20. Multivariable Linear Approximations, and Differentials

The concepts of differentials and linear approximations can be extended to functions of many variables.

20.1 Linear Approximations The linear approximation to a function $z = f(x, y)$ at point $(x, y) = (x_0, y_0)$ is the tangent plane to the function at that point. This tangent plane is the plane that passes through the point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$, and has the same derivatives there as the function in every direction (and in particular the *x*- and *y*- directions).

The equation of a plane is, in general,

$$p(x, y) = \alpha + \beta x + \delta y$$

We want the tangent to satisfy

$$\begin{aligned} p(x_0, y_0) &= f(x_0, y_0), \\ p_x'(x_0, y_0) &= f_x'(x_0, y_0), \\ p_y'(x_0, y_0) &= f_y'(x_0, y_0). \end{aligned}$$

Therefore

$$p_x' = \beta = f_x'(x_0, y_0) \text{ and } p_y' = \delta = f_y'(x_0, y_0).$$

Also, $p(x_0, y_0) = \alpha + f_x'(x_0, y_0)\, x_0 + f_y'(x_0, y_0)\, y_0 = f(x_0, y_0)$, which gives

$$\alpha = f(x_0, y_0) - f_x'(x_0, y_0)\, x_0 - f_y'(x_0, y_0)\, y_0$$

Substituting all this into $p(x, y)$ above gives

$$p(x, y) = f(x_0, y_0) + f_x{}'(x_0, y_0)(x - x_0) + f_y{}'(x_0, y_0)(y - y_0)$$

We use this function as the linear approximation to $f(x, y)$ at the point (x_0, y_0). That is to say, for all (x, y) near (x_0, y_0), we make the approximation

$$f(x, y) \approx p(x, y) = f(x_0, y_0) + f_x{}'(x_0, y_0)(x - x_0) + f_y{}'(x_0, y_0)(y - y_0).$$

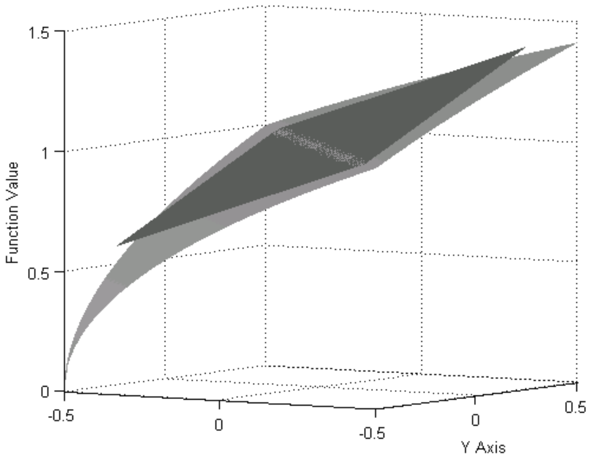
Example 20.1.1 Find the linear approx. for $f(x,y)=\sqrt{1+x+y}$ at the point $(x,y)=(0,0)$. We have $f(0,0)=1$, and as

$$f_x' = \frac{1}{2\sqrt{1+x+y}} = f_y' ,$$

so that $f_x'(0,0)=f_y'(0,0)=1/2$. The linear approximation to the function at $(0,0)$ is thus

$$z = 1+\frac{1}{2}(x-0)+\frac{1}{2}(y-0) = 1+\frac{1}{2}(x+y).$$

The function (curved surface) and the linear approximation (tangent plane) is shown below.



Remarks: The linear approximation to the function $f(x,y)$ at the point $(x,y)=(x_0,y_0)$ can be written in matrix notation as

$$\begin{aligned} f(x,y) \approx p(x,y) &= f(x_0,y_0)+f_x'(x_0,y_0)(x-x_0)+f_y'(x_0,y_0)(y-y_0) \\ &= f(x_0,y_0) + \begin{bmatrix} f_x'(x_0,y_0) & f_y'(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \\ &= f(x_0,y_0) + \mathbf{f}_0' (\mathbf{x}-\mathbf{x}_0) \end{aligned}$$

where $\mathbf{f}_0' = \begin{bmatrix} f_x' \\ f_y' \end{bmatrix}$ evaluated at the point (x_0,y_0), $\mathbf{x}=\begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{x}_0=\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. We can generalize to functions of more variables in the obvious way.

The <u>quadratic approximation</u> is given (in matrix notation) by

$$f(x,y) \approx q(x,y) = f(x_0,y_0) + \mathbf{f}_0' (\mathbf{x}-\mathbf{x}_0) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)'\mathbf{H}_0(\mathbf{x}-\mathbf{x}_0)$$

where $\mathbf{H}_0$ is the hessian matrix $\begin{bmatrix} f_{xx}'' & f_{xy}'' \\ f_{yx}'' & f_{yy}'' \end{bmatrix}$ evaluated at (x_0,y_0). *(Proof omitted)*

20.2 Differentials Recall that linear approximations can be expressed in terms of changes. Let $y = f(x)$ be the function to be approximated, and consider a small change in x from $x = x_0$ to $x = x_0 + dx$. The actual change in f is $f(x_0 + dx) - f(x_0)$. The linear approximation to this change is

$$p(x_0 + dx) - p(x_0) \ = f'(x_0)dx .$$

We denote the linear approximation to the actual change as "df", i.e.,

$$df = p(x_0 + dx) - p(x_0) = f'(x_0)\, dx .$$

For arbitrary x, we have $df = f'(x)\, dx$. The quantities 'df' and 'dx' are called the differentials of f and x respectively. If we write the function as $y = f(x)$, we can write $dy = f'(x)dx$.

For <u>functions of two variables</u>, taking the linear approximation of $f(x+dx, y+dy)$ at any point (x, y) gives

$$\begin{aligned} f(x+dx, y+dy) &\approx f(x,y) + f_x'(x+dx-x) + f_y'(y+dy-y) \\ &= f(x,y) + f_x' dx + f_y' dy \end{aligned}$$

Thus,

$$\begin{aligned} f(x+dx, y+dy) - f(x,y) &\approx f_x'(x+dx-x) + f_y'(y+dy-y) \\ &= f_x' dx + f_y' dy \end{aligned}$$

We call $f_x' dx + f_y' dy$ the <u>total differential</u> of f and give it the symbol df (or dz if the function is written $z = f(x, y)$). That is,

$$\boxed{dz \ = \ f_x' dx + f_y' dy}$$

This formula can be used to approximate changes in z as a result of small changes in x and y by the amounts dx and dy respectively.

NOTE again the conceptual difference between the (partial) derivatives and the (total) differential. The partial derivatives are the slopes of the function in particular directions. The differential is made up of four items, the two partials and the two quantities dx and dy. It is a formula for taking the linear approximation to the change in the function value when x and y are increased by dx and dy respectively.

Example 20.2.1 Earlier, we considered the function $z = f(x,y) = \sqrt{1+x+y}$, and found that

$$f_x' = \frac{1}{2\sqrt{1+x+y}} = f_y'$$

Therefore, the differential is

$$dz = \frac{dx}{2\sqrt{1+x+y}} + \frac{dy}{2\sqrt{1+x+y}} = \frac{1}{2\sqrt{1+x+y}}(dx+dy).$$

At the point $(x,y) = (0,0)$, we have $f_x'(0,0) = f_y'(0,0) = 1/2$, so

$$dz = \frac{1}{2}dx + \frac{1}{2}dy$$

At the point $(x,y) = (1,2)$, we have $f_x'(1,2) = f_y'(1,2) = \frac{1}{2\sqrt{1+1+2}} = \frac{1}{4}$, so

$$dz = \frac{1}{4}dx + \frac{1}{4}dy$$

As an illustration, consider a change in (x,y) from $(0,0)$ to $(0.1,0.1)$, i.e., $dx = dy = 0.1$. At $(0,0)$, $f(x,y) = f(0,0) = 1$. Because

$$f(0.1,0.1) = \sqrt{1+0.1+0.1} = 1.0954 \text{ (4 dec pl.)},$$

the actual change in z is 0.0954. Using the differential formula, at $(0,0)$, we have

$$dz = \frac{dx}{2\sqrt{1+x+y}} + \frac{dy}{2\sqrt{1+x+y}} = \frac{0.1}{2\sqrt{1+0+0}} + \frac{0.1}{2\sqrt{1+0+0}} = 0.1$$

which is our linear approximation to the actual change. The error is 0.0046.

For functions of n-variables, all these ideas remain valid: if $z = f(x_1, x_2, ..., x_n)$, then

$$dz = df = f_1' dx_1 + f_2' dx_2 + ... + f_n' dx_n.$$

Example 20.2.2 Determine the total differential for $z = x y^2 + x^3$. We have

$$\frac{\partial z}{\partial x} = y^2 + 3x^2, \text{ and } \frac{\partial z}{\partial y} = 2xy.$$

Therefore $dz = (y^2 + 3x^2)\, dx + 2xy\, dy$.

20.3 Rules for Differentials Although differentials and derivatives are different concepts, they contain the same information, and we can in fact differentiate expressions using the language of differentials. That is, we can use "rules for differentials" and differentiate expressions using these rules.

e.g. If $z = xy$, then $dz = y\,dx + x\,dy$.

Proof: $\partial z / \partial x = y$, $\partial z / \partial y = x$. Since $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$, we have $dz = y\,dx + x\,dy$.

Many of you are in fact familiar with this, and other similar expressions, and some of you were taught to memorize rules for differentiation using these expressions. (Now you know that these expressions are not simply mnemonics, i.e. memorization tricks, but have real mathematical meaning.)

For every differentiation rule, we can derive (and use) a corresponding "differential" rule.

1. If $z = ax + by$ where a and b are constants, then $dz = a\,dx + b\,dy$;

2. If $z = xy$, then $dz = y\,dx + x\,dy$;

3. If $z = \frac{x}{y}$, then $dz = \frac{y\,dx - x\,dy}{y^2}$;

4. If $z = x^r$, then $dz = rx^{r-1}dx$;

5. If $z = \ln x$, then $dz = \frac{dx}{x}$;

These rules can be used in conjunction with each other:

e.g. If $z = y \ln x$, then $dz = y\,d(\ln x) + dy \ln x = y\left(\frac{dx}{x}\right) + dy \ln x = \frac{y}{x}dx + \ln x\,dy$.

You can show this by computing the partial derivatives directly, and then constructing the differential from its definition:

since $\frac{\partial z}{\partial x} = \frac{y}{x}$, $\frac{\partial z}{\partial y} = \ln x$, and $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$, we have $dz = \frac{y}{x}dx + \ln x\,dy$.

<u>Example 20.3.1</u> Find dz in terms of dx and dy if $z = xy^2 + x^3$.

Using rules for differentials gives

$$\begin{aligned} dz &= d(xy^2) + d(x^3) \\ &= xd(y^2) + y^2 dx + 3x^2 dx \\ &= 2xydy + y^2 dx + 3x^2 dx \\ &= (3x^2 + y^2)dx + 2xydy \end{aligned}.$$

In simple examples, it is probably easier to simply take the partial derivatives, but there are times when it is 'cleaner' to "take differentials".

The convenience of "taking differentials" really comes into play with the chain rule:

Suppose $z = f(y)$, and $y = g(x)$. This means that $z = f(g(x))$, so the differential (wrt x) is $dz = [f(g(x))]' \, dx$. The chain rule gives us $[f(g(x))]' = f'(g(x))g'(x)$, so we have

$$dz = f'(g(x))g'(x)dx .$$

We can obtain this expression using differentials. From $z = f(y)$, we have $dz = f'(y)\, dy$, and $y = g(x)$ gives us $dy = g'(x)\, dx$. Simply substituting the latter into the former gives

$$dz = f'(y)g'(x)dx = f'(g(x))g'(x)dx .$$

The usefulness is in the fact that the expression $dz = f'(y)\, dy$ is valid regardless of whether y *is a function of any other variable.*

E.g. If $z = xy^2$. Then $dz = x\, d(y^2) + y^2 dx = y^2 dx + 2xy\, dy$.

If $x = s \ln t$ and $y = s + t^2$, then to find dz, simply take

$$dx = \ln t \, ds + \frac{s}{t} dt \quad \text{and} \quad dy = ds + 2t\, dt$$

and substitute into the previous expression to find dz with respect to ds and dt

$$\begin{aligned} dz &= y^2 dx + 2xy\, dy = (s+t^2)^2 [\ln t\, ds + \frac{s}{t} dt] + 2(s \ln t)(s+t^2)[ds + 2tdt] \\ &= (s+t^2)[(s+t^2)\ln t + 2s \ln t]ds + (s+t^2)[(s+t^2)\frac{s}{t} + 4st \ln t]\, dt \end{aligned}$$

In fact, from here you can read off the partial derivatives:

$$\frac{\partial z}{\partial s} = (s+t^2)[(s+t^2)\ln t + 2s\ln t] \quad \text{and} \quad \frac{\partial z}{\partial t} = (s+t^2)[(s+t^2)\frac{s}{t} + 4st\ln t].$$

We can apply all these ideas to systems of equations.

<u>Example 20.3.2</u> Suppose

$$u^2 v - u = x^3 + 2y^3$$

and

$$e^{ux} = vy$$

where u and v are endogenous, and x and y are exogenous. In other words, u and v are functions of x and y. You can easily show that the point $(x,y,u,v) = (0,1,2,1)$ satisfies both equations. What are the partial derivatives of u and v with respect to x and to y?

Taking differentials, we have

$$\begin{aligned} 2u\,du\,v + u^2 dv - du &= 3x^2 dx + 2(3y^2 dy) \\ e^{ux}(udx + xdu) &= vdy + ydv \end{aligned}$$

Rewriting gives

$$\begin{aligned} (v\,2u - 1)\ du \ + \ (u^2)\ dv &= 3x^2\ dx \ + \ 6y^2\ dy \\ xe^{ux}\ du \ - \ y\ dv &= -ue^{ux}\ dx \ + \ v\ dy \end{aligned}$$

In principle, we can solve these two equations in two unknowns (the unknowns are du and dv) in terms of dx, dy, u, v, x and y. The expressions will be, in this case, rather complicated (perhaps best expressed, and left, in matrix notation.

Instead of obtaining the general solution, which we are not looking for, we apply this to the point $(x,y,u,v) = (0,1,2,1)$. We get

$$\begin{aligned} 3\,du \ + \ 4\,dv &= 0\ dx \ + \ 6\,dy \\ 0\,du \ - \ 1\,dv &= -2\,dx \ + \ 1\,dy \end{aligned}$$

So $dv = 2dx - dy$ and

$$du = (1/3)[-4dv + 6dy] = (1/3)[-4(2dx - dy) + 6dy] = (1/3)[-8dx + 10dy]$$

In other words, at the point $(x,y,u,v) = (0,1,2,1)$,

$$\partial v/\partial x = 2\,,\ \partial v/\partial y = -1\,,\ \partial u/\partial x = -8/3\,, \text{ and } \partial u/\partial y = 10/3\ .$$

If we are interested in the general expression for the derivatives, but are interested only in differentiating with respect to x, we can simplify the general problem by setting $dy = 0$. For example, in the general case, we would simply solve

$$\begin{aligned} (v\,2u - 1)\ du \ + \ (u^2)\ dv &= 3x^2\ dx \\ xe^{ux}\ du \ - \ y\ dv &= -ue^{ux}\ dx \end{aligned}$$

Example 20.3.3   Consider the following macroeconomic model

$$C = c(Y,r),\ L = l(Y,r),\ I_V = h(Y,r)$$
$$Y = C + I + G,\ \ I = I_V + I_B,\ M = L$$

where f, g, and h are functions. The endogenous variables are Y (national income), r (interest rates), L (money demand), I_V (private investment), C (consumption), and I (total investment). The exogenous variables are M (money supply), I_B (public investment), G (public consumption)

If I_B and M are held fixed, what is the effect of an increase in G on the endogenous variables?

Differentiating gives

$$dC = c'_Y dY + c'_r dr, \qquad dL = l'_Y dY + l'_r dr, \qquad dI_V = h'_Y dY + h'_r dr,$$
$$dY = dC + dI + dG, \qquad dI = dI_V + dI_B, \qquad dM = dL.$$

Substituting, gives

$$\begin{aligned} dY &= [c'_Y dY + c'_r dr] + [(h'_Y dY + h'_r dr) + dI_B] + dG \\ &= (c'_Y + h'_Y)dY + (c'_r + h'_r)dr + dI_B + dG \end{aligned}$$

$$dM = l'_Y dY + l'_r dr$$

Because we are interested in an increase in G only, and are holding I_B and M fixed, let us set $dI_B = dM = 0$. Then the two equations become (with some rewriting)

$$[1-(c'_Y + h'_Y)]dY - (c'_r + h'_r)dr = dG \qquad \text{and} \qquad l'_Y dY + l'_r dr = 0$$

The second equation gives

$$dr = -\frac{l'_Y}{l'_r}dY,$$

and subs. into the first gives

$$[1-(c'_Y + h'_Y)]dY + (c'_r + h'_r)(l'_Y / l'_r)dY = dG \text{, so}$$

$$[1-(c'_Y + h'_Y) + (c'_r + h'_r)(l'_Y / l'_r)]dY = dG \text{, i.e.,}$$

$$dY = \left[\frac{l'_r}{l'_r(1-c'_Y - h'_Y) + l'_Y(c'_r + h'_r)}\right]dG$$

Subs into $dr = -(l'_Y / l'_r)dY$ gives $dr = -\left[\dfrac{l'_Y}{l'_r(1-c'_Y - h'_Y) + l'_Y(c'_r + h'_r)}\right]dG$.

The other effects can be calculated from

$dI = dI_V = h'_Y dY + h'_r dr$, $dL = l'_Y dY + l'_r dr$, and $dC = c'_Y dY + c'_r dr$.

Exercises

1. Suppose $Y = AK^{\alpha}L^{\beta}$. Differentiate (using differentials) to show that

$$\frac{dY}{Y} = \alpha\frac{dK}{K} + \beta\frac{dL}{L}.$$

2. Assume that the equation system

$$x^2 + sxy + y^2 - 1 = 0$$
$$x^2 + y^2 - s^2 + 3 = 0$$

defines x and y implicitly as differentiable functions of s.

(a) Differentiate the system (using differentials) and find the values of $\frac{dx}{ds}$ and $\frac{dy}{ds}$ when $x = 0$, $y = 1$, $s = 2$.

(b) Find an approximate value of the change in x if s increases from 2 to 2.1.

3. The system of equations

$$\ln(x+u) + uv - y^2e^{v+y} = 0$$
$$u^2 - x^v = v$$

defines u and v as differentiable functions of x and y around the point $P(x,y,u,v) = (2,1,-1,0)$.

(i) Differentiate the system (using differentials);

(ii) Find the values of the partial derivatives u'_x, u'_y, v'_x, v'_y;

(iii) Find an approximate value of $u(1.99,1.02)$, i.e., find the approximate value of u at the point $(x,y) = (1.99,1.02)$.

4. Let $Y = C + I$, $C = f(Y-T,r)$, and $I = h(r)$.

(a) Differentiate all three equations to obtain three equations relating dY, dC, dI, dT, and dr.

(b) Solve the three equations to obtain expressions for dY, dC, and dI, each in terms of dT and dr only.

(c) Write down expressions for $\partial Y/\partial T$, $\partial C/\partial T$, $\partial I/\partial T$, $\partial Y/\partial r$, $\partial C/\partial r$, and $\partial I/\partial r$.