19. Chain Rule for Functions of Many Variables, and Applications

19.1 The Chain Rule

Suppose we have a function of n variables

$$y = F(x_1, x_2, ..., x_n),$$

where each of the arguments is itself a function of m variables t_1, t_2, …, t_m, i.e.,

$$x_i = f_i(t_1, t_2, ..., t_m),\ i = 1, 2, \ldots, n.$$

What is the effect on y of a change in t_j, holding t_1, …, t_{j-1}, t_{j+1}, …, t_m fixed?

When there is a change in t_j, this results in changes in x_1, x_2, …, and x_n. Changes in each of these will in turn result in changes in y. The chain rule states that the overall effect on y is simply the sum of all these individual effects. More concisely,

$$\frac{\partial y}{\partial t_j} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial t_j} + ... + \frac{\partial y}{\partial x_n}\frac{\partial x_n}{\partial t_j}.$$

<u>Example 19.1.1</u> Let $z = F(x,y) = x + y^2$, $x = t$, $y = t^2$.

Then
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (1)(1) + (2y)(2t) = 1 + 4yt = 1 + 4t^3$$

<u>Example 19.1.2</u> Let $z = F(x,y) = x + y^2$, $x = t - s$, $y = ts$

Then
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (1)(1) + (2y)(s) = 1 + 2ys = 1 + 2ts^2$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (1)(-1) + (2y)(t) = -1 + 2yt = -1 + 2t^2 s$$

<u>Example 19.1.3</u> Suppose t denotes time, and $L(t) = L_0 e^{r_1 t}$ and $K(t) = K_0 e^{r_2 t}$ (in other words, L and K are growing at a constant rate over time, in particular,

$$\frac{d\ln L(t)}{dt} = \frac{dL(t)/dt}{L(t)} = r_1 \quad \text{and} \quad \frac{d\ln K(t)}{dt} = \frac{dK(t)/dt}{K(t)} = r_2.$$

Now suppose that $Y = F(L,K) = AL^{\alpha}K^{1-\alpha}$. At what rate is $Y(t)$ growing?

$$\ln Y = \ln F(L,K) = \ln A + \alpha \ln L + (1-\alpha)\ln K$$

Therefore,

$$\frac{d\ln Y}{dt} = \frac{\partial F(L,K)}{\partial \ln L}\frac{d\ln L}{dt} + \frac{\partial F(L,K)}{\partial \ln K}\frac{d\ln K}{dt} = \alpha r_1 + (1-\alpha) r_2.$$

We prove the chain rule for the simple case where $n = 2$, $m = 1$. The argument here is very similar to the chain rule for univariate functions that we proved in a previous class. Suppose $z = F(x, y)$, $x = f(t)$, $y = g(t)$. Write $\phi(t) = F(f(t), g(t))$. Then

$$\begin{aligned}\frac{d\phi}{dt} &= \lim_{\Delta t \to 0} \frac{\phi(t+\Delta t) - \phi(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(f(t+\Delta t), g(t+\Delta t)) - F(f(t), g(t))}{\Delta t}\end{aligned}$$

Define $\Delta x = f(t+\Delta t) - f(t) = f(t+\Delta t) - x$

$\Delta y = g(t+\Delta t) - g(t) = g(t+\Delta t) - y$,

and substitute this into the expression above.

$$\begin{aligned}\frac{d\phi}{dt} &= \lim_{\Delta t \to 0} \frac{F(x+\Delta x, y+\Delta y) - F(x, y)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(x+\Delta x, y+\Delta y) + F(x, y+\Delta y) - F(x, y+\Delta y) - F(x, y)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y)}{\Delta t} + \lim_{\Delta t \to 0} \frac{F(x, y+\Delta y) - F(x, y)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y)}{\Delta x} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \frac{F(x, y+\Delta y) - F(x, y)}{\Delta y} \frac{\Delta y}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y)}{\Delta x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \frac{F(x, y+\Delta y) - F(x, y)}{\Delta y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}\end{aligned}$$

Note that $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$ and

$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t} = \frac{dy}{dt}$.

Note also that $\Delta x \to 0$ and $\Delta y \to 0$ as $\Delta t \to 0$, therefore

$$\lim_{\Delta t \to 0} \frac{F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y)}{\Delta x} = F_1'(x, y)$$

and $\lim_{\Delta t \to 0} \frac{F(x, y+\Delta y) - F(x, y)}{\Delta y} = F_2'(x, y)$.

This gives us the chain rule.

Example 19.1.4 *(Second derivative with the Chain Rule)*

Let $z = f(x,y)$, $x = x(t)$, $y = y(t)$. Then

$$\frac{dz}{dt} = f_x' \frac{dx}{dt} + f_y' \frac{dy}{dt}$$

Note that f_x' and f_y' are themselves functions of x and y. Therefore

$$\frac{d^2z}{dt^2} = \underbrace{\frac{\partial f_x'}{dt}\frac{dx}{dt} + f_x' \frac{d(dx/dt)}{dt}}_{\text{applying product rule to } f_x' \frac{dx}{dt}} + \underbrace{\frac{\partial f_y'}{dt}\frac{dy}{dt} + f_y' \frac{d(dy/dt)}{dt}}_{\text{applying product rule to } f_y' \frac{dy}{dt}}$$

$$= \underbrace{\left[f_{xx}'' \frac{dx}{dt} + f_{xy}'' \frac{dy}{dt} \right]}_{\text{applying chain rule to } f_x'} \frac{dx}{dt} + f_x' \frac{d^2x}{dt^2} + \underbrace{\left[f_{yx}'' \frac{dx}{dt} + f_{yy}'' \frac{dy}{dt} \right]}_{\text{applying chain rule to } f_y'} \frac{dy}{dt} + f_y' \frac{d^2y}{dt^2}$$

$$= f_{xx}'' \left(\frac{dx}{dt} \right)^2 + 2 f_{xy}'' \frac{dy}{dt}\frac{dx}{dt} + f_{yy}'' \left(\frac{dy}{dt} \right)^2 + f_x' \frac{d^2x}{dt^2} + f_y' \frac{d^2y}{dt^2}$$

Example 19.1.5 (*Directional Derivatives*) We can use the chain rule to compute slopes in directions other than along the x- or y- axes.

Let $z = f(x,y)$, $x = x_0 + th$, $y = y_0 + tk$. Then dz/dt gives the derivative of z as x and y change according to the proportion

$$\frac{\text{change in } y}{\text{change in } x} = \frac{k}{h}.$$

We have

$$\frac{dz}{dt} = f_x' \frac{dx}{dt} + f_y' \frac{dy}{dt} = f_x' h + f_y' k \quad \text{and} \quad \frac{d^2z}{dt^2} = f_{xx}'' h^2 + 2 f_{xy}'' hk + f_{yy}'' k^2.$$

If $\sqrt{h^2 + k^2} = 1$, dz/dt is called the directional derivative.

Exercise: Show that $\frac{d^2z}{dt^2} = f_{xx}'' h^2 + 2 f_{xy}'' hk + f_{yy}'' k^2$ *can be written in matrix form as*

$$\frac{d^2z}{dt^2} = \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} f_{xx}'' & f_{xy}'' \\ f_{xy}'' & f_{yy}'' \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}.$$

19.2 Implicit Differentiation Revisited Using the chain rule, we can derive a general formula for implicit differentiation. Given an equation involving x and y, rewrite the equation in the form $F(x,y)=c$, where c is some constant. If the function $y=g(x)$ is a local solution to $F(x,y)=c$, then $F(x,g(x))=c$, and implicit differentiation gives

$$\frac{\partial F(x,y)}{\partial x}\frac{dx}{dx}+\frac{\partial F(x,y)}{\partial y}\frac{dy}{dx} = 0$$

so that $$\frac{dy}{dx} = -\frac{\frac{\partial F(x,y)}{\partial x}}{\frac{\partial F(x,y)}{\partial y}}. \qquad (*)$$

If $\partial F(x_0,y_0)/\partial y \neq 0$, and $F(x_0,y_0)$ is continuously differentiable over an open interval containing (x_0,y_0), then a local solution exists that passes through (x_0,y_0), and its derivative is given by (*).

<u>Example 19.2.1</u> The following is the graph of $y^2-x^3-x^2=0$. In an earlier chapter, we use implicit differentiation and found that $2yy'-3x^2-2x=0$, or

$$y'=(3x^2+2x)/2y.$$

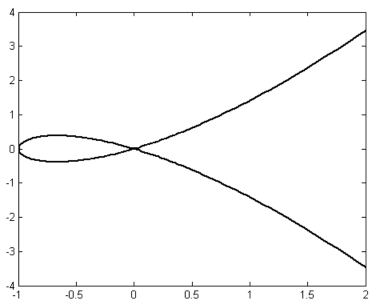
Using the formula (*) we have:

$$F(x,y) = y^2-x^3-x^2=0$$

$$\frac{\partial F(x,y)}{\partial y}=2y,\ \frac{\partial F(x,y)}{\partial x}=-3x^2-2x$$

which gives

$$y'=-\left(\frac{1}{2y}\right)\left(-3x^2-2x\right)=\frac{3x^2+2x}{2y}.$$



The derivative of this implicit function is defined at every point of the graph of the equation, except for the points (0,0) and (−1,0), i.e., where $y=0$.

Example 19.2.2 *Isoquants, Elasticity of Substitution*

One useful application of implicit derivation is in computing the slope of level curves, or contours. Suppose

$$z = F(x, y).$$

Consider the contour $F(x, y) = c$. Suppose this equation implicitly defines a function $y = f(x)$, that is, there is a function $f(x)$ such that $F(x, f(x)) = c$. Then,

$$0 = F_1'(x,y) + F_2'(x,y)\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{F_1'(x,y)}{F_2'(x,y)} \text{ as long as } F_2'(x,y) \neq 0.$$

This equation tells us, when x changes, how much y must change in order to keep F at the same level: y must change so that $F_2'(x,y)$ completely compensates $F_1'(x,y)$. The quantity

$$\frac{F_1'(x,y)}{F_2'(x,y)}$$

i.e., dy/dx *without* the negative sign, is often referred to as the Marginal Rate of Substitution (MRS) between y and x and is a very useful concept in economics.

Implicit differentiation readily extends to functions of two variables implicitly defined by an equation in three variables. Suppose

$$F(x, y, z) = c$$

and that $z = f(x, y)$ is a function that is implicitly defined by this equation, i.e.

$$F(x, y, f(x, y)) = c.$$

Then, we have

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial z}.\frac{\partial z}{\partial x} = 0 \qquad \text{[Note: we have changing } x\text{, holding } y \text{ fixed]}$$

and $$\frac{\partial F}{\partial y}\frac{dy}{dy} + \frac{\partial F}{\partial z}.\frac{\partial z}{\partial y} = 0 \qquad \text{[Note: here we change } y\text{, holding } x \text{ fixed]}$$

which gives

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\bigg/\frac{\partial F}{\partial z}.$$

The essential point is that to find $\partial z / \partial x$ where $z = f(x, y)$ is implicitly defined by the equation $F(x, y, z) = c$, we can simply treat y as a constant, and apply the formula (*).

Example 19.2.3 You want to buy x_0 of a product. You currently have an offer from someone to sell you the product at p_0 per unit, but you can search the market for a lower price. Suppose that by spending t units of time, you can get price $p(t)$, but that the opportunity cost is $c(t) = wt$. Suppose

$$p'(t) < 0, \; p''(t) > 0.$$

Your "profit" from spending t units searching for a lower price is

$$\pi(t) = (p_0 - p(t))x_0 - wt.$$

The first order condition for profit maximization is

$$-x_0 p'(t^*) - w = 0$$

i.e., the profit maximizing time spent searching is the t^* that satisfies this equation.

The second order condition which guarantees that t^* to be a strict global maximum is

$$\pi''(t) = -x_0 p''(t) < 0,$$

and is satisfied because of the assumption that $p''(t) > 0$.

Note that t^*, implicitly defined by the first order condition, are functions of w and x_0, i.e., $t^* = t^*(w, x_0)$. How does the optimal search time t^* change when w changes? When x_0 changes?

Let $F(x_0, w, t^*) = -p'(t^*)x_0 - w = 0$. Then

$$\frac{\partial F}{\partial w} = -1, \qquad \frac{\partial F}{\partial x_0} = -p'(t^*), \qquad \text{and} \quad \frac{\partial F}{\partial t^*} = -p''(t^*)x_0.$$

Therefore,

$$\frac{\partial t^*}{\partial w} = -\frac{\partial F}{\partial w}\Big/\frac{\partial F}{\partial t^*} = -\frac{1}{p''(t^*)x_0} < 0 \qquad \text{and}$$

$$\frac{\partial t^*}{\partial x_0} = -\frac{\partial F}{\partial x_0}\Big/\frac{\partial F}{\partial t^*} = -\frac{p'(t^*)}{p''(t^*)x_0} > 0.$$

Question: we said that $\frac{\partial F}{\partial w} = -1$. *We also pointed out that* t^* *is a function of* w *and* x_0. *Why is* $\frac{\partial F}{\partial w} \neq -p''(t^*)x_0 \frac{\partial t^*}{\partial w} - 1$*?*

Implicit Function Theorem for Systems of Equations A formula similar to (*) exists for functions implicitly defined in systems of equations. Suppose the variables x_1 and x_2 satisfies the equations

$$f_1(x_1, x_2, \alpha) = c_1$$
$$f_2(x_1, x_2, \alpha) = c_2$$

simultaneously; here α is a parameter. The solutions $x_1^*(\alpha)$ and $x_2^*(\alpha)$ are then implicitly defined by the two equations, i.e., $x_1^*(\alpha)$ and $x_2^*(\alpha)$ satisfies

$$f_1(x_1^*(\alpha), x_2^*(\alpha), \alpha) = c_1$$
$$f_2(x_1^*(\alpha), x_2^*(\alpha), \alpha) = c_2$$

(You might be able to find these two solutions explicitly. The point here is that you don't need to be able to solve the equations explicitly in order to find the derivatives. Also, note that the subscripts in f_1 and f_2 does not indicate derivatives, but are there to say that we have two different functions.)

Differentiating both equations by α gives

$$\frac{\partial f_1}{\partial x_1^*}\frac{dx_1^*}{d\alpha} + \frac{\partial f_1}{\partial x_2^*}\frac{dx_2^*}{d\alpha} + \frac{\partial f_1}{\partial \alpha} = 0$$
$$\frac{\partial f_2}{\partial x_1^*}\frac{dx_1^*}{d\alpha} + \frac{\partial f_2}{\partial x_2^*}\frac{dx_2^*}{d\alpha} + \frac{\partial f_2}{\partial \alpha} = 0$$

or in matrix notation

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1^*} & \frac{\partial f_1}{\partial x_2^*} \\ \frac{\partial f_2}{\partial x_1^*} & \frac{\partial f_2}{\partial x_2^*} \end{bmatrix} \begin{bmatrix} \frac{dx_1^*}{d\alpha} \\ \frac{dx_2^*}{d\alpha} \end{bmatrix} = - \begin{bmatrix} \frac{\partial f_1}{\partial \alpha} \\ \frac{\partial f_2}{\partial \alpha} \end{bmatrix}$$

which gives

$$\begin{bmatrix} \frac{dx_1^*}{d\alpha} \\ \frac{dx_2^*}{d\alpha} \end{bmatrix} = - \begin{bmatrix} \frac{\partial f_1}{\partial x_1^*} & \frac{\partial f_1}{\partial x_2^*} \\ \frac{\partial f_2}{\partial x_1^*} & \frac{\partial f_2}{\partial x_2^*} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f_1}{\partial \alpha} \\ \frac{\partial f_2}{\partial \alpha} \end{bmatrix}.$$

This formula extends to more than two equations, and more than two exogenous variables, and is known as the Implicit Function Theorem. The formula (*) is simply a special case with one endogenous variable and one equation.

Example 19.2.4 Here is an example from a previous section: consider the system of equations

$$u^2 + v = x$$

$$uv = 1 - x^2$$

which u and v satisfies simultaneously, for any given x. We wish to find how the solutions u^* and v^* changes with x, i.e. we wish to find $\partial u^* / \partial x$ and $\partial v^* / \partial x$. Write the equations as

$$f_1(u^*, v^*, x) = u^{*2} + v^* - x = 0$$

$$f_2(u^*, v^*, x) = u^* v^* - x^2 \quad = 1$$

Then

$$\begin{bmatrix} \dfrac{du^*}{dx} \\ \dfrac{dv^*}{dx} \end{bmatrix} = -\begin{bmatrix} \dfrac{\partial f_1}{\partial u^*} & \dfrac{\partial f_1}{\partial v^*} \\ \dfrac{\partial f_2}{\partial u^*} & \dfrac{\partial f_2}{\partial v^*} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial f_1}{\partial x} \\ \dfrac{\partial f_2}{\partial x} \end{bmatrix}$$

$$= \begin{bmatrix} 2u^* & 1 \\ v^* & u^* \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2x \end{bmatrix}$$

which you can solve easily using Cramer's rule. [*We will usually skip the asterisks in presentations of these sort – as we did in example 14.3.1 – in order to keep the notation clean*].

19.3 Homogenous Functions and Euler's Theorem Here are a few more applications of the chain rule in economics.

A function $f(x_1, x_2, ..., x_n)$ is homogenous of degree r if

$$f(tx_1, tx_2, ..., tx_n) = t^r f(x_1, x_2, ..., x_n).$$

Homogeneity of degree 1 is called linear homogeneity.

Example 19.3.1 $f(x, y) = 3x^2 y - y^3$ is homogenous of degree three, because

$$f(tx, ty) = 3(tx)^2(ty) - (ty)^3 = t^3(3x^2 y - y^3) = t^3 f(x, y)$$

Example 19.3.2 $f(x, y) = x^4 + x^2 y^2$ is homogenous of degree four, because

$$f(tx, ty) = (tx)^4 + (tx)^2(ty)^2 = t^4 x^4 + t^4 x^2 y^2 = t^4(x^4 + x^2 y^2) = t^4 f(x, y)$$

Example 19.3.3 Let $f(L, K) = L^\alpha K^{1-\alpha}$.

Then $f(tL, tK) = (tL)^\alpha (tK)^{1-\alpha} = tL^\alpha K^{1-\alpha} = tf(L, K)$.

If $f(K, L)$ represents a production function, then the property of linear homogeneity is referred to as "Constant Returns to Scale"

Example 19.3.4 Suppose $x_1^*(p_1, p_2, ..., p_n, M)$ represents the demand for a good where p_1 is the price of the good, and $p_2, ..., p_n$ is the price of other goods, and M is income. If prices and income increase by a factor k (e.g., if all prices *and* income both doubles), then we might expect the consumer's behavior to remain unchanged (i.e., there is no *money illusion*). That is, we would have

$$x_1^*(tp_1, tp_2, ..., tp_n, tM) = x_1^*(p_1, p_2, ..., p_n, M).$$

In terms of homogeneity, we would say that the demand function $x_1^*(p_1, p_2, ..., p_n, M)$ is homogenous of degree zero.

There are a number of results for homogenous functions that are very useful.

Theorem If $f(x_1, x_2, ..., x_n)$ is homogenous of degree r, then the first partials f_1', f_2', ..., f_n' are all homogenous of degree $r-1$.

Proof

Because f is homogeneous of degree r,

$$f(tx_1, tx_2, ..., tx_n) = t^r f(x_1, x_2, ..., x_n).$$

We can differentiate both sides with respect to x_i to get

$$f_i'(tx_1, tx_2, ..., tx_n)\frac{\partial(tx_i)}{\partial x_i} = t^r f_i'(x_1, x_2, ..., x_n)$$

$$f_i'(tx_1, tx_2, ..., tx_n)\, t = t^r f_i'(x_1, x_2, ..., x_n)$$

$$f_i'(tx_1, tx_2, ..., tx_n) = t^{r-1} f_i'(x_1, x_2, ..., x_n)$$

Example 19.3.4 If $f(L, K) = L^\alpha K^{1-\alpha}$ represents a production function, then the marginal products of K and L are homogenous of degree zero:

$$f(L, K) = L^\alpha K^{1-\alpha}$$

$$f_L'(L, K) = \alpha L^{\alpha-1} K^{1-\alpha} = \alpha(K / L)^{1-\alpha}$$

$$f_K'(L, K) = (1-\alpha) L^\alpha K^{-\alpha} = \alpha(K / L)^{-\alpha}$$

It is easy to verify the zero degree homogeneity of the marginal products.

Theorem (Euler's Theorem) If $f(x_1, x_2, ..., x_n)$ is homogenous of degree r, then

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + ... + x_n \frac{\partial f}{\partial x_n} = rf(x_1, x_2, ..., x_n)$$

Proof

Differentiate both sides of

$$f(tx_1, tx_2, ..., tx_n) = t^r f(x_1, x_2, ..., x_n)$$

with respect to t to get

$$f_1'(tx_1, tx_2 ... tx_n)\frac{\partial(tx_1)}{\partial t} + f_2'(tx_1, tx_2, ... tx_n)\frac{d(tx_2)}{dt} + ... + f_n'(tx_1, tx_2 ... tx_n)\frac{d(tx_n)}{dt}$$
$$= f(x_1, x_2, ..., x_n)\frac{d(t^r)}{dt}$$

$\Rightarrow$ $f_1'(tx_1, ... tx_n)x_1 + f_2'(tx_1, ... tx_n)x_2 + ... + f_n'(tx_1, ... tx_n)x_n = f(x_1, x_2, ..., x_n)rt^{r-1}$

$\Rightarrow$ $t^{r-1}x_1 f_1'(x_1, ... x_n) + t^{r-1}x_2 f_2'(x_1, ... x_n) + ... + t^{r-1}x_n f_n'(x_1, ... x_n) = rt^{r-1}f(x_1, x_2, ..., x_n)$

$\Rightarrow$ $x_1 f_1'(x_1, ... x_n) + x_2 f_2'(x_1, ... x_n) + ... + x_n f_n'(x_1, ... x_n) = rf(x_1, x_2, ..., x_n)$

Corollary If $r = 0$, then $\sum_{i=1}^{n} f_i'(x_1, x_2, ..., x_n)x_i = 0$.

It is also true that if

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + ... + x_n \frac{\partial f}{\partial x_n} = rf(x_1, x_2, ..., x_n)$$

then the function $f(x_1, x_2, ..., x_n)$ is homogenous of degree r. The proof of this statement is omitted.

Example 19.3.5 If $f(L, K) = L^\alpha K^{1-\alpha}$, then

$$f_L'(L, K) = \alpha L^{\alpha-1}K^{1-\alpha} = \alpha(K/L)^{1-\alpha},$$

$$f_K'(L, K) = (1-\alpha)L^\alpha K^{-\alpha} = \alpha(K/L)^{-\alpha}, \text{ and}$$

$$Lf_L'(L, K) + Kf_K'(L, K) = \alpha L^\alpha K^{1-\alpha} + (1-\alpha)L^\alpha K^{1-\alpha} = L^\alpha K^{1-\alpha} = 1f(L, K)$$

Example 19.3.6 If $f(x_1, x_2, ..., x_n) = f(\mathbf{x})$ is homogenous of degree k then

$$\mathrm{El}_{x_1} f(\mathbf{x}) + \mathrm{El}_{x_2} f(\mathbf{x}) + ... + \mathrm{El}_{x_n} f(\mathbf{x}) = k.$$

This follows immediately from Euler's Theorem, dividing throughout by f.

Exercises

1. Find $\frac{dz}{dt}$ using the chain rule when

   (a) $z = e^{1-xy}$, $x = t^{1/3}$, $y = t^3$

   (b) $z = \ln(2x^2 + y)$, $x = \sqrt{t}$, $y = t^{2/3}$

   (c) $z = \sqrt{1 + x - 2wx^4 y}$, $x = \ln t$, $y = t^{2/3}$, $w = t^{-2}$

Confirm your answers by substituting $x(t)$, $y(t)$ and $w(t)$ into the expression for z and differentiating directly.

2. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using the chain rule when

   (a) $z = 8x^2 y - 2x + 3y$, $x = uv$, $y = u - v$

   (b) $z = e^{x^2 y}$, $x = u + v\ln u$, $y = u^2 - v\ln v$

Confirm your answers by substituting $x(u,v)$ and $y(u,v)$ into the expression for z and differentiating directly.

3. Find $\left.\frac{\partial f}{\partial u}\right|_{u=1,v=-2}$ when $f(x,y) = x^2 y^2 - x + 2y$, $x = \sqrt{u}$, $y = uv^3$.

4. Suppose each of the following equations implicitly defines y as a differentiable function of x. Find $\frac{dy}{dx}$

   (a) $x^3 - 3xy^2 + y^3 = 5$ (b) $e^{xy} + ye^y = 1$ (c) $x - \sqrt{xy} = 4 - 3y$

5. Each of the following equations defines z implicitly as a differentiable function of x and y. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

   (a) $x^3 - 3yz^2 + xyz - 2 = 0$ (b) $\ln(1+z) + xy^2 = 1 - z$

6. Suppose $z = g(u)$ and $u = g(x,y)$. Write down the appropriate chain rule for finding $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

   Find the appropriate expressions for $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, and $\frac{\partial^2 z}{\partial x \partial y}$.

7. Suppose $z = f(x^2 + y^2)$. Show that $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$.

8. Let $g(r) = f(r, 1-r, 1/(1-r))$. Find an expression for $g'(r)$ in terms of r, f_1', f_2', and f_3'.

9. Suppose $z = f(x, y)$, where $x = x(t, s)$, $y = y(t, s)$, $t = t(q)$, and $s = s(q)$. That is, z is a function of x and y, x and y are functions of both t and s, and both t and s are functions of q. Find an expression for dz/dq.

10. The equation

$$\ln x + 2(\ln x)^2 = \frac{1}{2}\ln K + \frac{1}{3}\ln L$$

defines x as a differentiable function of K and L. Find $\partial x/\partial K$ and $\partial x/\partial L$.

11. Show that if $z = f(x, y)g(x, y)$, then $\mathrm{El}_x z = \mathrm{El}_x f + \mathrm{El}_x g$.

12. Consider the equation system

$$\begin{aligned} xe^y + yf(z) &= a \\ xg(x, y) + z^2 &= b \end{aligned}$$

where $f(z)$ and $g(x, y)$ are differentiable functions, and a and b are constants. Suppose that the system defines x and y as differentiable functions of z. Find expressions for dx/dz and dy/dz.