18. Partial Derivatives

The concept of "ceteris paribus" is one that is well known even to beginning students of economics. It is a mental experiment where we consider the effect of a change in one variable on another variable *holding everything else fixed*. The concept of partial derivatives delivers exactly these sorts of statements.

Example 18.1 Let $y = f(k,l) = 2k^{0.3}l^{0.5}$ be a production function with inputs k and l.

Suppose l is fixed at $l = 4$. What is the effect on y of increasing k?

If $l = 4$, then $y = f(k,4) = 2k^{0.3}4^{0.5} = 4k^{0.3}$. Thus fixing $l = 4$ reduces the problem to one with a function of one variable, and we know how to proceed:

$$\frac{d}{dk}f(k,4) = 1.2k^{-0.7}.$$

If $k = 1$, then $df(1,4)/dk = 1.2$, i.e. when $k = 1$, a one-unit increase in k will increase output by approximately 1.2 units. If $k = 2$, then $df(2,4)/dk = 0.739$.

We can also consider higher derivatives. For instance,

$$\frac{d^2}{dk^2}f(k,4) = -0.84k^{-1.7} < 0,$$

so output increases at a decreasing rate as more capital is added to the process, if labor is held fixed at $l = 4$.

All these values would of course change if l were fixed at some other value:

If $l = 9$, then

$$f(k,9) = 6k^{0.3}, \quad \frac{d}{dk}f(k,9) = 1.8k^{-0.7} \text{ and } \frac{d^2}{dk^2}f(k,9) = -1.26k^{-1.7} < 0$$

For any value of k, an increase in k will result in a larger increase in output than in the previous case when $l = 4$. Additional capital is more productive when $l = 9$ than when $l = 4$, although this productivity is diminishing at a faster rate. Clearly the effect on y of increasing k depends not only on the value of k, but also on the value of l.

Partial Derivatives In practice, we do not need to repeat this process for specific values of l. Instead, we simply treat l as fixed (as in the *ceteris paribus* mental experiment) and treat the function as a function of one variable k. Then we can take derivatives in the usual way. We say we are taking partial derivatives of y with respect to k, and write

$$\frac{\partial y}{\partial k} = \frac{\partial f}{\partial k} = f_k' = f_1'(k,l) = 0.6k^{-0.7}l^{0.5}.$$

(I have given four commonly used notations for partial derivatives -- use whichever notation seems most appropriate or convenient in your given situation.)

Note the use of "∂" rather than "d" in the $\partial y/\partial k$ notation. This is merely convention: we use ∂ when the derivative is a partial derivative (some variable is being held constant) and "d" for derivatives of functions of one variable.

Our answer to the question "what is the effect on y of increasing k when l is held at $l=4$?" can then be answered by putting $l=4$ into the expression of the partial derivative

$$f_1'(k,4) = 0.6k^{-0.7}4^{0.5} = 1.2k^{-0.7}$$

In similar fashion, we can ask what the effect is on y of increasing l holding k fixed. We have

$$\frac{\partial y}{\partial l} = \frac{\partial f}{\partial l} = f_l' = f_2'(k,l) = k^{0.3}l^{-0.5}$$

Example 18.2 $z = x^3y^4$. Then $\frac{\partial z}{\partial x} = 3x^2y^4$, and $\frac{\partial z}{\partial y} = 4x^3y^3$.

There is another notation for partial derivatives: given $z = f(x,y)$, I can also write f_1' for f_x', and f_2' for f_y'. The '1' in the subscript of f_1' means differentiate with respect to the first variable which appears in the list of independent variables in $f(x,y)$, i.e., 'x'. The '2' in the subscript of f_2' means differentiate with respect to the first variable which appears in the list of independent variables in $f(x,y)$, i.e., 'y'. This is especially useful when we apply the chain rule.

We can also take higher derivatives. In particular, for $z = f(x, y)$ we define

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \equiv f_{xx}''$$

$$\frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \equiv f_{yy}''$$

The interpretation of these two higher partial derivatives should be obvious.

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv f_{yx}''$$

How does the "rate at which f changes with y" change when x changes?

$$\frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv f_{xy}''$$

How does the "rate at which f changes with x" change when y changes?

$\frac{\partial^2 f}{\partial l \partial k}$ and $\frac{\partial^2 f}{\partial k \partial l}$ are called cross-partials.

<u>Example 18.3</u> For $f(k,l) = Ak^a l^b$

$$\frac{\partial f}{\partial k} = Aak^{a-1}l^b \qquad \frac{\partial f}{\partial l} = Abk^a l^{b-1}$$

$$\frac{\partial^2 f}{\partial k^2} = Aa(a-1)k^{a-2}l^b \qquad \frac{\partial^2 f}{\partial l^2} = Ab(b-1)k^a l^{b-2}$$

$$\frac{\partial^2 f}{\partial l \partial k} = Aabk^{a-1}l^{b-1} \qquad \frac{\partial^2 f}{\partial k \partial l} = Aabk^{a-1}l^{b-1}$$

Notice that $\frac{\partial^2 f}{\partial l \partial k} = \frac{\partial^2 f}{\partial k \partial l}$.

This is not a coincidence. It turns out that the order with which you take cross partials is irrelevant. This is a result known as Young's Theorem, which we will not prove in this course.

<u>Example 18.4</u> Let $z = x / y$. Then

$$\frac{\partial z}{\partial x} = \frac{1}{y}; \qquad \frac{\partial z}{\partial y} = -\frac{x}{y^2};$$

$$\frac{\partial^2 z}{\partial x^2} = 0; \qquad \frac{\partial^2 z}{\partial y^2} = \frac{2x}{y^3};$$

$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{y^2}; \qquad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{y^2}.$$

Partial Derivatives of Functions of Many Variables If $z = f(x_1, x_2, ..., x_n)$, we can ask how z responds to a change in the variable x_i, holding all other variables fixed. To answer such questions, treat $x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n$ as constants, and differentiate z with respect to x_i.

<u>Example 18.5</u> If $f(x,y,z) = \dfrac{x^4}{yz}$, then

$$f_x' = \frac{4x^3}{yz}; \qquad f_y' = -\frac{x^4}{y^2 z}; \qquad f_z' = -\frac{x^4}{yz^2};$$

For each of the n partial derivatives of the function $z = f(x_1, x_2, ..., x_n)$, we can compute n second-order partial derivatives:

$$f_{ij}'' = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

In other words, for a function of n-variables, we have n^2 second-order partial derivatives. We often organize these n^2 partial derivatives into the following square array, called the “Hessian Matrix”:

$$f''(\mathbf{x}) = \begin{pmatrix} f_{11}''(\mathbf{x}) & f_{12}''(\mathbf{x}) & \cdots & f_{1n}''(\mathbf{x}) \\ f_{21}''(\mathbf{x}) & f_{22}''(\mathbf{x}) & \cdots & f_{2n}''(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}''(\mathbf{x}) & f_{n2}''(\mathbf{x}) & \cdots & f_{nn}''(\mathbf{x}) \end{pmatrix}$$

<u>Example 18.6</u> $f(x,y,z) = \dfrac{x^4}{yz}$

$$f_x' = \frac{4x^3}{yz}; \qquad f_y' = -\frac{x^4}{y^2 z}; \qquad f_z' = -\frac{x^4}{yz^2};$$

$$f_{xx}'' = \frac{12x^2}{yz} \qquad f_{xy}'' = -\frac{4x^3}{y^2 z} \qquad f_{xz}'' = -\frac{4x^3}{yz^2}$$

$$f_{yx}'' = -\frac{4x^3}{y^2 z} \qquad f_{yy}'' = \frac{2x^4}{y^3 z} \qquad f_{yz}'' = \frac{x^4}{y^2 z^2}$$

$$f_{zx}'' = -\frac{4x^3}{yz^2} \qquad f_{zy}'' = \frac{x^4}{y^2 z^2} \qquad f_{zz}'' = \frac{2x^4}{yz^3}$$

We can compute even higher-ordered derivatives. It turns out that the cross-derivatives are the same regardless of the order in which the differentiation was carried out. We have

$$f_{xy}'' = f_{yx}'', \quad f_{xz}'' = f_{zx}'', \quad f_{yz}'' = f_{zy}''$$

This is always true. For any function of n variables $z = f(x_1, x_2, ..., x_n)$, we always have

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Exercises

1. Given $f(x,y)$ below, find $\dfrac{f(x+h,y)-f(x,y)}{h}$ and $\dfrac{f(x,y+h)-f(x,y)}{h}$. Find

   (i) $\lim_{h\to 0}\dfrac{f(x+h,y)-f(x,y)}{h}$ and (ii) $\lim_{h\to 0}\dfrac{f(x,y+h)-f(x,y)}{h}$

   (*Comment: as you do these drills, try to get the intuition that you are, in the case of (i) differentiating the function with respect to x, treating y as though it were a constant, and in the case of (ii), differentiating the function with respect to y, treating x as though it were a constant.*)

   (a) $f(x,y)=3x+2y$;

   (b) $f(x,y)=5xy$;

   (c) $f(x,y)=x^2+y^2$;

   (d) $f(x,y)=4x^2y$;

   (e) $f(x,y)=3x^2$;

   (f) $f(x,y)=y^2/x$;

   In each case, find $\lim_{h\to 0}\dfrac{f(x+h,y)-f(x,y)}{h}$ and $\lim_{h\to 0}\dfrac{f(x,y+h)-f(x,y)}{h}$

2. For all of the functions $f(x,y)$ in question 1, find

   (a) f_1' and f_2'

   (b) f_{11}'', f_{22}'', f_{12}'', and f_{21}''. Verify in each case that $f_{12}''=f_{21}''$.

3. For all of the functions $f(x,y,z)$ in question 2 and 4, find

   (a) f_1', f_2', f_3'

   (b) f_{11}'', f_{22}'', f_{33}'', f_{12}'', f_{21}'', f_{13}'', f_{31}'', f_{23}'' and f_{32}''. Verify in each case that $f_{ij}''=f_{ji}''$.

4 (a) Given $f(x,y)=x^2+2xy+y^2$, find

   (i) $f(-1,2)$ (ii) $f(a,a)$ (iii) $f(1/x,1/y)$ (iv) $g(x)=f(x,1/x)$

   (b) Given $F(x,y,z)=y\,e^{xyz}$, find

   (i) $F(1,1,1)$ (ii) $F(x,x^2,x^3)$ (iii) $F(x,1,1)$

5. Let $f(x,y) = \ln(y - 2x)$.

(i) Find the largest possible domain,

(ii) Find the range of the function when the function is defined over the largest possible domain;

(iii) Find the largest possible domain if it is desired that $f(x,y) \geq 0$;

(iv) Sketch the level curves $f(x,y) = k$ for $k = -1,\ 0,\ 1,\ 2$.

6. Given $f(x,y)$ below, find $f_x'(x,y)$ and $f_y'(x,y)$.

(a) $f(x,y) = x^2 + 2xy + y^2$ (b) $f(x,y) = x^3e^{-y} + y^3$

(c) $f(x,y) = x^2ye^{xy}$ (d) $f(x,y) = \dfrac{x^2y^3}{\sqrt{x+y}}$

7. Given $f(x,y) = x^2 + 2xy + y^2$, find

(a) $[f(x+h,y) - f(x,y)]/h$

(b) $[f(x,y+h) - f(x,y)]/h$

In both cases, evaluate the expression when $h \to 0$.

8. The volume of a cone of height h and radius (at its base) r is given by

$$V = f(r,h) = \pi r^2 h / 3.$$

How would you interpret the equation $\partial V / \partial r = 2V / r$. Show that the change in the volume of the cone as you increase its height while holding the base radius constant is proportional to the area of the base.

9. Given $w = f(x,y,z)$ below, find $\partial w / \partial x$, $\partial w / \partial y$, and $\partial w / \partial z$

(a) $f(x,y,z) = ye^{xyz}$ (b) $f(x,y,z) = \dfrac{x^2 - y^2}{y^2 + z^2}$

(c) $f(x,y,z) = xyz$ (d) $f(x,y,z) = ye^z \log(xz)$