21. Unconstrained Multivariable Optimization

We begin with unconstrained optimization of functions of two variables, because this is the easiest case, and allows us to use geometric intuition to understand the results.

21.1 Definitions Optimal points for functions of two variables are defined similarly to that of functions of one variable. Suppose we have $z = f(x, y)$:

– the point (x_0, y_0) is a global maximum point for f if $f(x, y) \le f(x_0, y_0)$ for every (x, y) in the domain of the function. The value $f(x_0, y_0)$ is called the maximum value of the function.
– the point (x_0, y_0) is a strict global maximum point for f if $f(x, y) < f(x_0, y_0)$ for every (x, y) in the domain of the function, with $(x, y) \ne (x_0, y_0)$. The value $f(x_0, y_0)$ is called the strict maximum value of f.

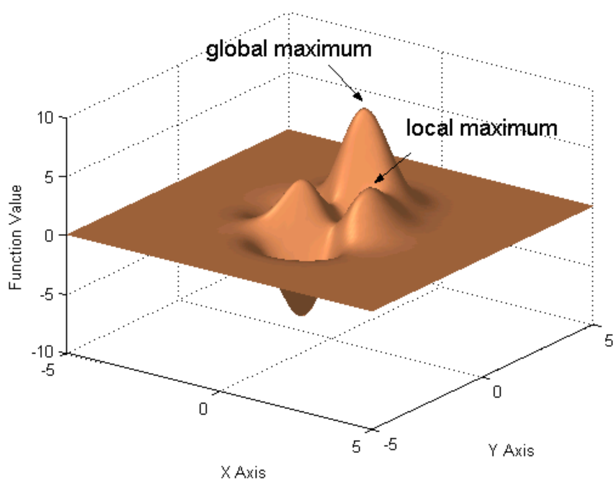
Minimum points are defined in similar fashion. When we do not need to distinguish between minimum points and maximum points, we can simply refer to them as optimal points or optima (sometimes also "extreme points"), and the corresponding values as optimal values

As in the univariate case, we can distinguish between global and local optimal points.

The point (x_0, y_0) is a local maximum point for f iff $f(x, y) \le f(x_0, y_0)$ for every (x, y) in some circle of radius r centered at (x_0, y_0). It is a local minimum point of the function iff

$$f(x, y) \ge f(x_0, y_0)$$

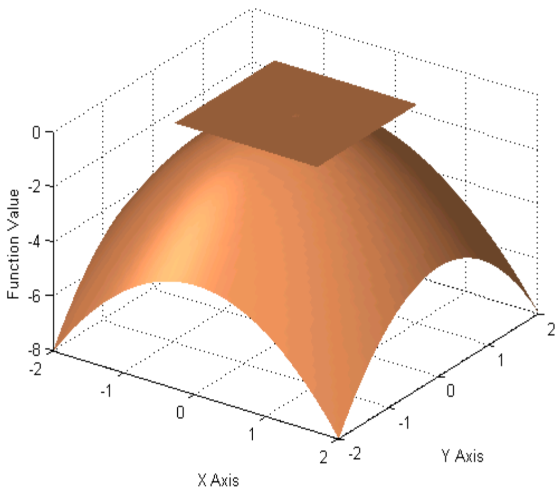
for every (x, y) in some circle of radius r centered at (x_0, y_0). Note that the radius is unspecified: all we need is that there be *some* r (it can be very small) such that (x_0, y_0) is optimal over a circle of radius r centered at (x_0, y_0).



21.2 <u>First and Second Order Conditions</u> Analogous to the one-variable case, we have necessary first-order conditions for optimal points of functions of two variables. In the one variable case, we have:

$$x_0 \text{ is an optimal point} \;\Rightarrow\; f'(x_0) = 0$$

That is, the slope of the tangent line should be zero at optimal points. The intuition for this is that if $f'(x_0) \neq 0$, then we will be able to attain a higher or lower value for $f(x)$ by taking larger or smaller values of x.



For functions of two variables, the intuition is similar. We require the directional derivative of the function to be zero in all directions. If this were not the case, then changing (x, y) along that direction (or in the opposite direction) will give a higher or lower value of $f(x, y)$. Put differently, if (x_0, y_0) is an optimum point, then the tangent plane at (x_0, y_0) should be zero-sloped in all directions.

The directional derivative in the "*h*-*k*" direction is $hf_1'(x_0, y_0) + kf_2'(x_0, y_0)$, and at the optimum point, we want this to be zero in all directions, meaning that

$$hf_1'(x_0, y_0) + kf_2'(x_0, y_0) = 0$$

for all possible values of h and k. Of course, it is impossible to literally check this conditions for each and every value of h and k. Instead, we note that

$$hf_1'(x_0, y_0) + kf_2'(x_0, y_0) = 0 \text{ for all } h, k \quad \Leftrightarrow \quad f_1'(x_0, y_0) = 0 \text{ and } f_2'(x_0, y_0) = 0$$

This should be obvious. If $f_1'(x_0, y_0) = 0$ and $f_2'(x_0, y_0) = 0$, then certainly $hf_1'(x_0, y_0) + kf_2'(x_0, y_0) = 0$ for any h and k. On the other hand, if either $f_1'(x_0, y_0) \neq 0$ or $f_2'(x_0, y_0) \neq 0$, say $f_1'(x_0, y_0) \neq 0$, then pick $h = 1$ and $k = 0$. Therefore:

Necessary First-Order Condition Suppose $z = f(x, y)$ is differentiable, and (x_0, y_0) is an interior point in the domain. Then

$$(x_0, y_0) \text{ is an optimal point} \Rightarrow f_1'(x_0, y_0) = f_2'(x_0, y_0) = 0.$$

[We will switch between $f_1'(x,y)$ and $f_x'(x,y)$ notations, whichever suits the occasion better. Points that satisfy $f_1'(x_0,y_0) = f_2'(x_0,y_0) = 0$ are often called stationary points of f .]

Example 21.2.1 Let $f(x,y) = 5 - x^2 + 6x - 2y^2 + 8y$. The stationary points are the points (x,y) such that

$$f_x' = -2x + 6 = 0 \qquad \text{and} \qquad f_y' = -4y + 8 = 0$$

i.e., The only candidate optimal point is $(x,y) = (3,2)$.

Example 21.2.2 Let $f(x,y) = x^3 + 2xy - 5x - y^2$. The candidate points are the points (x,y) such that

$$\text{(i)} \quad f_x' = 3x^2 + 2y - 5 = 0 \quad \text{and} \quad \text{(ii)} \quad f_y' = 2x - 2y = 0.$$

Solving we have from (ii) that $x = y$, so from (i)

$$3x^2 + 2x - 5 = 0 \Rightarrow (3x+5)(x-1) = 0 \Rightarrow x = -5/3 \text{ or } x = 1,$$

and the candidate optimal points are $(x,y) = (-5/3, -5/3)$ and $(x,y) = (1,1)$.

Example 21.2.3 Let $f(x,y) = x^2 - 2xy + y^2$. The stationary points are the points (x,y) such that

$$f_x' = 2x - 2y = 0 \text{ and } \qquad f_y' = -2x + 2y = 0.$$

Therefore, all points (x,y) such that $x = y$ are candidate optimum points.

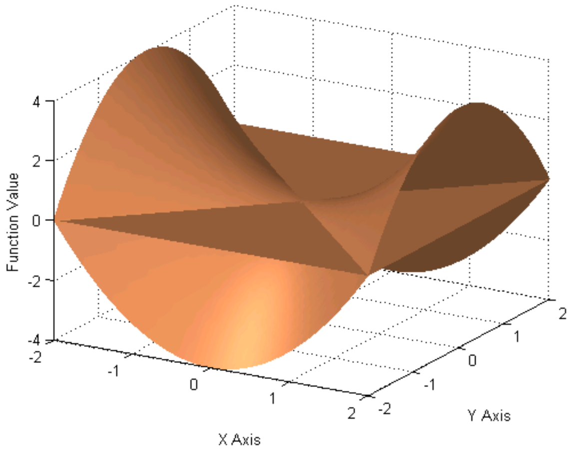
Example 21.2.4 Suppose a firm has production function $Q = F(K,L)$. Suppose the price of the product is p, the price of K is r, and price of L is w. The firm's profit function is therefore

$$\pi(K,L) = pF(K,L) - rK - wL.$$

Suppose F is differentiable and that π has a maximum with $K > 0$ and $L > 0$, then the necessary conditions for this maximum are

$$\pi_1'(K,L) = pF_1'(K,L) - r = 0 \quad \Rightarrow \quad pF_1'(K,L) = r$$
$$\pi_2'(K,L) = pF_2'(K,L) - w = 0 \quad \Rightarrow \quad pF_2'(K,L) = w$$

The candidate optimal points are any points (K,L) that satisfy the two equations above. We cannot solve for the candidate points explicitly; the candidate solutions are only characterized implicitly by the two equations.



As with functions of one variable, the first order conditions are only necessary conditions. For functions of one variable, we might have inflection points where $f'(x_0)=0$. The analogous case for functions of two (or more) variables are "saddle points" (see figure on the left for an example).

Example 21.2.5 Take $z = f(x,y) = x^2 - y^2$

$$\partial f / \partial x = 2x \text{ and } \partial f / \partial y = -2y,$$

so $(x,y)=(0,0)$ is a stationary point of the function. It is clear from its graph of the function, however, that this point is neither a maximum point nor a minimum point.

Local Second-Order Conditions Recall that for functions of one variable $f(x)$, we have

1) $f'(x_0)=0$ and $f''(x_0)<0 \quad \Rightarrow \quad f$ has a local maximum at $x=x_0$.
2) $f'(x_0)=0$ and $f''(x_0)>0 \quad \Rightarrow \quad f$ has a local minimum at $x=x_0$.
3) $f'(x_0)=0$ and $f''(x_0)=0 \quad \Rightarrow \quad ?$

The equivalent local second-order conditions for functions of two variables are:

Second Order Conditions for Local Optima Suppose that (x_0,y_0) is an interior stationary point of a twice-differentiable function $f(x,y)$. Then,

(a) If $f''_{11}(x_0,y_0)<0$ and $f''_{11}(x_0,y_0)f''_{22}(x_0,y_0)-f''_{12}(x_0,y_0)^2>0$, then (x_0,y_0) is a local max. point;

(b) If $f''_{11}(x_0,y_0)>0$ and $f''_{11}(x_0,y_0)''_{22}(x_0,y_0)-f''_{12}(x_0,y_0)^2>0$, then (x_0,y_0) is a local min. point;

(c) If $f''_{11}(x_0,y_0)f''_{22}(x_0,y_0)-f''_{12}(x_0,y_0)^2<0$, then (x_0,y_0) is a saddle point

(d) If $f''_{11}(x_0,y_0)f''_{22}(x_0,y_0)-f''_{12}(x_0,y_0)^2=0$, then ?

Note that the expression $f''_{11}(x_0,y_0)f''_{22}(x_0,y_0)-f''_{12}(x_0,y_0)^2$ is easy to remember as the determinant of the Hessian

$$\begin{bmatrix} f''_{11} & f''_{12} \\ f''_{12} & f''_{22} \end{bmatrix}$$

evaluated at the point (x_0,y_0).

We begin with (a): recall that for x_0 to be a maximum point of a function of one variable, we needed $f'(x_0) = 0$ and $f''(x_0) < 0$. For (x_0, y_0) to be a maximum point of a function of two variables, we also need the second-order derivative to be less than zero *in every direction*.

Let $g(t) = f(x_0 + th, y_0 + tk)$ for some fixed arbitrary values of h and k not both equal to zero. By the chain rule, we have

$$g''(0) = f''_{11}(x_0, y_0)h^2 + 2f''_{12}(x_0, y_0)hk + f''_{22}(x_0, y_0)k^2 .$$

This tells us the rate of change of the derivative along the direction (h,k). For a maximum point, that this second-order derivative has to be less than zero *in every direction* is therefore saying that $g''(0) < 0$ for every possible value of h and k (not both equal zero).

Sufficient conditions for $g''(0) < 0$ can be derived as follows: write $A = f''_{11}(x_0, y_0)$, $B = f''_{12}(x_0, y_0)$, and $C = f''_{22}(x_0, y_0)$, so that

$$\begin{aligned} g''(0) &= Ah^2 + 2Bhk + Ck^2 \\ &= A\left(h^2 + 2\frac{B}{A}hk + \frac{C}{A}k^2\right) \\ &= A\left[\left(h + \frac{B}{A}k\right)^2 + \frac{C}{A}k^2 - \frac{B^2}{A^2}k^2\right]. \\ &= A\left[\left(h + \frac{B}{A}k\right)^2 + \frac{CA - B^2}{A^2}k^2\right] \end{aligned}$$

This expression shows that $A < 0$ and $CA - B^2 > 0$, then the term in the square brackets is positive for all h and k, so that $g''(0) < 0$. The result in (b) follows similarly: if $A > 0$ and $CA - B^2 > 0$, then $g''(0) > 0$ for all h and k. Part (c) is trickier. Suppose $A > 0$ (the case $A < 0$ proceeds in a similar fashion.) If $CA - B^2 < 0$, then $g''(0) > 0$ when $h = 1$, $k = 0$, and $g''(0) < 0$ when h and k are chosen such that $h + Bk / A = 0$.

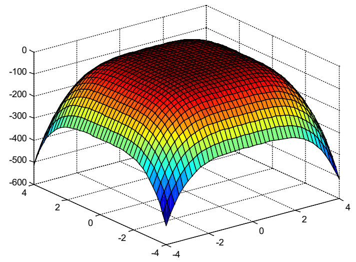
We'll give a counterexample to prove (d). Suppose

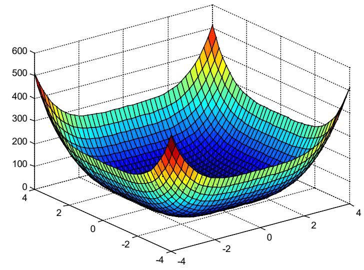
$$f(x,y) = -x^4 - y^4, \quad g(x,y) = x^4 + y^4, \quad h(x,y) = x^3 + y^3 .$$

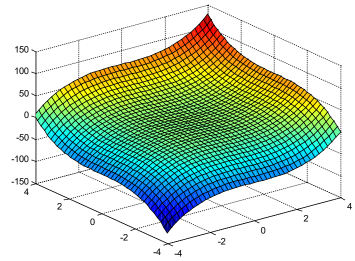
In all three cases, the only stationary point is at (0,0). We also have

$$f_{11}'' f_{22}'' - f_{12}''^2 = g_{11}'' g_{22}'' - g_{12}''^2 = h_{11}'' h_{22}'' - h_{12}''^2 = 0$$

at the point $(0,0)$. (Check these assertions on your own at home!) The figures below show that at (0,0), f has a maximum, g a mininum, and h has a saddle point.







Example 21.2.1 (continued) Let $f(x,y) = 5 - x^2 + 6x - 2y^2 + 8y$. The FOCs are

$$f'_x = -2x + 6 = 0 \text{ and } \quad f'_y = -4y + 8 = 0$$

so the only stationary point is $(x,y) = (3,2)$. The second order partial derivatives are $f''_{xx} = -2$, $f''_{yy} = -4$, and $f''_{xy} = f''_{yx} = 0$, therefore at the point $(3,2)$ we have

$$f''_{xx} = -2 < 0 \text{ and } \quad f''_{xx}f''_{yy} - f''^2_{xy} = (-2)(-4) = 8 > 0,$$

The point $(x,y) = (3,2)$ is a local maximum.

Example 21.2.2 (continued) Let $f(x,y) = x^3 + 2xy - 5x - y^2$. The FOCs are

$$\text{(i) } f'_x = 3x^2 + 2y - 5 = 0 \quad \text{and} \quad \text{(ii) } f'_y = 2x - 2y = 0,$$

and earlier we found the candidate optimal points are $(x,y) = (-5/3, -5/3)$ and $(x,y) = (1,1)$. The second order partial derivatives are $f''_{xx} = 6x$, $f''_{yy} = -2$, and $f''_{xy} = f''_{yx} = 2$, therefore at the point $(-5/3, -5/3)$ we have

$$f''_{xx} = 6(-5/3) = -10 < 0 \quad \text{and} \quad f''_{xx}f''_{yy} - f''^2_{xy} = (-10)(-2) - (2^2) = 16 > 0;$$

and at the point $(1,1)$ we have

$$f''_{xx} = 6(1) = 6 < 0 \quad \text{and} \quad f''_{xx}f''_{yy} - f''^2_{xy} = (6)(-2) - (2^2) = -16 < 0.$$

The point $(-5/3, -5/3)$ is a local maximum, whereas $(1,1)$ is a saddle point.

Example 21.2.3 (continued) Let $f(x,y) = x^2 - 2xy + y^2$. The FOCs are

$$f'_x = 2x - 2y = 0 \text{ and } \quad f'_y = -2x + 2y = 0$$

and all points (x,y) such that $x = y$ are candidate optimum points. The second order partial derivatives are $f''_{xx} = 2$, $f''_{xy} = -2$ and $f''_{yy} = 2$, so at the stationary points (in fact, at all points), we have

$$f''_{xx}(x,y)f''(x,y)_{yy} - f''_{xy}(x,y)^2 = 0.$$

The test is silent on the nature of these points. It should be obvious, nonetheless, that because $f(x,y) = x^2 - 2xy + y^2 = (x-y)^2$, all the stationary points are in fact global minimum points.

Example 21.2.5 (continued) Take $z = f(x,y) = x^2 - y^2$. The FOCs are

$$\partial f / \partial x \;=\; 2x \;=\; 0 \text{ and } \partial f / \partial y \;=\; -2y \;=\; 0,$$

so $(x, y) = (0,0)$ is a stationary point of the function. This point is a saddle point. The second order derivatives are $f''_{xx} = 2$, $f''_{yy} = -2$, and $f''_{xy} = 0$, so at the point $(0,0)$, (and in fact at all points) we have

$$f''_{xx}(x, y) f''_{yy}(x, y) - f''_{xy}(x, y)^2 = -4 < 0$$

<u>Example 21.2.6</u> Let $f(x, y) = x^3 - 3xy^2 + y^4$. The first order conditions are

$$f'_x = 3x^2 - 3y^2 = 0$$

$$f'_y = -6xy + 4y^3 = 0$$

You can show that the stationary points are $(x, y) = (0,0)$ or $(\frac{3}{2}, \frac{3}{2})$ or $(\frac{3}{2}, -\frac{3}{2})$.

Exercise: (a) Find all the second-order derivatives of the function

$$f(x, y) = x^3 - 3xy^2 + y^4$$

(b) Show that $(\frac{3}{2}, \frac{3}{2})$ and $(\frac{3}{2}, -\frac{3}{2})$ are minimum points, whereas the second order condition is silent over the point (0,0).

In some cases the shape of the function guarantees that the stationary point is either a global maximum or a global minimum. For example, consider $f(x, y) = -x^2 - y^2$ (pictured earlier). Functions like this are called "concave functions". For one-variable functions, we used the second derivative to characterize concavity and convexity. We can do the same for functions of two variables. For a twice-differentiable function:

$f''_{11}(x, y) \;<\; 0$, and $f''_{11}(x, y) f''_{22}(x, y) - (f''_{12}(x, y))^2 \;>\; 0$ for all x, y

$\Rightarrow$ f is a strictly concave function,

$f''_{11}(x, y) \;>\; 0$, and $f''_{11}(x, y) f''_{22}(x, y) - (f''_{12}(x, y))^2 \;>\; 0$ for all x, y

$\Rightarrow$ f is a strictly convex function,

$f''_{11}(x, y) \;\le\; 0$, $f''_{22}(x, y) \;\le\; 0$, and $f''_{11}(x, y) f''_{22}(x, y) - (f''_{12}(x, y))^2 \;\ge\; 0$ for all x, y

$\Leftrightarrow$ f is a concave function,

$f''_{11}(x, y) \;\ge\; 0$, $f''_{22}(x, y) \;\ge\; 0$, and $f''_{11}(x, y) f''_{22}(x, y) - (f''_{12}(x, y))^2 \;\ge\; 0$ for all x, y

$\Leftrightarrow$ f is a convex function,

In terms of second order conditions, this implies:

Second Order Conditions for Global Maximum and Minimum Points Suppose that (x_0, y_0) is an interior stationary point for a twice-differentiable function $f(x, y)$ defined over a convex set. Then if for all (x, y),

(a) $f''_{11}(x,y) < 0$, and $f''_{11}(x,y)f''_{22}(x,y) - (f''_{12}(x,y))^2 > 0$, then (x_0, y_0) is a strict global max point.

(b) $f''_{11}(x,y) > 0$, and $f''_{11}(x,y)f''_{22}(x,y) - (f''_{12}(x,y))^2 > 0$, then (x_0, y_0) is a strict global min. point.

(c) $f''_{11}(x,y) \le 0$, $f''_{22}(x,y) \le 0$, and $f''_{11}(x,y)f''_{22}(x,y) - (f''_{12}(x,y))^2 \ge 0$, then (x_0, y_0) is a global max. point.

(d) $f''_{11}(x,y) \ge 0$, $f''_{22}(x,y) \ge 0$, and $f''_{11}(x,y)f''_{22}(x,y) - (f''_{12}(x,y))^2 \ge 0$, then (x_0, y_0) is a global min. point.

Example 21.2.7 Suppose $z = f(x, y) = x^2 + y^2$. Find the global minimum point. The stationary points satisfy

$$f'_1(x,y) = 2x = 0 \text{ and } f'_2(x,y) = 2y = 0$$

so $(x, y) = (0,0)$ is the only stationary point of the function. We also have

$$f''_{11}(x,y) = 2,\ f''_{22}(x,y) = 2,\ f''_{12}(x,y) = 0.$$

Therefore for all (x, y),

$$f''_{11}(x,y) = 2 > 0, \text{ and } f''_{11}(x,y)f''_{22}(x,y) - f''_{12}(x,y)^2 = 4 > 0,$$

so this function is strictly convex, and $(x, y) = (0,0)$ is a strict global minimum point.

Example 21.2.1 (continued) Let

$$f(x,y) = 5 - x^2 + 6x - 2y^2 + 8y.$$

Earlier we found the only stationary point to be the point $(x, y) = (3,2)$. We found the second order partial derivatives to be $f''_{xx} = -2$, $f''_{yy} = -4$, and $f''_{xy} = f''_{yx} = 0$, therefore at all points (x, y) we have

$$f''_{xx} = -2 < 0 \quad \text{and} \quad f''_{xx}f''_{yy} - f''^2_{xy} = (-2)(-4) = 8 > 0.$$

This function is strictly concave, so $(x, y) = (3,2)$ is a strict global maximum point.

Example 21.2.2 (continued) Let

$$f(x,y) = x^3 + 2xy - 5x - y^2.$$

We found earlier that the stationary points are $(x,y)=(-5/3,-5/3)$ and $(x,y)=(1,1)$. The former is a local maximum, the latter is a saddle point. We also found the second order partial derivatives to be $f''_{xx}=6x$, $f''_{yy}=-2$, and $f''_{xy}=f''_{yx}=2$, so that

$$f''_{xx}f''_{yy}-f''^{2}_{xy} \ = \ -12x-4 \, .$$

This function is not concave or convex throughout its domain. This function is strictly concave only over the region where $x<-1/3$.

<u>Example 21.2.3 (continued)</u> Let $f(x,y)=x^2-2xy+y^2$. The stationary points are the points (x,y) such that

$$f'_x \ = \ 2x-2y \ = \ 0 \ \text{and} \qquad f'_y \ = \ -2x+2y \ = \ 0$$

i.e., all points (x,y) such that $x=y$ are candidate optimum points. The second order partial derivatives are $f''_{xx}=2$, $f''_{xy}=-2$ and $f''_{yy}=2$, so the functions satisfies

$$f''_{xx}\geq 0 \, , \ f''_{yy}\geq 0 \ \ f''_{xx}(x,y)f''_{yy}(x,y)-f''_{xy}(x,y)^2 \geq 0 \ \text{ for all } \ x,y$$

so the function is convex, but not strictly so [what does it look like?] Because the function is convex, all the stationary points are global minimum points.

Exercise Let $f(x,y)=e^{x^2+y^2}$. Determine the convexity/concavity of this function.

21.3 Multivariable Optimization: n Variables Consider now a function $y=f(x_1,x_2,...,x_n)$ of n -variables. We have

Necessary First-Order Condition Suppose $y=f(x_1,x_2,...,x_n)$ is differentiable, and $(x_1^*,x_2^*,...,x_n^*)$ is an interior point of the domain, then

$$(x_1^*,x_2^*,...,x_n^*) \text{ is an optimum point} \quad \Rightarrow$$

$$\left.\frac{\partial f}{\partial x_1}\right|_{(x_1^*,x_2^*,...,x_n^*)} \ = \ 0 \, , \ \left.\frac{\partial f}{\partial x_2}\right|_{(x_1^*,x_2^*,...,x_n^*)} \ = \ 0 \, , \ ..., \ \left.\frac{\partial f}{\partial x_n}\right|_{(x_1^*,x_2^*,...,x_n^*)} \ = \ 0$$

[I've switched notations on you again, $\left.\frac{\partial f}{\partial x_1}\right|_{(x_1^*,x_2^*,...,x_n^*)}$ refers to the first partial derivative of f with respect to x_1, evaluated at the point $(x_1^*,x_2^*,...,x_n^*)$.]

Second Order Conditions Recall that the second order condition for local optima involved the sign of

$$\begin{aligned} g''(0) &= f_{11}''(x_0, y_0)h^2 + 2f_{12}''(x_0, y_0)hk + f_{22}''(x_0, y_0)k^2 \\ &= \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} f_{11}'' & f_{12}'' \\ f_{12}'' & f_{22}'' \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} \qquad (*) \end{aligned}$$

where $g(t) = f(x(t), y(t))$, $x = x_0 + th$, $y = y_0 + tk$.

The conditions that guarantee a local minimum are conditions that guarantee that this expression is greater than zero for all nontrivial h and k. The conditions for a local maximum are conditions that guarantee that this expression is less than zero for all nontrivial h and k.

For instance, for a local minimum in the two variable case, we require

$$f_{11}''(x_0, y_0) > 0, \text{ and } f_{11}''(x_0, y_0)f_{22}''(x_0, y_0) - (f_{12}''(x_0, y_0))^2 > 0,$$

This can be written

$$\left| f_{11}'' \right| > 0, \text{ and } \begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{12}'' & f_{22}'' \end{vmatrix} > 0$$

where the partials are all evaluated at the point (x_0, y_0).

For a local maximum, we can likewise write the FOC as

$$\left| f_{11}'' \right| < 0, \text{ and } \begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{12}'' & f_{22}'' \end{vmatrix} > 0$$

where the partials are evaluated at the point (x_0, y_0).

Incidentally, matrix expressions such as that in (*) are called quadratic forms. If (*) is strictly greater that zero for all nontrivial h and k (i.e., not both zero), we say that the quadratic form is positive definite. If it is less that zero for all nontrivial h and k, we say that the quadratic form is negative definite.

The conditions in the n-variable case is the following: for $f(x_1, x_2, ..., x_n) = f(\mathbf{x})$, the Hessian is

$$f''(\mathbf{x}) = \begin{bmatrix} f_{11}''(\mathbf{x}) & f_{12}''(\mathbf{x}) & \cdots & f_{1n}''(\mathbf{x}) \\ f_{21}''(\mathbf{x}) & f_{22}''(\mathbf{x}) & \cdots & f_{2n}''(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}''(\mathbf{x}) & f_{n2}''(\mathbf{x}) & \cdots & f_{nn}''(\mathbf{x}) \end{bmatrix}$$

Let

$$|H_1| = f_{11}'', \quad |H_2| = \begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{12}'' & f_{22}'' \end{vmatrix}, \ |H_3| = \begin{vmatrix} f_{11}'' & f_{12}'' & f_{13}'' \\ f_{12}'' & f_{22}'' & f_{23}'' \\ f_{13}'' & f_{23}'' & f_{33}'' \end{vmatrix}, \text{ etc.}$$

These are called "leading principal minors" of the Hessian matrix.

Second Order Conditions for Local Optima If at the stationary point $\mathbf{x}_0$,

$$|H_1| < 0, \ |H_2| > 0, \ |H_3| < 0, ..., \ \begin{cases} |H_n| > 0 & \text{if } n \text{ is even} \\ |H_n| < 0 & \text{if } n \text{ is odd} \end{cases}$$

then $f(\mathbf{x})$ has a local maximum at $\mathbf{x}_0$; if at the stationary point $\mathbf{x}_0$

$$|H_1| > 0, \ |H_2| > 0, \ |H_3| > 0, ..., \ |H_n| > 0,$$

then $f(\mathbf{x})$ has a local minimum at the stationary point $\mathbf{x}_0$.

Second Order Conditions for Global Optima If for all $\mathbf{x}$,

$$|H_1| < 0, \ |H_2| > 0, \ |H_3| < 0, ..., \ \begin{cases} |H_n| > 0 & \text{if } n \text{ is even} \\ |H_n| < 0 & \text{if } n \text{ is odd} \end{cases}$$

then $f(\mathbf{x})$ is strictly concave, and achieves its strict global maximum at the stationary point $\mathbf{x}_0$; if for all $\mathbf{x}$,

$$|H_1| > 0, \ |H_2| > 0, \ |H_3| > 0, ..., \ |H_n| > 0,$$

then $f(\mathbf{x})$ is strictly convex, and achieves its strict global minimum at the stationary point $\mathbf{x}_0$.

We can rewrite the strict concavity condition as $(-1)^r |H_r| > 0$ for $r = 1, ..., n$.

<u>Example 21.3.1</u> Let $f(x, y, z) = 2x - x^2 + 10y - y^2 + 3 - z^2$. Then

$$f_x' = 2 - 2x, \ f_y' = 10 - 2y, \text{ and } f_z' = -2z,$$

so the only stationary point is $(x, y, z) = (1, 5, 0)$. Furthermore, the Hessian matrix is

$$\begin{bmatrix} f_{xx}'' & f_{xy}'' & f_{xz}'' \\ f_{xy}'' & f_{yy}'' & f_{yz}'' \\ f_{xz}'' & f_{yz}'' & f_{zz}'' \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and we have

$$|f_{xx}''| = -2 < 0, \quad \begin{vmatrix} f_{xx}'' & f_{xy}'' \\ f_{xy}'' & f_{yy}'' \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0, \begin{vmatrix} f_{xx}'' & f_{xy}'' & f_{xz}'' \\ f_{xy}'' & f_{yy}'' & f_{yz}'' \\ f_{xz}'' & f_{yz}'' & f_{zz}'' \end{vmatrix} = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -8 < 0$$

for all values of (x, y, z). The function f is therefore strictly concave, and $(1, 5, 0)$ is a strict global maximum point.

The version that allows for possibly non-strict concavity and non-strict convexity is as follows: Given an $(n \times n)$ matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

a principal minor of order r, denoted by $\Delta_r(\mathbf{A})$, is the determinant of the $(r \times r)$ matrix that remains when the same $n-r$ rows and columns are deleted. For instance, for a (3×3) matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The order 1 principal minors are $\Delta_1(\mathbf{A}) = |a_{11}|,\ |a_{22}|,\ |a_{33}|$

The order 2 principal minors are $\Delta_2(\mathbf{A}) = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

The order 3 principal minor is $\Delta_3(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

i.e. it is the determinant of the matrix itself.

The leading principal minors of a matrix are thus a subset of the principal minors.

Consider $f(x_1, x_2, ..., x_n)$, and its Hessian

$$f''(\mathbf{x}) = \begin{bmatrix} f_{11}''(\mathbf{x}) & f_{12}''(\mathbf{x}) & \cdots & f_{1n}''(\mathbf{x}) \\ f_{21}''(\mathbf{x}) & f_{22}''(\mathbf{x}) & \cdots & f_{2n}''(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}''(\mathbf{x}) & f_{n2}''(\mathbf{x}) & \cdots & f_{nn}''(\mathbf{x}) \end{bmatrix}$$

We have the following:

f is convex $\Leftrightarrow$ all $\Delta_r\left(f''(\mathbf{x})\right) \ge 0$ for all $\mathbf{x}$, $r = 1, 2, ..., n$

f is concave $\Leftrightarrow$ all $(-1)^r \Delta_r\left(f''(\mathbf{x})\right) \ge 0$ for all $\mathbf{x}$, $r = 1, 2, ..., n$

<u>Example 21.3.2</u> Let $f(x_1, x_2, x_3) = -x_1^2 + 6x_1x_2 - 9x_2^2 - 2x_3^2$. The Hessian matrix is

$$f''(x_1,x_2,x_3) = \begin{bmatrix} -2 & 6 & 0 \\ 6 & -18 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$$

The order 1 principal minors are

$$\Delta_1(f'') = |-2| < 0,\ |-18| < 0,\ |-4| < 0$$

The order 2 principal minors are

$$\Delta_2(f'') = \begin{vmatrix} -18 & 0 \\ 0 & -4 \end{vmatrix} > 0,\ \begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix} > 0,\ \begin{vmatrix} -2 & 6 \\ 6 & -18 \end{vmatrix} = 0$$

The order 3 principal minor is

$$\Delta_3(f'') = \begin{vmatrix} -2 & 6 & 0 \\ 6 & -18 & 0 \\ 0 & 0 & -4 \end{vmatrix} = 0.$$

We have

$$\text{all } (-1)\Delta_1(f'') \geq 0, \text{ all } (-1)^2\Delta_2(f'') \geq 0, \text{ and all } (-1)^3\Delta_3(f'') \geq 0,$$

therefore the function is concave. However, the leading principal minors are

$$|H_1| = |-2| < 0,\ |H_2| = \begin{vmatrix} -2 & 6 \\ 6 & -18 \end{vmatrix} = 0, \text{ and } H_3 = \begin{vmatrix} -2 & 6 & 0 \\ 6 & -18 & 0 \\ 0 & 0 & -4 \end{vmatrix} = 0$$

they do not meet the requirement for strict concavity. Does this mean that the function is not strictly concave?

21.4 The Envelope Theorem Suppose a firm chooses N to maximize profit

$$\pi(N,P,q) = PF(N) - qN$$

where P and q are prices, and N, the choice variable, is the quantity of an input to be used. Note that this is a univariate optimization problem, not a multivariate one. Even though the profit function is written as a function of N, p, and q, we are choosing N, taking p and q as fixed. To simplify matters further, assume $P = 1$, and $F(N) = \sqrt{N}$, so

$$\pi(N,q) = N^{1/2} - qN$$

In general, the problem to be solved will depend on a number of parameters. In our simplified example, the only parameter is q. Solving this problem for a fixed value of q we have

FOC N^* satisfies $\left.\frac{\partial \pi}{\partial N}\right|_{N^*} = (1/2)N^{*-1/2} - q = 0$, or $N^* = (2q)^{-2}$

$$\text{SOC} \quad \frac{\partial^2 \pi}{\partial N^2} = -(1/4)N^{-3/2} < 0$$

so $N^* = (2q)^{-2}$ maximizes profits. If the firm chooses this optimal level of input, then its profits will be

$$\pi^*(q) = \pi(N^*(q), q) = \sqrt{N^*} - qN^* = \frac{1}{2q} - \frac{1}{4q} = \frac{1}{4q}.$$

This is the firm's maximum profits, for any given level of q. It cannot do better. The optimal level N^* depends on the parameter q, and therefore the maximum profit levels also depend on q.

In comparative statics, we are interested in two questions: how do the optimal choices change when the parameters change, and how the optimal value of the objective function changes when the parameter changes. In the context of this example, it is how N^* changes with q, and how π^* changes with q. In this example, we can simply take derivatives. We have for N^*:

$$\frac{dN^*}{dq} = -2(2q)^{-3}(2) = -\frac{1}{2q^3} < 0,$$

and for π^*, take $d\pi^*(q)/dq$, i.e., take $d\pi(N^*(q), q)/dq$:

$$\frac{d\pi^*(q)}{dq} = \frac{d\pi(N^*(q), q)}{dq} = -\frac{1}{4q^2} < 0.$$

(as q increases, the maximum profit falls, as expected).

In more general examples, simply taking derivatives might not be feasible. When we use general functions, for example, we often find that we cannot solve for the optimal choices explicitly (for instance, if we did not specify $F(N) = \sqrt{N}$, but left it simply as $F(N)$. We have spent a considerable amount of time developing methods to implicitly differentiate the optimal choices with respect to the parameters.

This section is concerned with a method to simplify (in some cases, make feasible) the differentiating of the optimal value of the objective function with respect to the parameters.

Coming back to our specific example, what we did to get $d\pi^*(q)/dq$ was to substitute the optimal solution into the objective function, then differentiate, i.e., we did $d\pi^*(q)/dq = d\pi(N^*(q), q)/dq$. We computed the value function, then differentiated that. Note that we would have obtained the same expression for $d\pi^*(q)/dq$ if we had first differentiated the objective (profit) function $\pi(N,q) = N^{1/2} - qN$ with respect to q, and then evaluated the resulting expression at the optimal level of N^*. We have

$$\frac{\partial \pi(N,q)}{\partial q} = -N .$$

Evaluating this expression at N^*, we obtain

$$\frac{\partial \pi(N^*,q)}{\partial q} = -N^* = -\frac{1}{4q^2}$$

That is, $\dfrac{\partial \pi(N^*,q)}{\partial q} = \dfrac{d\pi^*(q)}{dq}$.

This is not a coincidence, but a consequence of the "Envelope Theorem", and it applies in general. The result is perhaps surprising, and worth considering in greater detail.

Applying the chain rule to $\dfrac{d\pi(N^*(q),q)}{dq}$ gives:

$$\frac{d\pi(N^*(q),q)}{dq} = \pi_1'(N^*(q),q)\frac{dN^*}{dq} + \pi_2'(N^*(q),q)\,\cancel{\frac{dq}{dq}}$$

There are two effects: even if I were to hold my optimal choice fixed at the pre-change value of q, when q changes I should expect my profit levels to change (this is the second term). But because my optimal choice depends on q, when q changes, my optimal choice changes, which should then also change profits (this is the first term).

However, we know that $\partial \pi / \partial N^*$, i.e., the derivative of π evaluated at the optimum, is zero. Therefore the first term is zero, and we have

$$\frac{d\pi(N^*(q),q)}{dq} = \pi_2'(N^*(q),q)$$

What happens is this: because $\partial \pi / \partial N = 0$ at the optimum N^*, the effect of a change in N^* on π becomes negligible as we consider smaller changes in q. In the limit, we can ignore this 'indirect' effect, and we only need to compute the direct effect of the parameter of the objective function.

The envelope theorem is very useful in that it can greatly simplify the computation of the derivative of the value function. Sometimes, there is no other way except to appeal to the envelop theorem. If our problem was to

$$\max \quad \pi(N,P,q) = PF(N) - qN$$

where $F(N)$ was not specified, then we cannot explicitly solve for N^*. If we cannot solve for N^* explicitly, we cannot substitute it into the objective function to get an explicit value function to differentiate. However, with the envelope theorem (and if the requisite assumptions for the existence of a solution are met), we can still say that

$$\frac{d\pi^*(q)}{dq} = \frac{\partial \pi(N^*,q)}{\partial q} = -N^*.$$

[Incidentally] why is the Envelope Theorem called the Envelope Theorem?

… because of a neat geometric feature of the value function. Consider the example

$$\min_x f(x,a) = x^2 - ax + a^2$$

[Note that I put $f(x,a)$ to emphasize the presence of the parameter, and because I want later to differentiate with respect to a. The notation "$\max_x$" means maximize with respect to x. As far as the maximization problem is concerned, the problem is exactly the same as if I had written $\max f(x) = x^2 - ax + a^2$]

The FOC is $\partial\pi / \partial x = 2x - a = 0$ so the solution is $x^* = a/2$.

The SOC is $\partial^2\pi / \partial x^2 = 2 > 0$ so x^* is a global minimum point.

The minimum value of the function is then

$$f(x^*(a),a) = (a/2)^2 - a(a/2) + a^2 = 3a^2/4.$$

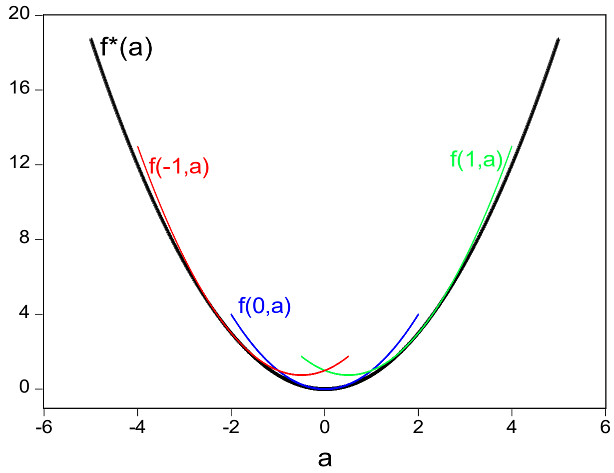
As noted before, this is called the 'value function', and we write $f^*(a) = f(x^*(a),a)$.

The envelope theorem says that $df^*(a)/da = \partial f(x^*(a),a)/\partial a$. In words, the slope of the value function (wrt a) is the same as the slope of the partial derivative of the objective function $f(x,a)$ wrt a, evaluated at the maximum $x = x^*(a)$.

In the following diagram, I plot $f^*(a)$ as well as $f(x,a)$ for three values of x (-1, 0, and 1). The envelope theorem is so named because the value function $f^*(a)$ (bold line) 'envelopes' $f(x,a)$ for any given value of x (three are shown, as thin lines -- red, blue, and green if reading in color -- corresponding to $x=-1$, $x=0$ and $x=1$).

Note that when $a = 0$, $x^* = 0$. The envelope theorem says that the slope of the value function (bold line) at $a = 0$ is the same as the slope of the objective function $f(x, a)$ wrt a (thin line) when $x = x^* = 0$.

Similar statements can be made for the cases $a = 2$ ($x^* = 1$), $a = -2$ ($x^* = -1$), and so on



The envelope theorem can be stated for the general multivariable optimization problem: Suppose we have

$$\text{Maximize } f(x_1, \dots, x_n, p_1, \dots, p_k)$$

where the x's are the choice variables.

We carry out the maximization, and get $x_1^*(p_1, \dots, p_k)$, …, $x_n^*(p_1, \dots, p_k)$ as our solution. Putting the x^*'s into our objective function gives the value function

$$f^*(p_1, \dots, p_k) = f(x_1^*(p_1, \dots, p_k), \dots, x_n^*(p_1, \dots, p_k), p_1, \dots, p_k)$$

The envelope theorem states that

$$\frac{\partial f^*(p_1, \dots, p_k)}{\partial p_i} = \left. \frac{\partial f(x_1(p_1, \dots, p_k), \dots, x_n(p_1, \dots, p_k), p_1, \dots, p_k)}{\partial p_i} \right|_{x_1^*, x_2^*, \dots, x_n^*}.$$

Example 21.4.1 Let the production function be $Q = F(K, L)$, Q is output, K is capital input, L is labor input. Price per unit of Q is p, price per unit of K is r, price per unit of L is w.

Suppose that all the assumptions are made that ensure that a profit maximizing solution exists, and let the solutions be K^* and L^*. We want to see how

$$\pi^*(p, w, r) = \pi(K^*, L^*, p, w, r)$$

changes with p, w, and r (i.e. how does the firms maximized profits change with prices of the good or of the factors change). Of course, we will also be interested in how K^* and L^* changes with prices p, w, and r, but we focus on profits in this example).

Profit function is $\pi(K,L,p,r,w) = pF(K,L) - rK - wL$, which gives

$$\frac{\partial \pi(K,L,p,r,w)}{\partial p} = F(K,L),$$

$$\frac{\partial \pi(K,L,p,r,w)}{\partial r} = -K,$$

$$\frac{\partial \pi(K,L,p,r,w)}{\partial w} = -L,$$

The envelope theorem then tells us that

$$\frac{\partial \pi^*(p,r,w)}{\partial p} = F(K^*,L^*) = Q^*,$$

$$\frac{\partial \pi^*(p,r,w)}{\partial r} = -K^*,$$

$$\frac{\partial \pi^*(p,r,w)}{\partial w} = -L^*.$$

Note that we didn't make any assumptions about $F(K,L)$ (except to assume that there is a solution to the maximization problem.) Notice also that didn't even solve the optimization problem, and yet we can say something sensible about how maximized profits change with prices.

Exercises

1. For each of the following functions, find and characterize all optima points

(a) $f(x,y)=(x-2)^2+(y+1)^2$

(b) $f(x,y)=1-x^2-y^2$

(c) $f(x,y)=e^{xy}$

(d) $f(x,y)=xy-x^3-y^3$

2. Let $f(x,y)=\ln(1+x^2y)$, with $x\in\mathbb{R}$ and $y>0$. Find all the stationary points of $f(x,y)$ and characterize them (max or min or saddle, local or global).

3. The function

$$f(x,y,z) \;=\; 100-2x^2-y^2-3z-xy+e^{x+y+z}.$$

has one stationary point. Find it, and show that it is a global maximum point.

4. Show that the function $f(x,y)=(1+y)^3x^2+y^2$ has a local minimum at $(x,y)=(0,0)$, but that it has no global minimum.

The goal of the next five questions is to help you understand the Envelope Theorem. The optimization problems here are all optimization of functions of one variable.

5. a. Let $f(x)=x+1/x$, $x>0$. What is the value of x, x^*, that minimizes $f(x)$? What is the value of $f(x)$ at x^*?

b. Let $f(x)=2x+1/x$, $x>0$. What is the value of x, x^*, that minimizes $f(x)$? What is the value of $f(x)$ at x^*?

c. Let $f(x,b)=bx+1/x$, $x>0$, and where b is a positive constant. What is the value of x, x^*, that minimizes $f(x)$ for any given b? What is $f(x,b)$ at x^*?

(Comment: The answer to the first question in part (c) should be a function of b, that is, $x^*=x^*(b)$. The answer to the second question in part (c) should also be a function of b, in particular, you are finding $f(x^*(b),b)$. The function $f^*(b)=f(x^*(b),b)$ is called the value function).

d. Evaluate $x^*(b)$ and $f^*(b)$ at $b=1$ and at $b=2$. Verify that you get the same answers as in parts (a) and (b) respectively.

e. Compute $\frac{dx^*}{db}$ and $\frac{df^*}{db}$. Interpret your results

f. Given $f(x,b)=bx+1/x$, find $\partial f/\partial b$. Evaluate this partial derivative at $x=x^*(b)$, i.e, find $f_2'(x^*(b),b)$. Verify that $f_2'(x^*(b),b)=df^*/db$.

6. Let $f(x,p)=(100x-px^2)^{1/2}$.

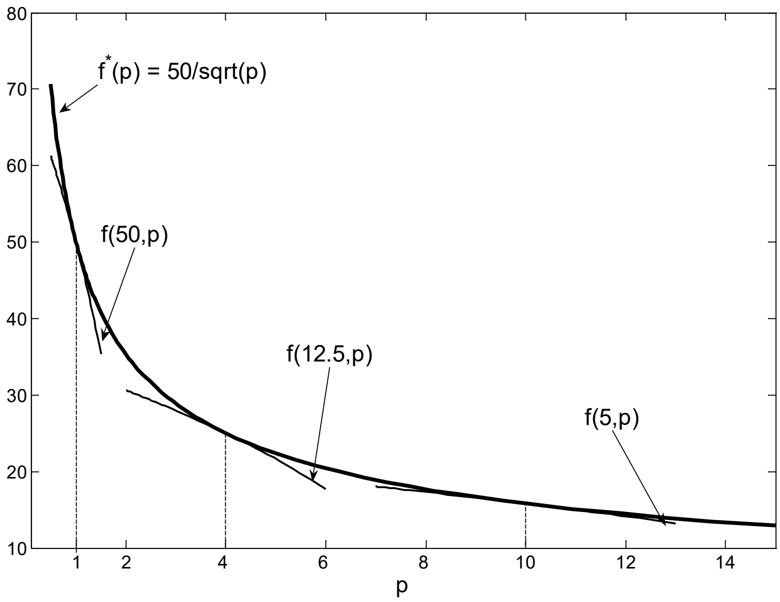
a. Find x^* such that x^* maximizes $f(x,p)$ for any given $p>0$.

b. Find dx^*/dp. Interpret this derivative.

c. Find $f^*(p)=f(x^*(p),p)$. Find df^*/dp. Interpret this derivative.

d. Find $\partial f(x,p)/\partial p$. Evaluate this derivative at $x=x^*(p)$. Verify that you get the same expression as in part (c). Explain why this is the case, by differentiating $f(x^*(p),p)$ using the chain rule.

e. Draw on the same figure the functions $f^*(p)$, and $f(x,p)$ for $x=50$, 12.5, 5. (This is done for you below).



Note that $x^*=50$ when $p=1$, $x^*=12.5$ when $p=4$, $x^*=5$ when $p=10$.

7. A farmer has the production function $Y = \sqrt{N}$, $N > 0$, where N is the amount of fertilizer put into the production process, and Y is the farmer's output. Fertilizer costs 1 dollar per pound, and each unit of output fetches a price of p dollars. Given p, the farmer chooses N to maximize profit

$$\pi(N,q) = p\sqrt{N} - N.$$

(a) Find the profit maximizing level of N, $N^*(p)$. Find the value function $\pi^*(p) = \pi(N^*(p), p)$. How much fertilizer should the farmer use if $p = 4$? If $p = 8$? What is the farmer's profit when $p = 4$ and when $p = 8$, assuming the farmer maximizes profits?

(b) Find $d\pi^*(p)/dp$.

(c) Find $\partial\pi(N,p)/\partial p$. Evaluate this derivative at $N = N^*(p)$. Compare this derivative with the derivative in part (b).

8. A farmer has the production function $Y = \sqrt{N}$, $N > 0$, where N is the amount of fertilizer put into the production process, and Y is output. Fertilizer costs q dollar per pound, and each unit of output fetches a price of p dollars. Given p and q, the farmer chooses N to maximize profit

$$\pi(N,p,q) = p\sqrt{N} - qN.$$

(a) Find the profit maximizing level of N, $N^*(p,q)$. Find the value function

$$\pi^*(p,q) = \pi(N^*(p,q), p, q).$$

How much fertilizer should the farmer use if $p = 4$ and $q = 1$? If $p = 8$ and $q = 1$? If $p = 8$ and $q = 2$? What are the farmer's profits at those values of p and q, assuming the farmer maximizes profits?

(b) Find $\partial\pi^*(p,q)/\partial p$ and $\partial\pi^*(p,q)/\partial q$.

(c) Find $\partial\pi(N,p,q)/\partial p$ and $\partial\pi(N,p,q)/\partial q$. Evaluate these derivatives at $N = N^*(p,q)$. Compare these derivatives with the derivatives obtained in part (b).

9. A farmer has the production function $Y = f(N)$, $N > 0$, $f'(N) > 0$, and $f''(N) < 0$, where N is the amount of fertilizer put into the production process, and Y is the farmer's output. Fertilizer costs q dollar per pound, and each unit of output fetches a price of p dollars. Given p and w, the farmer chooses N to maximize profits

$$\pi(N, p, q) = pf(N) - qN .$$

a. Show that the profit maximizing choice of N, $N^*(p,q)$ satisfies the condition

$$f'(N^*(p,q)) = \frac{q}{p}$$

b. Determine the sign of $\partial N^* / \partial p$ and $\partial N^* / \partial q$.

c. Find $\partial \pi^*(p,q)/\partial p$ and $\partial \pi^*(p,q)/\partial q$ in terms of N^*.

10. Let $f(x,y)$ be such that $f'_x > 0$, $f'_y > 0$, $f''_{xx} < 0$, $f''_{yy} < 0$, and $f''_{xx}f''_{yy} - f''^2_{xy} > 0$. Consider the problem

$$\max\ \pi(x,y) = p\, f(x,y) - q_1 x - q_2 y$$

where p, q_1, q_2 are all positive. Let $x^*(p,q_1,q_2)$ and $y^*(p,q_1,q_2)$ be the stationary points of the function $\pi(x,y)$.

(a) Explain why the stationary points of the function solve the maximization problem (show that the SOC for maximization is satisfied).

(b) Find $\partial x^* / \partial q_1$ and $\partial y^* / \partial q_1$. Show that given our assumptions, that the former is negative, but the sign of the latter cannot be determined (what does it depend on?).

A function $g(x,y)$ is said to be homogenous of degree k if $g(tx,ty) = t^k g(x,y)$ for any $t > 0$.

(c) Show that if $g(x,y)$ is homogenous of degree k, then

$$xg'_1(x,y) + yg'_2(x,y) = kg(x,y)$$

(d) Show that both $x^*(p,q_1,q_2)$ and $y^*(p,q_1,q_2)$ are homogenous of degree zero, i.e., show that for any $t > 0$,

$$x^*(tp,tq_1,tq_2) = x^*(p,q_1,q_2) \quad \text{and} \quad y^*(tp,tq_1,tq_2) = y^*(p,q_1,q_2).$$

(e) Is the value function $\pi^*(p,q_1,q_2)$ also homogenous? If so, to what degree?

(f) Find $\frac{\partial \pi^*}{\partial p}$, $\frac{\partial \pi^*}{\partial q_1}$, and $\frac{\partial \pi^*}{\partial q_2}$. Use the latter two expressions to show that $\frac{\partial x^*}{\partial q_2} = \frac{\partial y^*}{\partial q_1}$.

11. Suppose a firm chooses x_1 and x_2 to maximize

$$\pi = pf(x_1, x_2) - w_1 x_1 - w_2 x_2 \text{, where } p > 0, w_1 > 0, w_2 > 0,\ x_1 > 0,\ x_2 > 0.$$

Assume that $f(x_1, x_2) > 0$, $f_1' > 0$, $f_2' > 0$, $f_{11}'' < 0$, $f_{22}'' < 0$, $f_{11}'' f_{22}'' - (f_{12}'')^2 > 0$ for all x_1, x_2. Let the optimum solution to this maximization problem be x_1^* and x_2^*.

(i) Write down the first order conditions for this problem. Are the second order conditions for a global maximum satisfied?

(ii) Show that the derivatives of x_1^* and x_2^* with respect to p satisfies

$$pf_{11}'' \frac{\partial x_1^*}{\partial p} + pf_{12}'' \frac{\partial x_2^*}{\partial p} = -f_1'$$

$$pf_{21}'' \frac{\partial x_1^*}{\partial p} + pf_{22}'' \frac{\partial x_2^*}{\partial p} = -f_2'$$

Write this system of equations in matrix form.

(iii) Solve the equations in (ii) using Cramer's rule to obtain expressions for

$$\frac{\partial x_1^*}{\partial p} \text{ and } \frac{\partial x_2^*}{\partial p}.$$

Explain why their signs depend on the value of f_{12}''.

(iv) Let $\pi^*(p, w_1, w_2) = \pi(x_1^*, x_2^*, p, w_1, w_2)$. What are the signs of $\partial \pi^* / \partial p$ and $\partial \pi^* / \partial w_1$?

12. Suppose a firm is able to choose the price P of his product. Furthermore, the firm can influence the demand for its product by spending A on advertising. Suppose demand for the product is

$$Q = (\alpha - P)A^{\beta} \text{, with } \alpha > 0,\ 0 < \beta < 1.$$

(i) Find the elasticity of demand Q with respect to A.

Suppose that the cost of producing one unit of the good is c. (We will assume that $c < \alpha$.) The profit function of the firm is therefore

$$\pi = (P - c)(\alpha - P)A^{\beta} - A.$$

(ii) Find A^* and P^* (the profit maximizing levels of A and P) and show that the second-order condition for local minimum is satisfied at this solution.

(iii) Find the derivatives of A^*, P^*, and π^* with respect to β.

13(a) Show that the function

$$f(x,y) = 6x^2e^y - 4x^3 - e^{6y}$$

has only one stationary point, and that this stationary point is a local maximum, but not a global maximum.

(b) Find and classify the stationary points of the function

$$f(x,y,z) = x^2y + 2y^2z - 4z - 2x\,.$$

(c) Find and classify all the stationary points of the function

$$f(x,y,z) = 5x^2 + 5y^2 + 9z^2 - 6xz - 12yz\,.$$

14. Find the stationary points of

$$f(x,y) = (x^2 - axy)e^y$$

and classify them. Let (x^*, y^*) be a stationary point of $f(x,y)$ and let $f^*(a) = f(x^*, y^*)$. Find $df^*(a)/da$ by

(i) differentiating $f(x^*, y^*)$ directly (i.e. without using the Envelope Theorem) and

(ii) using the Envelope Theorem.