16. Introduction to Differential Equations

A differential equation is an equation of the form

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0 .$$

Some examples: $y'' - 5y' + 6y = 0$, where $y = f(x)$, $y' = \frac{d}{dx} f(x)$

$y' = 2x + 5$,

$\frac{d}{dt} k(t) = rk(t)$, sometimes written $\dot{k}(t) = rk(t)$

etc.

Many statements in economics (and science in general) are naturally translated into equations of this form. For instance, to say that "capital k appreciates at a constant (percentage) rate over time" we would write

$$\frac{\dot{k}(t)}{k(t)} = r .$$

If we say that the rate of absorption of a new technology in an industry is approximately constant at first but goes to zero as more and more firms in the industry adopts the technology, this would translate into a statement like

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{K}\right)$$

where K represents some sort of 'maximum' capacity.

"Finding a solution" to a given differential equation $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$ means finding functions $y = f(x)$ that satisfy the equation. The problem is thus an extension of the idea of integration, where the objective is to find functions that have a given derivative. Here the goal is to find functions that satisfy an equation that defines the derivative *implicitly*.

For example, in integration, one is given an explicit derivative such as $y' = 2x + 5$, where the derivative is given in terms of x only. The objective is to find all functions that have this derivative: using the usual integration techniques, we have

$$f(x) = \int f'(x)dx = x^2 + 5x + C$$

The differential equations

$$\frac{d}{dt}k(t) = rk(t) \text{ and } y'' + y' - 12y = 0,$$

however, only define the derivative implicitly. The objective is, given such an equation, to find the class of functions that satisfy the equation.

<u>Example 16.1</u> The function $k(t) = x_0 e^{rt}$, for any x_0, is a solution to the differential equation

$$\frac{d}{dt}k(t) = rk(t)$$

This can be seen by differentiating $k(t) = x_0 e^{rt}$, which gives

$$k'(t) = rx_0 e^{rt} = rk(t)$$

<u>Example 16.2</u> The functions $y = c_1 e^{3x} + c_2 e^{-4x}$ are solutions to the differential equation

$$y'' + y' - 12y = 0.$$

Again, (twice) differentiating $y = c_1 e^{3x} + c_2 e^{-4x}$ gives

$$y' = 3c_1 e^{3x} - 4c_2 e^{-4x} \text{ and } y'' = 9c_1 e^{3x} + 16c_2 e^{-4x}$$

so

$$\begin{aligned} y'' + y' - 12y &= (9c_1 e^{3x} + 16c_2 e^{-4x}) + (3c_1 e^{3x} - 4c_2 e^{-4x}) - 12(c_1 e^{3x} + c_2 e^{-4x}) \\ &= 0 \end{aligned}$$

It is harder, of course, to find a solution rather than to merely show that a given function is a solution. In this section, we discuss how to find the solutions to two small classes of differential equations: (i) separable differential equations, which are differential equations that can be written in the form

$$\frac{dy}{dx} = f(x)g(y),$$

and (ii) certain *first-order linear differential equations*. Linear differential equations are differential equations of the form

$$a_n(x)\frac{d^n y}{dx} + a_{n-1}(x)\frac{d^{n-1} y}{dx} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

The <u>order</u> of a differential equation is the order of the highest derivative in the equation. So first-order linear differential equations are those of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Separable Equations The first order differential equation is separable if it can be written as

$$\frac{dy}{dx} = f(x)g(y)$$

General Solution: Write this equation as $\frac{1}{g(y)}\frac{dy}{dx} = f(x)$. Integrating both sides with respect to x gives $\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx + C$. This is equivalent to

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C.$$

An easy way to remember this is to write $\frac{dy}{dx} = f(x)g(y)$ as $\frac{1}{g(y)}\,dy = f(x)\,dx$.

Example 16.3 The differential equation

$$\frac{dk(t)}{dt} = rk(t)$$

is separable and can be solved in this manner. We have

$$\frac{dk}{dt} = rk \quad \Rightarrow \quad \frac{1}{k}\frac{dk}{dt} = r$$

$$\Rightarrow \quad \int \frac{1}{k}\frac{dk}{dt}\,dt = \int r\,dt$$

$$\Rightarrow \quad \int \frac{1}{k}\,dk = \int r\,dt$$

$$\Rightarrow \quad \ln k = rt + C_0 \quad \text{(putting both constants of integration together)}$$

So $k = e^{rt+c_0}$, or $k = x_0 e^{rt}$ where $x_0 = e^{c_0}$.

Example 16.4 If $x^2\frac{dy}{dx} = \frac{x^2+1}{3y^2+1}$, then $(3y^2+1)\frac{dy}{dx} = \frac{x^2+1}{x^2} = 1 + \frac{1}{x^2}$, so

$$\int (3y^2+1)\,dy = \int 1 + \frac{1}{x^2}\,dx$$

$$y^3 + y = x - \frac{1}{x} + C$$

Note that the solution here is also only implicitly defined.

In both examples, we computed general solutions, i.e. we found entire classes of functions that satisfy the differential equations. If we are told that the solution passes through a specific point, we will be able to find a specific solution. For instance, in Example 16.2.1, if we are told that

$$\frac{dk(t)}{dt} = rk(t) \quad \text{and} \quad k(0) = 2,$$

then we know that $k(0) = x_0 e^{r0} = 2$, or $x_0 = 2$, and the specific solution is $k = 2e^{rt}$. The condition $k(0) = 2$ is often called an initial condition, because it is often given at $t = 0$, but in principle any point can be given.

Example 16.5 If interest r on a deposit is continuously compounded, this means that $dy/dt = ry$. We have shown that this implies $y = y_0 e^{rt}$ where $y_0 = y(0)$ is the initial deposit. If \$5000 is deposited at time $t = 0$, and draws interest of 8% per annum continuously compounded, then in eighteen years, we have

$$y(18) = 5000e^{0.08(18)} = \$21103.48.$$

Linear First Order Equations A first-order linear differential equation is one with the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

General Solution:

If we multiply throughout by $e^{\int P(x)dx}$, then we get

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$

The left-hand side of the equation is equivalent to $\frac{d}{dx}\left[ye^{\int P(x)dx}\right]$, so we have

$$\frac{d}{dx}\left[ye^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx}$$

Integrating both sides yields $ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx + C$ so the solution is

$$y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + C\right].$$

The function $e^{\int P(x)dx}$ is called the integrating factor. Do not memorize the solution. Memorize the form of the integrating factor, and what to do with it.

Example 16.6 If $y' + ay = b$, then

$$y = Ce^{-at} + \frac{b}{a}$$

for some arbitrary constant C. The integrating factor is $e^{\int P(x)dx} = e^{\int a\,dx} = e^{ax}$. Multiplying throughout gives

$$\frac{d}{dx}\left[ye^{ax}\right] = be^{ax} \Rightarrow ye^{ax} = \int be^{ax}\,dx = \frac{b}{a}e^{ax} + C \Rightarrow y = e^{-ax}\left[\frac{b}{a}e^{ax} + C\right] = \frac{b}{a} + Ce^{-ax}.$$

Example 16.7 Let $y' + ay = by^2$.

Although this equation is not a first-order linear differential equation, we can make the substitution $v = 1/y$, which gives $dv/dx = -(1/y^2)\, dy/dx$. Rewriting the differential equation as

$$(1/y^2)y' + a(1/y) = b$$

we have

$$v' - av = -b$$

Multiplying throughout by the integrating factor $e^{\int -a\, dx} = e^{-ax}$, we have

$$\frac{d}{dx}\left[ve^{-ax}\right] = -be^{-ax} \quad \Rightarrow \quad ve^{-ax} = \frac{b}{a}e^{-ax} + C_0$$

$$\Rightarrow \quad v = \frac{b}{a} + C_0 e^{ax} = \frac{b + aC_0 e^{ax}}{a}$$

Converting back to y, we have $y = \dfrac{a}{b + aC_0 e^{ax}} = \dfrac{a}{b + C_1 e^{ax}}$, where $C_1 = aC_0$.

This differential equation is usually applied in the following form, which is more easily interpretable. Let $y = y(t)$, and

$$\dot{y} = ry\left(1 - \frac{y}{K}\right) = ry - \frac{r}{K}y^2$$

Typically, K is positive. If $y < K$, and if y is near zero, then this y/K is near zero, and y grows exponentially. As y approaches K, $\dot{y}$ approaches zero (no growth). This model has been used to describe population growth toward some limit set by K. Initially, the population grows exponentially. As it reaches some capacity (e.g., because of limited food supply) growth tapers off towards zero. If $y > K$, then the population declines until $\dot{y} = 0$.

Substituting $a = -r$ and $b = -r/K$ into the solution $y = a/(b + C_1 e^{ax})$, we get

$$y(t) = \frac{-r}{-(r/K) + C_1 e^{-rt}} = \frac{K}{1 + C_2 e^{-rt}}.$$

If we let $y(0) = y_0$, then $e^{-rt} = 1$, and we can solve for C_2 in terms of y_0. Doing so and substituting back into $y(t)$ leads to the solution

$$y(t) = \frac{K}{1 + \dfrac{K - y_0}{y_0} e^{-rt}}.$$

Exercises

1. Verify that the given function is a solution of the given differential equation:

   (i) $\frac{dy}{dx} - 2y = e^{3x}$; $y = e^{3x} + 10e^{2x}$ (ii) $\frac{dy}{dt} + 20y = 24$; $y = \frac{6}{5} - \frac{6}{5}e^{-20t}$

   (iii) $(y')^3 + xy' = y$; $y = x + 1$ (iv) $y''' - 3y'' + 3y' - y = 0$; $y = x^2 e^x$

   Which of the above are linear differential equations? Which of the above are first order differential equations?

2. Solve the following separable differential equations

   (i) $xy' = 4y$ (ii) $\frac{dy}{dx} = \frac{y^2}{x^2}$ (iii) $\frac{dy}{dx} = e^{3x+2y}$

   (iv) $\frac{dy}{dx} = -\frac{x}{y}$ (v) $(1+x)\frac{dy}{dx} - y = 0$ (vi) $\frac{dy}{dx} = y^2 - 4$

   Hint for part (vi): $\frac{4}{y^2 - 4} = \frac{1}{y-2} - \frac{1}{y+2}$.

   For part (vi), what is your answer if in addition, you are told that $y(0) = 4$? How about if $y(0) = 2$?

3. Solve the following differential equations

   (i) $\frac{dy}{dx} - 3y = 0$ (ii) $x\frac{dy}{dx} - 4y = x^6 e^x$ (iii) $(x^2 + 9)\frac{dy}{dx} + xy = 0$

   (iv) $\frac{dy}{dx} + 2xy = x$; $y(0) = 3$ (v) $x^2 y' + xy = 1$ (vi) $\frac{dy}{dx} = y + e^x$

4. Show that the class of functions

$$y(t) = \frac{K}{1 + C_2 e^{-rt}}$$

   are solutions to the differential equation $\dot{y} = ry\left(1 - \frac{y}{K}\right)$, $K > 0$. Let $y_0 = y(0)$ and solve for C_2 to show that the solution can be written in the form

$$y(t) = \frac{K}{1 + \frac{K - y_0}{y_0} e^{-rt}}.$$

   Plot this function for $y \geq 0$ for the cases

   (i) $y_0 > K$ (ii) $y_0 = K$ (iii) $0 < y_0 < K$.