17. Functions of Many Variables

We have been studying functions of one variable $y = f(x)$. More realistic situations usually require modeling a variable as being determined by two or more other variables.

The following is an example of a function of two variables:

$$z = 2x^{0.3}y^{0.6}.$$

This equation maps pairs of x-y values to z values.

(x,y)		z
$(0,0)$	$\rightarrow$	0
$(1,1)$	$\rightarrow$	2
$(1.5,1)$	$\rightarrow$	2.26
$(1,1.5)$	$\rightarrow$	2.55
$(1.5,1.5)$	$\rightarrow$	2.88
$(2,2)$	$\rightarrow$	3.73
	etc…	

Recall that a function f from set A to set B is a rule that maps each element of A to some element in B. Functions of two variables are functions where the domain A is a set of pairs of real numbers. In set notation, we say that $A \subset \mathbb{R}^2$. Functions of three variables are functions where the domain A is a set of triples of real numbers, i.e., $A \subset \mathbb{R}^3$.

As in the case of functions of one variable, it is important to be aware of the domain of definition of a function.

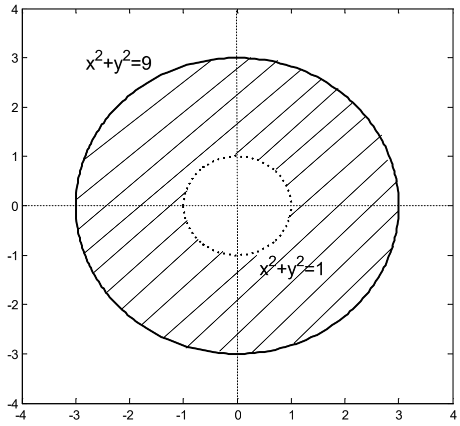
Example 17.1

$$f(x,y) = \frac{2}{\sqrt{x^2+y^2-1}} + \sqrt{9-(x^2+y^2)}$$

This function requires $x^2+y^2-1>0$ and $9-(x^2+y^2)\geq 0$, i.e.

$$x^2+y^2>1 \text{ and } x^2+y^2\leq 9$$

The largest possible domain is illustrated in the this diagram.

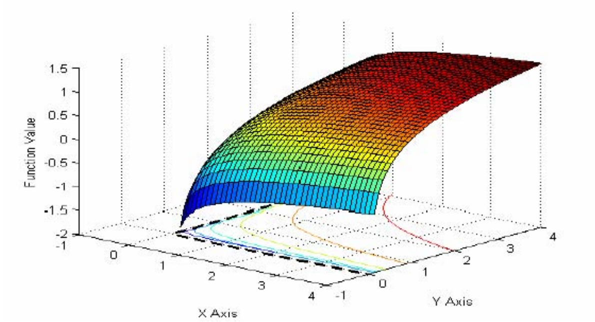


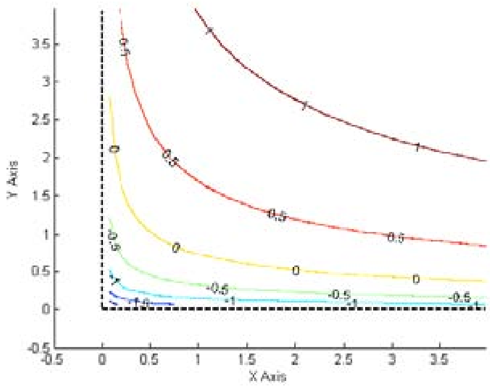
Visualization of Functions of Two Variables There are two main ways of visualizing functions $z = f(x,y)$ of two variables.

(A) A three dimensional diagram, using the vertical axis to represent the function values (because we write the function as $z = f(x,y)$, we call this the *z*-axis), and the two horizontal axes to represent the arguments of the function (the *x*-*y* axes).

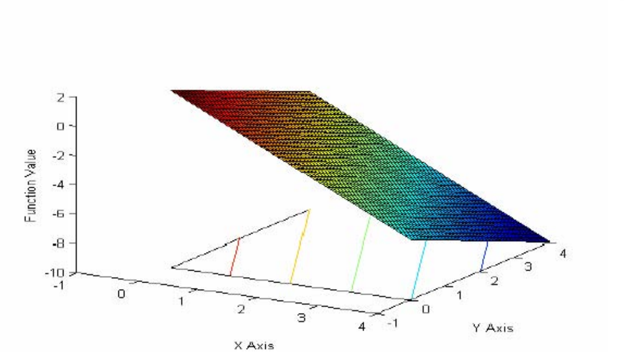
(B) Level Curves, or Contour Plots. Show only the *x*-*y* axes, and plot (x,y) pairs for which the function value is c, for preselected values of c.

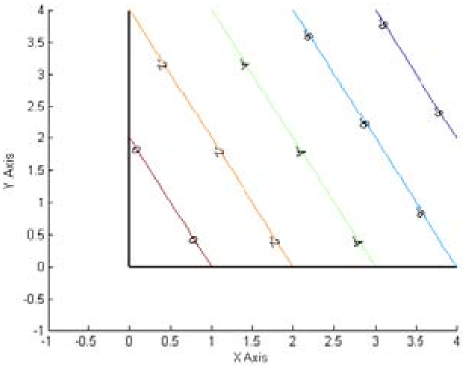
Example 17.2 $z = F(x,y) = \log(1.2) + 0.3\log(x) + 0.6\log(y)$, $x>0$, $y>0$.



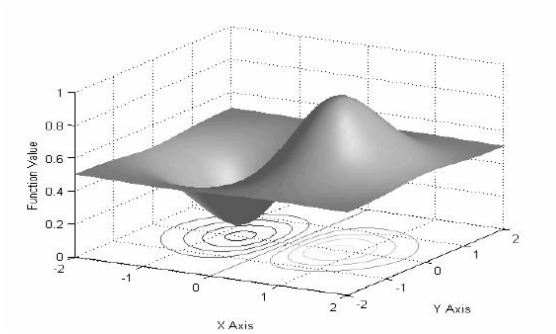


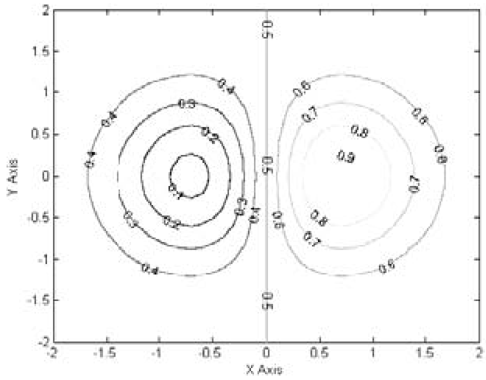
Example 17.3 $z = 2 - 3x - y$, $x \geq 0$, $y \geq 0$.



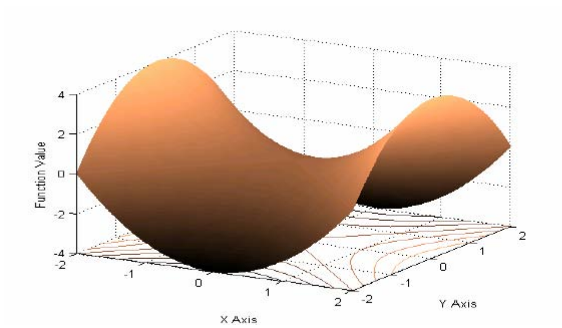


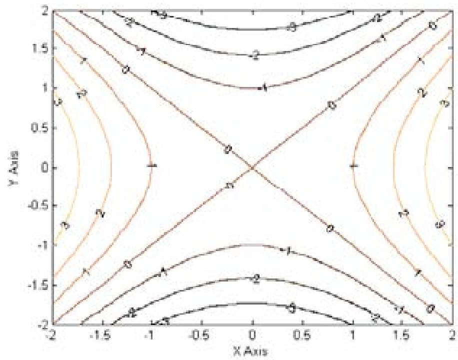
Example 17.4 $z = 0.5 + x\exp(-x^2 - y^2)$, $x, y \in \mathbb{R}$





Example 17.5 $z = f(x, y) = x^2 - y^2$





There are many fanciful ways for visualizing functions of three and more variables, but we won't be concerned with those methods here. Contour plots of functions of two variables are, however, used often in economics tests, and we will learn how to draw them. The contour plots in the first example could very well have been drawings of utility curves, or of isoquants.

To draw level curves: For a function $z = f(x, y)$, fix a value of z, say z_0. Solve the equation for y, i.e., write $z_0 = f(x, y)$ as $y = g(x)$. Plot this on the *x*-*y* axis. Repeat for some other value of z_0.

Example 17.6 Let $z = 2x^{0.3}y^{0.5}$.

Then $y = \left(\dfrac{1}{2x^{0.3}}\right)^2$ is the level curve corresponding to $z = 1$,

and $y = \left(\dfrac{7}{2x^{0.3}}\right)^2$ is the level curve corresponding to $z = 7$.

To show that an equation is a level curve:

Example 17.7 To show that $x^2 + y^2 = 6$ is a level curve of the function

$$f(x, y) = \sqrt{x^2 + y^2} - x^2 - y^2 + 2,$$

show that $x^2 + y^2 = 6$ “solves” $f(x, y) = c$ for some constant c. We have

$$\begin{aligned} f(x, y) &= \sqrt{x^2 + y^2} - x^2 - y^2 + 2 \\ &= \sqrt{x^2 + y^2} - (x^2 + y^2) + 2 \\ &= \sqrt{6} - 6 + 2 \\ &= \sqrt{6} - 4 \end{aligned}$$

so $x^2 + y^2 = 6$ corresponds to the level curve $f(x, y) = \sqrt{6} - 4$.

Exercises

1. For each of the following functions $f(x,y)$, find (i) the largest possible domain; (ii) sketch the largest possible domain;

(a) $\dfrac{x^2+y^3}{y-x+2}$

(b) $\sqrt{2-(x^2+y^2)}$. Can you also figure out the range, when the function is defined over the largest possible domain?

(c) $\sqrt{(4-x^2-y^2)(x^2+y^2-1)}$

(d) $\dfrac{1}{e^{x+y}-3}$

(e) $\ln(x-a)^2+\ln(y-b)^2$ where a and b are constants.

(f) $2\ln(x-a)+2\ln(y-b)$ where a and b are constants.

(g) $3xy^3-45x^4-3y$

(h) $\sqrt{1-xy}$

(i) $\ln(2-(x^2+y^2))$

(j) $\dfrac{1}{\sqrt{x+y-1}}$

(k) $\sqrt{x^2-y^2}+\sqrt{x^2+y^2-1}$

(l) $\sqrt{y-x^2}-\sqrt{\sqrt{x}-y}$

(m) $\ln(1-x^2-y^2)$

(n) $\sqrt{x^2+y^2-4}$

(o) $\dfrac{1}{x-y^2}$

(p) $\ln xy$

(q) $xe^{-\sqrt{y+2}}$

(r) $\dfrac{\sqrt{4-x^2}}{y^2+3}$

Domains and Range: Functions of Many Variables

2. For each of the following functions $f(x,y,z)$, find the largest possible domain.

(a) $\sqrt{25-x^2-y^2-z^2}$

(b) $\dfrac{xyz}{x+y+z}$.

Using the f(…) notation

3(a) Given $f(x,y)=x+2y$, find $f(0,1)$, $f(a,a)$, $f(a+h,b)-f(a,b)$, $f(y,x)$

(b) Given $f(x,y)=xy^2$, find $f(-1,2)$, $f(a,a)$, $f(a+h,b)$, $f(a,b+k)-f(a,b)$.

(c) Given $f(x,y)=3x^2-2xy+y^3$, find $f(1,1)$, $f(-2,3)$, $f(1/x,1/y)$, and

$[f(x+h,y)-f(x,y)]/h$, $[f(x,y+k)-f(x,y)]/k$.

4. (a) Given $f(x,y,z)=xy^2z^3+3$, find $f(2,1,2)$, $f(a,a,a)$, $f(t,-t^2,t)$

(b) Given $f(x,y,z)=zxy+x$, find $f(x+y,x-y,x^2)$, $f(xy,y/x,xz)$.

5. Let $f(x,y)=x^2+2xy+y^2$ and $g(x,y)=3x^2-2xy+y^3$. Show that

(a) $f(2x,2y)=2^2 f(x,y)$ (b) $f(tx,ty)=t^2 f(x,y)$

(c) $g(tx,ty)\neq t^k f(x,y)$ for any k.

Visualization: Level Curves, and Beyond

6. Sketch the level curve $f(x,y)=k$ for the specified values of k

(a) $f(x,y)=x^2+y^2, k=0,1,2,3,4$;

(b) $f(x,y)=x^2+y$, $k=-2,-1,0,1,2$;

(c) $f(x,y)=x^2+9y^2$, $k=0,1,2,3,4$;

7. Sketch the level curve $f(x,y)=k$ for the specified values of k, and sketch the 3D graph of the function

(a) $f(x,y)=\sqrt{x^2+y^2}$, $k=0,1,2,3,4$;

(b) $f(x,y)=x^2$, $k=0,1,2,3,4$;

(c) $f(x,y)=y+1$, $k=-2,-1,0,1,2$;

(d) $f(x,y)=2+3x+6y$, $k=1,2,3,4$.