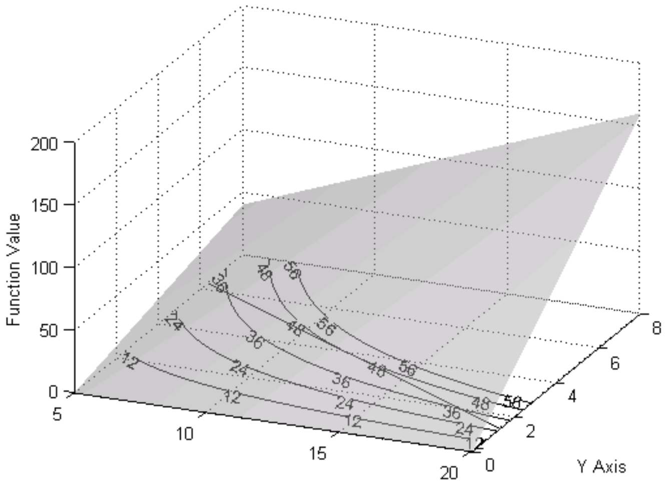
gives $x^* = 25.25$, $y^* = 50.5$, so that $x^* y^* = 1275.125$.

The actual change in the value of the objective function is $25.125$. The value $\lambda^*$ calculated in the first problem is a linear approximation of this value.

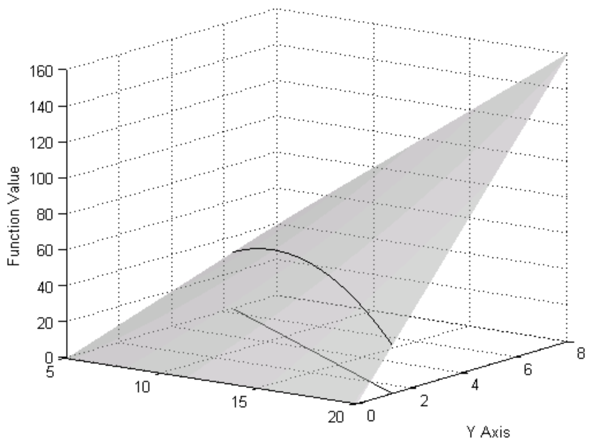
*22.4 Why the Lagrange Multiplier method works*

<u>Example 22.4.1</u> Maximize $xy$ subject to $x + 3y = 24$.

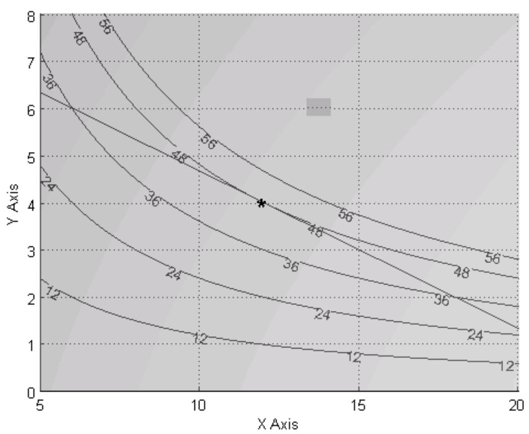
Recall that $y^* = 4$, and $x^* = 12$. The maximum value of $xy = 48$.




This is a plot of $xy$ for $x \in [5,20]$, $y \in [0,8]$. Several contour lines are drawn on the $x$-$y$ plane. The straight line on the $x$-$y$ plane is the constraint $x + 3y = 24$.




Our maximization problem basically asks: what is the point along this line that maximizes the function $xy$?




At the maximum point, the constraint line should be tangent to the contour for $f(x, y)$ at the maximum value. If the constraint line intersects a contour, then moving along the constraint can bring us to a higher value of the objective function.

At the tangent point, the slope of the contour is $-f_1'(x,y)\big/ f_2'(x,y)$, and that of the constraint line is $-g_1'(x,y)\big/ g_2'(x,y)$, i.e.,

$$\frac{f_1'(x,y)}{f_2'(x,y)} = \frac{g_1'(x,y)}{g_2'(x,y)} \quad \text{or} \quad \frac{f_1'(x,y)}{g_1'(x,y)} = \frac{f_2'(x,y)}{g_2'(x,y)}$$

which is what we obtain from the F.O.C. of the Lagrangian function.

There are situations where the Lagrange Multiplier method does not apply:

**Theorem** Suppose that $f(x,y)$ and $g(x,y)$ have continuous partial derivatives in some domain $A$ of the $x$ - $y$ plane, and that $(x_0, y_0)$ is both an interior point of $A$ and a local optimal point for $f(x,y)$ subject to the constraint $g(x,y)=c$. Suppose further that $g_1'(x,y)$ and $g_2'(x,y)$ are not both zero. Then there exists a unique number $\lambda$ such that the Lagrangian

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-c)$$

has a stationary point at $(x_0, y_0)$. [See SH, Section 14.4].

Suppose $(x_0, y_0)$ solves the problem

$$\max\ f(x,y) \text{ subject to the constraint } g(x,y)=c.$$

Then the theorem tells us that (subject to some conditions holding) $(x_0, y_0)$ is a stationary point of the Lagrangian. However, $(x_0, y_0)$ might not maximize the *Lagrangian*.

**Example** [SH 14.4 Problem No 1] The point $(1,1)$ solves the problem

$$\max\ xy \text{ subject to } x+y=2$$

with $\lambda=1$. However, $(1,1)$ does not solve the problem

$$\max\ \mathcal{L}(x,y) = xy - 1(x+y-2)$$

since $\mathcal{L}(1,1) < \mathcal{L}(2,2)$.

This is fine: the objective is not to maximize the Lagrangian, it is to maximize the function $f(x,y)$ subject to the constraint $g(x,y)=c$. But in some situations, the stationary point $(x_0, y_0)$ in fact maximizes the *Lagrangian* for all $(x,y)$. In such cases, and if the constraint is met, then the point $(x_0, y_0)$ must also solve the constrained maximization problem. A similar argument holds for minimization problems. This leads to the following result:

**Theorem** Suppose the problem is

$$\text{max/min } f(x,y) \text{ subject to the constraint } g(x,y) = c.$$

and that $(x_0, y_0)$ is a stationary point of the Lagrangian. If the Lagrangian is concave in $x$ and $y$, then $(x_0, y_0)$ solves the maximization problem. If the Lagrangian is convex in $x$ and $y$, then $(x_0, y_0)$ solves the minimization problem.

<u>Example</u> max $10x^{1/2}y^{1/3}$ subject to $2x + 4y = 12$

The Lagrangian is

$$\max \ \mathcal{L} = 10x^{1/3}y^{1/2} - \lambda(2x + 4y - 10)$$

The FOCs are

$$\frac{10}{3}x^{-2/3}y^{1/2} - 2\lambda = 0,\ 5x^{1/3}y^{-1/2} - 4\lambda = 0, \text{ and } 2x + 4y = 10.$$

The first two equations lead to

$$\lambda = \frac{5}{3}x^{-2/3}y^{1/2} \text{ and } \lambda = \frac{5}{4}x^{1/3}y^{-1/2}.$$

or $3x = 4y$, so the budget constraint becomes $5x = 10$, or $x = 2$, $y = 3/2$.

The Lagrangian is concave:

$$\mathcal{L}_{xx} = -\frac{20}{9}x^{-5/3}y^{1/2} < 0$$

$$\mathcal{L}_{xy} = \frac{5}{3}x^{-2/3}y^{-1/2},\ \mathcal{L}_{yy} = -\frac{5}{2}x^{1/3}y^{-3/2}, \text{ so}$$

$$\mathcal{L}_{xx}\mathcal{L}_{yy} - \mathcal{L}_{xy}^{\ 2} = \frac{100}{18}x^{-4/3}y^{-1} - \frac{25}{9}x^{-4/3}y^{-1} > 0$$

so $(x, y) = (2, 3/2)$ solves the constrained maximization problem.

*Local Second Order Conditions*

For the two-variable one-constraint case, let

$$D(x,y) = (f_{xx}'' - \lambda g_{xx}'')(g_y')^2 - 2(f_{xy}'' - \lambda g_{xy}'')g_x'g_y' + (f_{yy}'' - \lambda g_{yy}'')(g_x')^2$$

and let $(x_0, y_0)$ be a stationary point of the Lagrangian,

(a) If $D(x_0, y_0) < 0$, then $(x_0, y_0)$ solves the local maximization problem.

(b) If $D(x_0, y_0) > 0$, then $(x_0, y_0)$ solves the local minimization problem.

The expression $D(x,y)$ is more easily remembered as the determinant of the "bordered Hessian":

$$D(x,y) = -\begin{vmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{xy} & \mathcal{L}_{yy} \end{vmatrix} = -\begin{vmatrix} 0 & g_x & g_y \\ g_x & f''_{xx} - \lambda g''_{xx} & f''_{xy} - \lambda g''_{xy} \\ g_y & f''_{xy} - \lambda g''_{xy} & f''_{yy} - \lambda g''_{yy} \end{vmatrix}$$

Generalization to more variables in more constraints will utilize this form of the expression.

<u>Example</u> Maximize/Minimize $3xy$ subject to $x^2 + y^2 = 8$. Earlier we found that the stationary points are $(2,2)$ and $(-2,-2)$ with $\lambda = 3/2$, and $(2,-2)$ and $(-2,2)$ with $\lambda = -3/2$.

We have

$$\begin{aligned} D(x,y) &= (-2\lambda)(4y^2) - 2(3)(2x)(2y) + (-2\lambda)(4x^2) \\ &= -8\lambda y^2 - 24xy - 8\lambda x^2 \end{aligned}$$

At $(2,-2)$ and $(-2,2)$, with $\lambda = -3/2$, $D(x,y) > 0$, so these are local minimums.

At $(2,2)$ and $(-2,-2)$, with $\lambda = 3/2$, $D(x,y) < 0$, so these are local maximums.

## 22.5 The Lagrange Multiplier Method ($n$-variables, $m$-equality constraints)

The basic ideas presented here apply to optimization problems involving more than two variables, and more than one constraint. Suppose the problem is

$$\begin{aligned} &\text{max / min} \quad f(x_1, x_2, \ldots, x_n) \\ &\text{subject to} \quad g_1(x_1, x_2, \ldots, x_n) = c_1, \\ &\qquad\qquad \ldots, \\ &\qquad\qquad g_m(x_1, x_2, \ldots, x_n) = c_m \end{aligned}$$

First, write down the Lagrangian

$$\mathcal{L}(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) - \sum_{j=1}^{m} \lambda_j (g_j(x_1, x_2, \ldots, x_n) - c_j)$$

and the F.O.C.,

$$\mathcal{L}'_1(x_1, x_2, \ldots, x_n) = f'_1(x_1, x_2, \ldots, x_n) - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x_1, x_2, \ldots, x_n)}{\partial x_1}$$

$$\ldots$$

$$\mathcal{L}'_n(x_1, x_2, \ldots, x_n) = f'_n(x_1, x_2, \ldots, x_n) - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x_1, x_2, \ldots, x_n)}{\partial x_n}$$

together with the constraints and solve to get potential optimum points.

Example 22.5.1 max/min $f(x,y,z) = x^2 + y^2 + z^2$

st $g_1(x,y,z) = x + 2y + z = 30$ and $g_2(x,y,z) = 2x - y - 3z = 10$

The Lagrangian is

$$\mathcal{L}(x,y,z) = x^2 + y^2 + z^2 - \lambda_1(x + 2y + z - 30) - \lambda_2(2x - y - 3z - 10)$$

The first order conditions are the five equations

$$\mathcal{L}_1'(x,y,z) = 2x - \lambda_1 - 2\lambda_2 = 0$$

$$\mathcal{L}_2'(x,y,z) = 2y - 2\lambda_1 + \lambda_2 = 0$$

$$\mathcal{L}_3'(x,y,z) = 2z - \lambda_1 + 3\lambda_2 = 0$$

$$x + 2y + z = 30$$

$$2x - y - 3z = 10$$

You can find the solution to be $x = 10$, $y = 10$, $z = 0$, $\lambda_1 = 12$, and $\lambda_2 = 4$.

*Second Order Conditions* The conditions for the general $n$-variable, $m$-constraints case will be taken as optional for this course. These are given to you in the appendix to these notes.

## *22.6 The Envelope Theorem*

We explicitly allow for other parameters by writing down the problem in the following way. Let $\mathbf{r} = (r_1, r_2, \ldots r_k)$

max (min) $f(\mathbf{x},\mathbf{r},\mathbf{c})$ subject to $g_1(\mathbf{x},\mathbf{r}) = c_1$, …, $g_m(\mathbf{x},\mathbf{r}) = c_m$

The Lagrangian is

$$\mathcal{L} = f(\mathbf{x},\mathbf{r},\mathbf{c}) - \lambda_1(g_1(\mathbf{x},\mathbf{r}) - c_1) - \lambda_2(g_2(\mathbf{x},\mathbf{r}) - c_2) - \ldots - \lambda_m(g_m(\mathbf{x},\mathbf{r}) - c_m)$$

The solutions are then written as $\mathbf{x}^*(\mathbf{r},\mathbf{c})$ and the value function is

$$f^*(\mathbf{r},\mathbf{c}) = f(\mathbf{x}^*(\mathbf{r},\mathbf{c}),\mathbf{r},\mathbf{c}).$$

The Envelope Theorem (proof omitted) is

$$\frac{\partial f^*(\mathbf{r},\mathbf{c})}{\partial r_i} = \frac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{r},\mathbf{c}),\mathbf{r},\mathbf{c})}{\partial r_i}, \text{ for all } i = 1, 2, \ldots, k.$$

$$\frac{\partial f^*(\mathbf{r},\mathbf{c})}{\partial c_j} = \frac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{r},\mathbf{c}),\mathbf{r},\mathbf{c})}{\partial c_j}, \text{ for all } j = 1, 2, \ldots, m.$$

Note that the envelope theorem implies that $\dfrac{\partial f^*(\mathbf{r},\mathbf{c})}{\partial c_j} = \dfrac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{r},\mathbf{c}),\mathbf{r},\mathbf{c})}{\partial c_j} = \lambda_j^*$.

Economic Interpretation of the Lagrange Multiplier:

As a first application, we consider again the interpretation of the Lagrange Multiplier. Consider

$$\text{max (min) } f(\mathbf{x},\mathbf{r}) \text{ subject to } g_1(\mathbf{x},\mathbf{r}) = c_1,\ \ldots,\ g_m(\mathbf{x},\mathbf{r}) = c_m$$

solved to obtain $\mathbf{x}^*(\mathbf{r},\mathbf{c})$ and the value function

$$f^*(\mathbf{r},\,\mathbf{c}) = f(\mathbf{x}^*(\mathbf{r},\,\mathbf{c}),\,\mathbf{r},\,\mathbf{c}) = f(x_1{}^*(\mathbf{r},\,\mathbf{c}),\,x_2{}^*(\mathbf{r},\,\mathbf{c}),\,\ldots,\,x_n{}^*(\mathbf{r},\,\mathbf{c}),\,\mathbf{r},\,\mathbf{c}\,)$$

The Lagrangian is

$$\mathcal{L} = f(\mathbf{x},\mathbf{r},\mathbf{c}) - \lambda_1(g_1(x,r)-c_1) - \lambda_2(g_2(x,r)-c_2) - \ldots - \lambda_m(g_m(x,r)-c_m)$$

so that

$$\frac{\partial f^*(\mathbf{r},\mathbf{c})}{\partial c_i} = \frac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{r},\mathbf{c}),\mathbf{r},\mathbf{c})}{\partial c_i} = \lambda_i^* \text{ for all } i = 1, 2, .., m.$$

The differential of $f^*(\mathbf{r},\mathbf{c})$ with respect to $\mathbf{c}$ is thus

$$df^* = \frac{\partial f^*(\mathbf{c})}{\partial c_1}dc_1 + \frac{\partial f^*(\mathbf{c})}{\partial c_2}dc_2 + \ldots + \frac{\partial f^*(\mathbf{c})}{\partial c_m}dc_m = \lambda_1{}^*dc_1 + \lambda_2{}^*dc_2 + \ldots + \lambda_m{}^*dc_m$$

which can be used to approximate the change in $f^*$ as a result of a change in $\mathbf{c}$ by

$$d\mathbf{c} = [dc_1 \quad dc_2 \quad \cdots \quad dc_m]$$

so each change $dc_i$ is "valued" at $\lambda_i$ , $i = 1, 2, \ldots, m$.

Example 22.6.1 min $C = rK + wL$ subject to $F(K, L) = Q$

The Lagrangian is $\mathcal{L}(r,w,\lambda) = rK + wL - \lambda(F(K,L) - Q)$ so that

$$\frac{\partial \mathcal{L}}{\partial r} = K,\ \frac{\partial \mathcal{L}}{\partial w} = L,\text{ and } \frac{\partial \mathcal{L}}{\partial Q} = \lambda\,.$$

so that

$$\frac{\partial\, C^*(K,L,r,w,Q)}{\partial r} = K^*,$$

$$\frac{\partial\, C^*(K,L,r,w,Q)}{\partial w} = L^*,$$

and

$$\frac{\partial\, C^*(K,L,r,w,Q)}{\partial Q} = \lambda^*$$

## Exercises

1. Using the Lagrange Multiplier Method, find $(x^*, y^*)$ that maximizes the function $U(x,y) = xy$ subject to the constraint $2x + y = m$. Find also $\lambda^*$. Use FOC only, as we have showed graphically in class that the solution is a global max. Show that $U^*(m) = U(x^*(m), y^*(m)) = m^2/8$. What is the change in the value of $U^*(m)$ when $m$ changes from $m = 2$ to $m = 3$?

Plot $U^*(m)$ and mark out the point on the graph where $m = 2$. Draw the tangent to the graph at the point $(2, U^*(2))$. Consider a change in the value of $m$ from $m = 2$ to $m = 3$. Mark out the distance indicating the actual change in the value of $U^*(m)$ when $m$ changes from $m = 2$ to $m = 3$. On the same diagram, mark out the linear approximation to the actual change. What is the value of this linear approximation to the actual change?

2. Find the values of $x$ and $y$ that maximizes the function $U(x,y) = x^2 y$, such that $3x + 4y = 72$, $x \geq 0$, $y \geq 0$. Do this by (a) writing the constraint as $y = 18 - 3x/4$ and substituting into $U(x,y)$, (b) using the Lagrange Multiplier Method.

In part (b), find also $\lambda^*$, the value of the Lagrange Multiplier corresponding to the max. pt, and also $U^* = U(x^*, y^*)$ where $(x^*, y^*)$ is your solution to the maximization problem.

Repeat the problem for $U(x,y) = x^2 y$, such that $3x + 4y = 73$, $x \geq 0$, $y \geq 0$, and find in particular $U^*$ in this case. What is the difference between this value of $U^*$ and the value of $U^*$ that you found with the constraint $3x + 4y = 72$? How does this difference compare with $\lambda^*$ that you found earlier?

3. Find the values of $x$ and $y$ that maximizes $f(x_1, x_2) = 2x_1 x_2 + 3x_1$ subject to $x_1 + 2x_2 = 83$. Do this by (a) substitution, and

(b) using the Lagrange Multiplier Method.

In part (b), find also $\lambda^*$, the value of the Lagrange Multiplier corresponding to the max. pt, and also $f^* = f(x_1^*, x_2^*)$ where $(x_1^*, x_2^*)$ is your solution to the maximization problem. What is the (linear) approximation to the change in $f^*$ if the constraint is changed to $x_1 + 2x_2 = 84$? If the constraint is changed to $x_1 + 2x_2 = 85$? If the constaint is changed to $x_1 + 2x_2 = 83.2$?

Repeat the problem with the constraint changed to $x_1 + 2x_2 = 84$. What is the actual change, and find in particular $U^*$ in this case. What is the difference between this value of $U^*$ and the value of $U^*$ that you found with the constraint $3x + 4y = 72$? How does this difference compare with $\lambda^*$ that you found earlier?

4. Solve the utility maximization problem $u(x, y) = 100x^{1/2}y^{1/4}$ subject to $px + qy = m$. Show that your Lagrangian function is concave, so your solution is a global maximum. Find $\partial x^* / \partial p$, $\partial x^* / \partial q$, $\partial x^* / \partial m$. Find $u^* = u(x^*, y^*)$. Find $\partial u^* / \partial p$, $\partial u^* / \partial q$, $\partial u^* / \partial m$.

Find $\partial \mathcal{L} / \partial p$, $\partial \mathcal{L} / \partial q$, and $\partial \mathcal{L} / \partial m$. Evaluate these partial derivatives at the optimum values of $x$ and $y$. Compare your results with $\partial u^* / \partial p$, $\partial u^* / \partial q$, $\partial u^* / \partial m$.

5. Solve the maximization problem

max $\quad x + a \ln y \quad$ subject to $px + qy = m$ (assume, $0 \le a < m / p$)

Find the function $f^*(a, p, q, m)$ and compute its partial derivatives wrt all four variables. Check if the results accord with the envelope theorem.

*Remark*: The optimized value of an objective function (i.e. $f^*$, $U^*$, $u^*$ in our previous questions) is called the <u>value function</u>. Thus, in the problem max/min $f(x, y)$, $f(x, y)$ is the objective function, $f^* = f(x^*, y^*)$ is called the value function.

6. Solve the maximization problem $x^{1/5}y^{2/5}z^{1/5}$ subject to $px + qy + rz = m$. (FOC only.)

7. Consider the problem

$$\text{minimize } x^2 + y^2 + z \text{ subject to } \begin{cases} x^2 + 2xy + y^2 + z^2 = a \\ x + y + z = 1 \end{cases}$$

where $a$ is a constant.

(a) Use the Lagrange Multiplier method to set up necessary conditions for a minimum.

(b) Find the solution when $a = 5/2$ (You can take it that the minimum exists.)

(c) The minimum value of $x^2 + y^2 + z$ depends on $a$. Let's call it $V(a)$. What is $V'(5/2)$?

8. A theory of demand for money says that an individual will choose $M$ (average money holdings per period) and $n$ (number of withdrawals per period) to minimize the cost of holding money subject to the constraint that the individual is able to cover all her expenses over the period. This can be formalized as the problem of choosing $M$ and $n$ to

minimize $nPf + iM$ subject to the constraint that $2nM = Py$

where $P$ are prices, $y$ is her planned consumption over the next period, $f$ is a real fixed cost per withdrawal, and $i$ is interest. (Total expenses for the period is $Py$.) All variables are positive.

(a) Write down the Lagrangian for this problem.

(b) Write down the FOC and solve it for the optimal $M^*$ (the "money demand").

(c) Show that the SOC for a (local) minimization problem is satisfied.

(d) Show that the interest elasticity of money demand, i.e. the (partial) elasticity of $M^*$ with respect to $i$, is equal to $-1/2$.

(e) Give an interpretation to the Lagrange Multiplier.

9. A consumer faces the following utility maximization problem

$$\max_{x,y} \; x^a + y \quad \text{subject to } px + y = m, \text{ with } 0 < a < 1$$

(a) Find the demand functions $x^*(p,m,a)$ and $y^*(p,m,a)$, and show using the second-order conditions for local optimality that you have a maximum.

(b) Find the partial derivatives of the demand functions with respect to $p$ and $m$ and check their signs.

(c) For the case $a = 1/2$, find an expression for the value function (i.e. the value of the utility function at the optimum level of consumption). Denote this by $V(p,m)$. Find $\partial V(p,m)/\partial p$ by differentiating the expression you found and verify that $\partial V(p,m)/\partial p = -x^*(p,m)$. Again, verify this using the Envelope Theorem.

10. Your production function is $K^{1/2}L^{1/2}$. Your cost function is $rK + wL$.

(a) How much $K$ and $L$ should you hire if you must produce $Q$ units at minimum cost? That is, find $K^*$ and $L^*$ to minimize

$$rK + wL \text{ subject to } K^{1/2}L^{1/2} = Q$$

Find also $\lambda^*$, the value of $\lambda$ when producing at the cost minimizing value of $K$ and $L$. Give an economic interpretation of $\lambda^*$.

(b) Find $\partial C^* / \partial r$, $\partial C^* / \partial w$, and $\partial C^* / \partial Q$. Do this two ways:

i. by substituting $K^*$ and $L^*$ into $C^* = rK^* + wL^*$, and then differentiating directly;

ii. using the Envelope Theorem.

**Appendix   Second Order Conditions**

For the general $n$-variable, $m$-constaints case, define the "Bordered Hessian" as

$$\left[\begin{array}{ccc:ccc} 0 & \cdots & 0 & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \\ \hdashline \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} & \mathcal{L}''_{11} & \cdots & \mathcal{L}''_{1n} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} & \mathcal{L}''_{1n} & \cdots & \mathcal{L}''_{nn} \end{array}\right]$$

Define, for $r = m+1, m+2, ..., m+n$

$$\overline{H}_r = \left|\begin{array}{ccc:ccc} 0 & \cdots & 0 & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_r} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_r} \\ \hdashline \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} & \mathcal{L}''_{11} & \cdots & \mathcal{L}''_{1r} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_r} & \cdots & \frac{\partial g_m}{\partial x_r} & \mathcal{L}''_{1r} & \cdots & \mathcal{L}''_{rr} \end{array}\right|$$

If $x^*$ is a stationary point, and if

(a) $(-1)^m \overline{H}_r > 0$ for $r = m+1, ..., n$ then $x^*$ is a local minimum;

(b) $(-1)^r \overline{H}_r > 0$ for $r = m+1, ..., n$ then $x^*$ is a local maximum.

where each of the $\overline{H}_r$ is evaluated at the point $x^*$.

For the 2-variable 1-constraint case

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda\,(g(x,y) - c)$$

the "Bordered" Hessian works out to be

$$\bar{H} = \begin{pmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{xy} & \mathcal{L}_{yy} \end{pmatrix} = \begin{pmatrix} 0 & g_x & g_y \\ g_x & f''_{xx} - \lambda g''_{xx} & f''_{xy} - \lambda g''_{xy} \\ g_y & f''_{xy} - \lambda g''_{xy} & f''_{yy} - \lambda g''_{yy} \end{pmatrix}$$

The second order condition then reduces to checking the sign of $\bar{H}_2$ which is the determinant

$$\bar{H}_2 = \begin{vmatrix} 0 & g_x & g_y \\ g_x & f''_{xx} - \lambda g''_{xx} & f''_{xy} - \lambda g''_{xy} \\ g_y & f''_{xy} - \lambda g''_{xy} & f''_{yy} - \lambda g''_{yy} \end{vmatrix}$$

If $(-1)^1 \bar{H}_2 > 0$, or equivalently $\bar{H}_2 < 0$, then $x^*$ is a local minimum.
If $(-1)^2 \bar{H}_2 > 0$, or equivalently $\bar{H}_2 > 0$, then $x^*$ is a local maximum.

Equivalently, let

$$D(x,y) = (f_{xx}'' - \lambda g_{xx}'')(g_y')^2 - 2(f_{xy}'' - \lambda g_{xy}'')g_x' g_y' + (f_{yy}'' - \lambda g_{yy}'')(g_x')^2$$

and let $(x_0, y_0)$ be a stationary point of the Lagrangian,

(a) If $D(x_0, y_0) < 0$, then $(x_0, y_0)$ solves the local maximization problem.
(b) If $D(x_0, y_0) > 0$, then $(x_0, y_0)$ solves the local minimization problem.

**Example** In the 4-variable, 1-constraint case ($n = 4$, $m = 1$), the Lagrangian is

$$\mathcal{L}(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3, x_4) - \lambda(g(x_1, x_2, x_3, x_4) - c)$$

For the second order condition, we have,

for $x^*$ to be a local minimum:

$$(-1)^1 \bar{H}_2 > 0, \quad (-1)^1 \bar{H}_3 > 0, \quad (-1)^1 \bar{H}_4 > 0,$$

i.e., $\bar{H}_2 < 0$, $\bar{H}_3 < 0$, $\bar{H}_4 < 0$;

for $x^*$ to be a local maximum:

$$(-1)^2 \bar{H}_2 > 0, \quad (-1)^3 \bar{H}_3 > 0, \quad (-1)^4 \bar{H}_4 > 0,$$

i.e., $\bar{H}_2 > 0$, $\bar{H}_3 < 0$, $\bar{H}_4 > 0$;

where

$$\bar{H}_2 = \begin{vmatrix} 0 & g_1' & g_2' \\ g_1' & \mathcal{L}_{11}'' & \mathcal{L}_{12}'' \\ g_2' & \mathcal{L}_{12}'' & \mathcal{L}_{22}'' \end{vmatrix},$$

$$\bar{H}_3 = \begin{vmatrix} 0 & g_1' & g_2' & g_3' \\ g_1' & \mathcal{L}_{11}'' & \mathcal{L}_{12}'' & \mathcal{L}_{13}'' \\ g_2' & \mathcal{L}_{12}'' & \mathcal{L}_{22}'' & \mathcal{L}_{23}'' \\ g_3' & \mathcal{L}_{13}'' & \mathcal{L}_{23}'' & \mathcal{L}_{33}'' \end{vmatrix},$$

$$\bar{H}_3 = \begin{vmatrix} 0 & g_1' & g_2' & g_3' & g_4' \\ g_1' & \mathcal{L}_{11}'' & \mathcal{L}_{12}'' & \mathcal{L}_{13}'' & \mathcal{L}_{14}'' \\ g_2' & \mathcal{L}_{12}'' & \mathcal{L}_{22}'' & \mathcal{L}_{23}'' & \mathcal{L}_{24}'' \\ g_3' & \mathcal{L}_{13}'' & \mathcal{L}_{23}'' & \mathcal{L}_{33}'' & \mathcal{L}_{34}'' \\ g_4' & \mathcal{L}_{14}'' & \mathcal{L}_{24}'' & \mathcal{L}_{34}'' & \mathcal{L}_{44}'' \end{vmatrix}$$

**Example** In the 4-variable, 2-constraint case ( $n = 4$ , $m = 2$ ), the Lagrangian is

$$\mathcal{L}(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3, x_4) - \lambda_1(g_1(x_1, x_2, x_3, x_4) - c_1) - \lambda_2(g_2(x_1, x_2, x_3, x_4) - c_2)$$

For the second order condition, we have,

for $x^*$ to be a local minimum:

$$(-1)^2 \bar{H}_3 > 0, \quad (-1)^2 \bar{H}_4 > 0, \quad \text{i.e.,} \quad \bar{H}_3 > 0, \quad \bar{H}_4 > 0;$$

for $x^*$ to be a local maximum:

$$(-1)^3 \bar{H}_3 > 0, \quad (-1)^4 \bar{H}_4 > 0, \quad \text{i.e.,} \quad \bar{H}_3 < 0, \quad \bar{H}_4 > 0;$$

where

$$\bar{H}_3 = \begin{vmatrix} 0 & 0 & \partial g_1 / \partial x_1 & \partial g_1 / \partial x_2 & \partial g_1 / \partial x_3 \\ 0 & 0 & \partial g_2 / \partial x_1 & \partial g_2 / \partial x_2 & \partial g_2 / \partial x_3 \\ \partial g_1 / \partial x_1 & \partial g_2 / \partial x_1 & \mathcal{L}_{11}'' & \mathcal{L}_{12}'' & \mathcal{L}_{13}'' \\ \partial g_1 / \partial x_2 & \partial g_2 / \partial x_2 & \mathcal{L}_{12}'' & \mathcal{L}_{22}'' & \mathcal{L}_{23}'' \\ \partial g_1 / \partial x_3 & \partial g_2 / \partial x_3 & \mathcal{L}_{13}'' & \mathcal{L}_{23}'' & \mathcal{L}_{33}'' \end{vmatrix},$$

$$\bar{H}_3 = \begin{vmatrix} 0 & 0 & \partial g_1 / \partial x_1 & \partial g_1 / \partial x_2 & \partial g_1 / \partial x_3 & \partial g_1 / \partial x_4 \\ 0 & 0 & \partial g_2 / \partial x_1 & \partial g_2 / \partial x_2 & \partial g_2 / \partial x_3 & \partial g_2 / \partial x_4 \\ \partial g_1 / \partial x_1 & \partial g_2 / \partial x_1 & \mathcal{L}_{11}'' & \mathcal{L}_{12}'' & \mathcal{L}_{13}'' & \mathcal{L}_{14}'' \\ \partial g_1 / \partial x_2 & \partial g_2 / \partial x_2 & \mathcal{L}_{12}'' & \mathcal{L}_{22}'' & \mathcal{L}_{23}'' & \mathcal{L}_{24}'' \\ \partial g_1 / \partial x_3 & \partial g_2 / \partial x_3 & \mathcal{L}_{13}'' & \mathcal{L}_{23}'' & \mathcal{L}_{33}'' & \mathcal{L}_{34}'' \\ \partial g_1 / \partial x_4 & \partial g_2 / \partial x_4 & \mathcal{L}_{14}'' & \mathcal{L}_{24}'' & \mathcal{L}_{34}'' & \mathcal{L}_{44}'' \end{vmatrix}.$$

**Exercise** Write out the Bordered Hessian for a constrained optimization problem with

(a) 3 choice variables and 1 constraint

(b) 4 choice variables and 3 constraints

**Example** min $(y+z-3)^2$ such that $x^2+y+z=2$ and $x+y^2+2z=2$

The Lagrangian is $\mathcal{L}=(y+z-3)^2-\lambda_1(x^2+y+z-2)-\lambda_2(x+y^2+2z-2)$

The FOCs are

(1) $\dfrac{\partial\mathcal{L}}{\partial x} = -2\lambda_1 x-\lambda_2 = 0$,

(2) $\dfrac{\partial\mathcal{L}}{\partial y} = 2(y+z-3)-\lambda_1-2\lambda_2 y = 0$,

(3) $\dfrac{\partial\mathcal{L}}{\partial z} = 2(y+z-3)-\lambda_1-2\lambda_2 = 0$,

(4) $\dfrac{\partial\mathcal{L}}{\partial\lambda_1} = x^2+y+z-2 = 0$.

(5) $\dfrac{\partial\mathcal{L}}{\partial\lambda_2} = x+y^2+2z-2=0$

You would solve these 5 equations for the 5 unknowns in your problem.

From (2) and (3), get $y=1$.

Substituting $y=1$ into (4) and (5), solve to get $x=1$ or $x=-1/2$.

From $(5)$, $y=1$, $x=1$, implies $z=0$.

From (5), $y=1$, $x=-1/2$, implies $z=3/4$.

When $(x,y,z)=(1,1,0)$, (1) and (2) implies $(\lambda_1,\lambda_2)=(4/3,-8/3)$.

When $(x,y,z)=(-1/2,1,3/4)$, (1) and (2) implies $(\lambda_1,\lambda_2)=(5/6,5/6)$.

For the second order condition: $m=2$, $n=3$. We only need to check the sign of

$$\overline{H}_3 = \begin{vmatrix} 0 & 0 & \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \partial g_1/\partial x_3 \\ 0 & 0 & \partial g_2/\partial x_1 & \partial g_2/\partial x_2 & \partial g_2/\partial x_3 \\ \partial g_1/\partial x_1 & \partial g_2/\partial x_1 & \mathcal{L}''_{11} & \mathcal{L}''_{12} & \mathcal{L}''_{13} \\ \partial g_1/\partial x_2 & \partial g_2/\partial x_2 & \mathcal{L}''_{12} & \mathcal{L}''_{22} & \mathcal{L}''_{23} \\ \partial g_1/\partial x_3 & \partial g_2/\partial x_3 & \mathcal{L}''_{13} & \mathcal{L}''_{23} & \mathcal{L}''_{33} \end{vmatrix}$$

We have $\overline{H}_3 = \begin{vmatrix} 0 & 0 & 2x & 1 & 1 \\ 0 & 0 & 1 & 2y & 2 \\ 2x & 1 & -2\lambda_1 & 0 & 0 \\ 1 & 2y & 0 & 2-2\lambda_2 & 2 \\ 1 & 2 & 0 & 2 & 2 \end{vmatrix}$

When $(x, y, z) = (1,1,0)$, and $(\lambda_1, \lambda_2) = (4/3, -8/3)$, this is

$$\bar{H}_3 \quad = \quad \begin{vmatrix} 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & -8/3 & 0 & 0 \\ 1 & 2 & 0 & 22/3 & 2 \\ 1 & 2 & 0 & 2 & 2 \end{vmatrix} \quad = \quad 48$$

so $(-1)^2 \bar{H}_3 > 0$ and $(-1)^3 \bar{H}_3 < 0$. This is a local minimum.

When $(x, y, z) = (-1/2, 1, 3/4)$, (1) and (2) implies $(\lambda_1, \lambda_2) = (5/6, 5/6)$.

$$\bar{H}_3 \quad = \quad \begin{vmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ -1 & 1 & -5/3_1 & 0 & 0 \\ 1 & 2 & 0 & 1/3 & 2 \\ 1 & 2 & 0 & 2 & 2 \end{vmatrix} \quad = \quad -15$$

so $(-1)^2 \bar{H}_3 < 0$ and $(-1)^3 \bar{H}_3 > 0$. This is a local maximum.