15. Integration

The first methods used to measure the area between a graph and the x-axis were based on approximations, and taking the approximations to the limit. (We call areas computed this way 'definite integrals'.) Later, the connection between areas and derivatives was discovered and used to greatly simplify the solution to the area problem.

It should be mentioned that approximations remain an important part of the integration story, and there remain many 'area' problems that can only be solved using approximations (now with the help of computers, of course).

15.1 The area between a graph and the x-axis over the interval $[a,b]$ can be approximated by a sum of the areas of rectangles:

$$\text{Approx. Area} = \sum_{i=1}^{n} f(x_i^*)\Delta x\,,\ \Delta x = \frac{b-a}{n},\ x_i^* \in [x_{i-1}, x_i],\ x_0 = a\,,\ x_i = x_{i-1} + \Delta x\,,\ i = 1,\ldots,n\,.$$

The larger n is, the more accurate is the approximation. The **definite integral** of f from a to b is defined as

$$\int_a^b f(x)dx \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

is the limit exists.

Remark The **integral sign** $\int$ was first introduced by Liebniz, and is intended to be a stylized 'S' for 'sum', reflecting the idea that the integral is the limit of a sum. When writing

$$\int_a^b f(x)dx\,,$$

the function $f(x)$ is called the **integrand**, and a and b are the lower and upper **limits of integration** respectively. The symbol dx is there to indicate the independent variable, and is there to mimic the 'Δx' on the right-hand side. The process of calculating an integral is called **integration**. The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called the **Riemann sum**.

Example 15.1.1 Find the definite integral $\int_1^2 x^3\, dx$.

The Riemann sum is

$$\sum_{i=1}^{n}\left(x_i^*\right)^3 \Delta x\,,\ \ \Delta x = \frac{2-1}{n} = \frac{1}{n}\,,\ \ x_i^* \in [x_{i-1}, x_i]\,,\ \ x_0 = 1\,,\ \ x_i = x_{i-1} + \Delta x\,,\ \ i = 1,\ldots,n\,.$$

We take x_i^* to be the right end point of the interval $[x_{i-1}, x_i]$, i.e., we take $x_i^* = 1 + i/n$. Therefore, the Riemann sum is

$$\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3 \frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{3i}{n}+\frac{3i^2}{n^2}+\frac{i^3}{n^3}\right) = 1+\frac{3}{n^2}\sum_{i=1}^{n} i+\frac{3}{n^3}\sum_{i=1}^{n} i^2+\frac{1}{n^4}\sum_{i=1}^{n} i^3$$

Note that

$$\frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\frac{n(n+1)}{2} \to \frac{1}{2} \text{ as } n\to\infty$$

$$\frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \to \frac{1}{3} \text{ as } n\to\infty \quad \text{(why?)}$$

$$\frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\left(\sum_{i=1}^{n} i\right)^2 = \frac{1}{n^4}\frac{n^2(n+1)^2}{4} \to \frac{1}{4}$$

Therefore

$$\int_1^2 x^3\, dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\frac{1}{n} = \lim_{n\to\infty}\left(1+\frac{3}{n^2}\sum_{i=1}^{n} i+\frac{3}{n^3}\sum_{i=1}^{n} i^2+\frac{1}{n^4}\sum_{i=1}^{n} i^3\right) = 1+\frac{3}{2}+1+\frac{1}{4} = \frac{15}{4}$$ ð

15.1.2 Exercise Find the definite integral $\int_0^2 x^3\, dx$ by taking the limit of the Riemann sum choosing x_i^* to be the left endpoint of $[x_{i-1}, x_i]$. Draw a diagram to illustrate the Riemann sum.

15.1.3 Exercise Find $\int_0^2 x^2\, dx$ by taking the limit of the Riemann sum.

Remark If $f(x)$ takes positive and negative values, then the definite integral gives the net area.

15.14 Exercise Find $\int_{-1}^2 x^3\, dx$ by taking the limit of the Riemann sum. Compare your answer to that in Exercise 15.1.2. Draw a diagram to illustrate the Riemann sum, and the area computed.

There are cases where the limit of the Riemann sum does not exist. If the limit exists, we say that the function in **integrable** over the interval $[a,b]$. Fortunately, most of the functions we will deal with are integrable:

Theorem If f is continuous over $[a,b]$, or if it has only a finite number of jump discontinuities, then f is integrable over $[a,b]$.

(Proof omitted)

15.2 Properties of the definite integral:

(a) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(b) $\int_a^a f(x)dx = 0$

(c) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ where c is any constant

(d) $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

(e) $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

(f) If $f(x) \geq 0$ over $[a,b]$, then $\int_a^b f(x)dx \geq 0$

(g) If $f(x) \geq g(x)$ over $[a,b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

(h) If $m \leq f(x) \leq M$ over $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Proofs are omitted.

<u>15.2.1 Example</u> Find $\int_1^2 (1+2x^3)\,dx$.

We have

$$\int_1^2 (1+2x^3)\,dx = \int_1^2 1\,dx + 2\int_1^2 x^3\,dx = 1+2\frac{15}{4} = \frac{17}{2}$$

<u>15.2.2 Exercise</u> Find $\int_0^2 (3+4x^2+2x^3)\,dx$.

<u>15.2.3 Exercise</u> Show that $\int_{-2}^2 f(x)\,dx = 0$ for any odd function $f(x)$.

<u>15.2.4 Exercise</u> Use property 15.2(h) to show that

$$0.367 \leq \int_0^1 e^{-x^2}\,dx \leq 1$$

Even with the help of the properties of definite integrals, finding areas by taking the limits of Riemann sums is not easy except in the simplest examples. The following results, called the fundamental theorems of calculus, make finding areas easy. In some cases, previously intractable problems become trivial.

15.3 The Fundamental Theorem of Calculus I If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$, differentiable on (a,b), and $g'(x) = f(x)$.

Proof

If x and $x+h$ are in (a,b), then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \left(\int_a^x f(t)\,dt - \int_x^{x+h} f(t)\,dt\right) - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

so for $h \neq 0$,

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Suppose for now that $h > 0$. Since f is continuous on $[x, x+h]$, the Extreme Value Theorem says that there exists $u, v \in [x, x+h]$ such that $f(u)$ and $f(v)$ are the global minimum and global maximum values of f over $[x, x+h]$. We have

$$f(u)h \le \int_x^{x+h} f(t)\,dt \le f(v)h$$

$$f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)$$

$$f(u) \le \frac{g(x+h)-g(x)}{h} \le f(v)$$

A similar expression can be obtained for $h < 0$. Now let $h \to 0$

$$\lim_{h\to 0} f(u) \le \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \le \lim_{h\to 0} f(v)$$

Since $u, v \in [x, x+h]$, $u, v \to x$ as $h \to 0$. Also, the middle expression is $g'(x)$ by definition. Therefore we can write the above as

$$\lim_{u\to x} f(u) \le g'(x) \le \lim_{v\to x} f(v)$$

Since f is continuous at x, $\lim_{u\to x} f(u) = \lim_{v\to x} f(v) = f(x)$. It follows from the Squeeze Theorem that

$$g'(x) = f(x)$$

For $x = a$ or $x = b$, we can apply one-sided limits to the argument above to show that $g(x)$ is continuous at those points .

The FTC1 shows us the exact sense in which integration and differentiation are 'inverse' operations. using Liebniz notation for derivatives, we can express FTC1 as:

$$\text{if } f \text{ is continuous on } [a,b], \text{ then } \frac{d}{dx}\int_a^x f(t)\,dt \;=\; f(x).$$

15.4 Given a function $f(x)$, another function $F(x)$ satisfying $F'(x) = f(x)$ is called an antiderivative of $f(x)$. For example, it is clear that for any C,

$$F(x) \;=\; \frac{1}{3}x^3 + C$$

has derivative x^2. That is, for any constant C, $F(x) = \frac{1}{3}x^3 + C$ is an **antiderivative** of the function $f(x) = x^2$.

15.4.1 Example Because

$$\frac{d}{dx}\left(\frac{1}{a+1}x^{a+1} + C\right) \;=\; x^a,\; a \neq -1, \text{ for any constant } C,$$

therefore $\frac{1}{a+1}x^{a+1} + C$ is an the antiderivative of x^a, for any $a \neq -1$ and any constant C.

15.4.2 Example Because

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \text{ for } x \neq 0,$$

therefore $\ln|x| + C$ is an antiderivative of $\frac{1}{x}$, $x \neq 0$, for any C.

The Fundamental Theorem of Calculus II If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a)$$

where F is any antiderivative of f.

Proof

Let $g(x) = \int_a^x f(x)\,dx$. FTC1 says that $g'(x) = f(x)$, so g is an antiderivative of f. This allows us to write

$$F(x) = g(x) + C$$

for $x \in (a,b)$. Since g is continuous over $[a,b]$, this equality also holds over $[a,b]$. Therefore

$$\begin{aligned} F(b) - F(a) &= (g(b) + C) - (g(a) + C) \\ &= g(b) - g(a) \\ &= g(b) \qquad\qquad \text{since } g(a) = \int_a^a f(t)\, dt = 0 \\ &= \int_a^b f(t)\, dt \text{ ð} \end{aligned}$$

The second fundamental theorem of calculus therefore moves us from computing integrals as a limit of sums, to computing definite integrals by finding antiderivatives. The theorem also shows that differentiation and integration are inverse processes.

Another useful way of writing the FTC1 is

$$\int_a^b F'(x)\, dx = F(b) - F(a).$$

Because of the close connection between the definite integral and the antiderivative, we use the notation

$$\int f(x)\, dx$$

to denote the antiderivative. Note that the antiderivative is a family of functions, whereas the definite integral is a number.

<u>15.4.3 Example</u> We write

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

15.5 The FTC highlights the importance of developing systematic ways of finding antiderivatives. These generally come from 'reversing' the rules for finding derivatives.

The substitution rule

e.g Find $\int \frac{x}{\sqrt{1-4x^2}}\, dx$.

Suppose we define $u = 1 - 4x^2$. Then the differential of u is $du = -8x\, dx$. If we treated the 'dx' in the integral as though it is the differential dx in $du = -8x\, dx$, and write

$$\int \frac{x}{\sqrt{1-4x^2}}\, dx = \int \frac{1}{\sqrt{1-4x^2}}\, x\, dx = \int \frac{1}{\sqrt{u}}\left(-\frac{1}{8}\right) du,$$

then the integral becomes very easy to solve.

$$\int \frac{1}{\sqrt{u}}\left(-\frac{1}{8}\right) du = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\, du = -\frac{1}{8}2\sqrt{u}+C = -\frac{1}{4}\sqrt{1-4x^2}+C .$$

To check that this is correct:

$$\frac{d}{dx}\left(-\frac{1}{4}\sqrt{1-4x^2}+C\right) = -\frac{1}{4}\frac{1}{2}\left(1-4x^2\right)^{-1/2}(-8x) = \frac{x}{\sqrt{1-4x^2}} .$$

Why does this work? Remember that the symbol 'dx' in the integral is simply notation (albeit a cleverly chosen one). The rationale for the method is the chain rule:

$$[F(g(x))]' = F'(g(x))g'(x)$$

from which it follows, if we write $u = g(x)$ and $F'(x) = f(x)$, that

$$\int F'(g(x))g'(x)\, dx = \int f(g(x))g'(x)\, dx = F(g(x))+C = F(u)+C = \int F'(u)\, du = \int f(u)\, du .$$

The substitution rule can be used whenever you are faced with an integral of the form

$$\int f(g(x))g'(x)\, dx .$$

The argument above says that to integrate an expression of the form $f(g(x))g'(x)$, make the substitutions $u = g(x)$, replacing $g'(x)\, dx$ with du, and integrate $f(u)$ with respect to u, then convert back to x. The trick is to recognize when the integrand takes the form $f(g(x))g'(x)$.

The following mnemonic is helpful for implementing integration by substitution: given an integration problem

$$\int f(g(x))g'(x)\, dx$$

make the substitutions $u = g(x)$ and $du = g'(x)\, dx$ and rewrite the problem as

$$\int f(\underbrace{g(x)}_{u})\underbrace{g'(x)\, dx}_{du} = \int f(u)\, du$$

15.5.1 Example [SH 9.6.2(a)] Evaluate $\int x(2x^2+3)^5\,dx$

Let $u = g(x) = 2x^2 + 3$, then $du = 4x\,dx$. Rewrite

$$x(2x^2+3)^5 = \frac{1}{4}[(2x^2+3)^5(4x)]$$

Then

$$\int x(2x^2+3)^5\,dx = \frac{1}{4}\int \underbrace{(2x^2+3)}_{u}{}^5\,\underbrace{4x\,dx}_{du} = \frac{1}{4}\int u^5\,du = \frac{1}{4}\frac{1}{6}u^6 = \frac{1}{24}(2x^2+3)^6 + C$$

Applying this to definite integrals, since $[F(g(x))]' = F'(g(x))g'(x)$, we have

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du$$

15.5.2 Example SH 9.6.3(a) Evaluate $\int_0^1 x\sqrt{1+x^2}\,dx$.

Let $u = g(x) = 1 + x^2$ so that $du = 2x\,dx$. Rewrite $x\sqrt{1+x^2} = \frac{1}{2}[2x\sqrt{1+x^2}\,]$.

Furthermore, $g(0) = 1$ and $g(1) = 2$. Then

$$\int_0^1 x\sqrt{1+x^2}\,dx = \frac{1}{2}\int_{g(0)}^{g(1)} \sqrt{\underbrace{1+x^2}_{u}}\,\underbrace{2x\,dx}_{du} = \frac{1}{2}\int_1^2 \sqrt{u}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{1}{2}\left(\frac{2}{3}2\sqrt{2} - \frac{2}{3}\right) = \frac{1}{3}(2\sqrt{2}-1)$$

Integration by Parts Just as the substitution rule for integration comes from the chain rule for differentiation, integration by parts is the integration counterpart of the product rule for differentiation. The product rule states that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of integration, this says that

$$\int [f'(x)g(x) + f(x)g'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

This rule is easier to remember if we use the language of differentials: letting $u = f(x)$ and $v = g(x)$, we can remember this rule as

$$\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)dx}_{dv} = [\underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v}] - \int \underbrace{g(x)}_{v}\,\underbrace{f'(x)\,dx}_{du}$$

$$\text{i.e., } \int u\,dv = uv - \int v\,du.$$

For definite integrals, we have

$$\int_a^b [f'(x)g(x) + f(x)g'(x)]\,dx \;=\; \big[f(x)g(x)\big]_a^b$$

from which follows

$$\int_a^b f(x)g'(x)\,dx \;=\; \big[f(x)g(x)\big]_a^b - \int_a^b f'(x)g(x)\,dx$$

<u>15.5.3 Example</u> To compute $\int \frac{1}{x}\ln x\,dx$, take $u = \ln x$, $dv = \frac{1}{x}\,dx$. Then $du = (1/x)\,dx$, and $v = \ln x$.

$$\begin{aligned}
\int \underbrace{(\ln x)}_{u}\underbrace{\left(\frac{1}{x}\right)dx}_{dv} &= \int u\,dv \\
&= uv - \int v\,du \\
&= \ln(x)\ln(x) - \int (\ln x)\frac{1}{x}dx \\
2\int \frac{1}{x}\ln x\,dx &= [\ln(x)]^2
\end{aligned}$$

Putting in the constant of integration, we have $\int \frac{1}{x}\ln x\,dx = \frac{1}{2}[\ln(x)]^2 + C$.

<u>15.5.4 Example</u> $\int \ln x\,dx$. Let $u = \ln x$, $du = (1/x)\,dx$. Let $dv = dx$, $v = x$.

Then $$\int \ln x\,dx \;=\; x\ln x - \int x(1/x)dx \;=\; x\ln x - \int dx \;=\; x\ln x - x + C.$$

<u>15.5.5 Example [SH 9.5.2(a)]</u> Find $\int_{-1}^{1} x\ln(x+2)\,dx$.

Let $f(x) = \ln(x+2)$, so that $f'(x) = \dfrac{1}{x+2}$. Let $g'(x) = x$.

Then

$$\begin{aligned}
\int_{-1}^{1} x\ln(x+2)\,dx &= \Bigg|_{-1}^{1} \frac{1}{2}x^2\ln(x+2) - \int_{-1}^{1}\frac{1}{2}\frac{x^2}{x+2}\,dx \\
&= \frac{1}{2}\left(\ln 3 - \ln 1\right) - \frac{1}{2}\int_{-1}^{1} x - 2 + \frac{4}{x+2}\,dx \\
&= \frac{1}{2}\ln 3 - \frac{1}{2}\left[\frac{1}{2}x^2 - 2x + 4\ln(x+2)\right]_{-1}^{1} \\
&= \frac{1}{2}\ln 3 - \frac{1}{2}\left[\left(\frac{1}{2} - 2 + 4\ln 3\right) - \left(\frac{1}{2} + 2 + 4\ln 1\right)\right] \\
&= \frac{1}{2}\ln 3 - \frac{1}{2}\left[-4 + 4\ln 3\right] \\
&= 2 - \frac{3}{2}\ln 3
\end{aligned}$$

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Despite the ability of the FTC to simplify the area problem, there are many functions $f(x)$ have no closed form integrals, meaning that we cannot write down an explicit expression $F(x)$ such that

$$\int f(x)\,dx = F(x) + C .$$

For example, $f(x) = e^{-x^2}$ has no closed form integral. These have to be computed by approximation, in a manner similar to taking the Riemann sum for large enough n.

15.6 Unbounded Intervals of Integration

Consider the function $f(x) = \lambda e^{-\lambda x}$, $\lambda > 0$, $x \in [0, \infty)$. Although the value of the function is strictly positive for every x, we can sensibly speak of the area bounded by $f(x)$, the *x*-axis, and the vertical line $x = 0$. This is so even though the area is not bounded on the right. It turns out that this area exists because $f(x)$ falls towards zero "fast enough" as x increases.

How do we compute this area?

We know that to compute the area bounded by $f(x)$, the *x*-axis, and the vertical lines $x = 0$ and $x = a$, $a > 0$, we would compute the definite integral

$$\int_0^a \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x} \right]_0^a = -e^{-\lambda a} - (-1) = (1 - e^{-\lambda a}) .$$

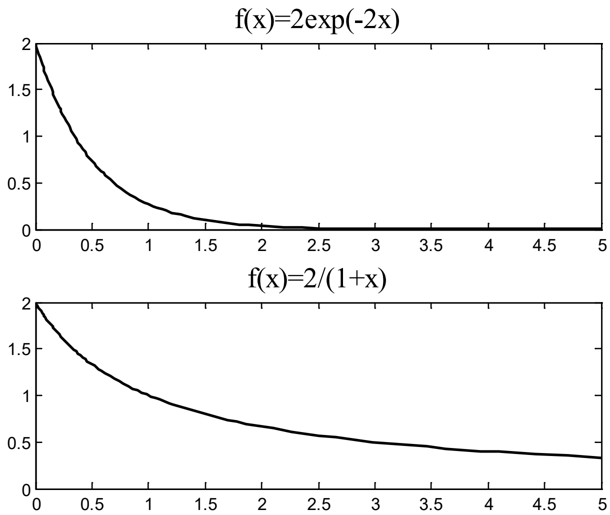
As we increase a the area bounded by $f(x)$, the *x*-axis, and the vertical lines $x = 0$ and $x = a$ gets closer to the desired area A. So A is obtained by taking

$$A = \lim_{a \to \infty} (1 - e^{-\lambda a}) = 1$$

Note that not all such integrals can be evaluated. For instance, this process would have failed if we had applied it to the function $g(x) = 1/(1+x)$, $x \ge 0$.

$$\int_0^a 1/(1+x)\,dx = \left[\ln(1+x) \right]_0^a = \ln(1+a) \text{, but } \ln a \to \infty \text{ as } a \to \infty .$$

One integral exists, while the other does not, even though both integrands converge to zero as $x \to \infty$.



Integrals of Unbounded Functions

Consider the function $f(x) = 1/\sqrt{x}$. This function increase without bound as $x \to 0$. Can we sensibly speak of the area under the curve and over the interval $(0,1]$.

Suppose we first find, for $0 < h < 1$, the area

$$\int_h^1 1/\sqrt{x}\, dx = \left[2\sqrt{x}\right]_h^1 = 2 - 2\sqrt{h}.$$

As we take a sequence of h values decreasing towards zero, we get closer to the area under the curve and over the interval $(0,1]$, and in fact,

$$\lim_{h \to 0^+} 2 - 2\sqrt{h} = 2.$$

Again there are some functions for which this procedure will not work, e.g., $g(x) = 1/x$. We have

$$\int_h^1 1/x\, dx = [\ln x]_h^1 = -\ln h, \text{ but } -\ln h \to \infty \text{ as } h \to 0^+.$$

Exercises

1. (a) Confirm that the following statement is correct, then give a corresponding integration formula:

$$\frac{d}{dx}\left[\sqrt{1+x^2}\right] = \frac{x}{\sqrt{1+x^2}}$$

(b) Prove by differentiating : $\int \frac{dx}{(1-x)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$

2. Evaluate:

(a) $\int 2\,dx$ (b) $\int 0\,dx$ (c) $\int x^2\,dx$ (d) $\int x^{\pi}\,dx$ (e) $\int \frac{1}{x}\,dx$

(f) $\int \frac{1}{2x^3}\,dx$ (g) $\int 1+x+x^3\,dx$ (h) $\int \frac{x^4}{2}+x^2\,dx$ (i) $\int \frac{x^3+y+z}{2}\,dx$

3. Let $f(x)$ and $F(x)$ be such that $\int f(x)\,dx = F(x)+C$. Find $F(x)$ if

(a) $f(x)=x^2$, $F(0)=2$ (b) $f(x)=\frac{x^2}{2}+3x$, $F(4)=1$.

4. Evaluate (a) $\int_{-3}^{0}(x^2-4x+7)\,dx$ (b) $\int_{-1}^{2}x(1+x^2)\,dx$ (c) $\int_{1}^{2}\frac{1}{x^6}\,dx$

5. Find the area between $f(x)$ and the x-axis over the stated intervals:

(a) $f(x)=\sqrt{x}$, $[0,9]$ (b) $f(x)=x^4$, $[-1,1]$ (c) $f(x)=x^2-4x-5$, $[-2,6]$

6. Sometimes we want to differentiate integrals with respect to the limits. This is, of course, what the FTCs do: using Liebniz notation for derivatives, we can express FTC1 as:

if f is continuous on $[a,b]$, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

(a) Show that $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))\,b'(x) - f(a(x))\,a'(x)$

(b) Find $\frac{d}{dt}\int_{-t}^{t}\frac{1}{\sqrt{x^4+1}}\,dx$

7. Find (a) $\frac{d}{dx}\int_1^x \frac{1}{1+t+t^2}\,dt$ (b) $\frac{d}{dx}\int_x^1 \frac{1}{1+t+t^2}\,dt$ (c) $\frac{d}{dx}\int_1^{x^2} \frac{1}{1+t+t^2}\,dt$

8. Evaluate using integration by parts:

(a) $\int xe^{-x}dx$ (b) $\int \sqrt{x}\ \ln x\, dx$ (c) $\int_{-2}^{2} \ln(x-3)\, dx$

9. Evaluate $\int x^2 \sqrt{x-1}\, dx$ by

(a) substitution, with $u = x-1$;

(b) repeated integration by parts, choosing in turn $u = x^2$, $u = x$, $u = 1$.

Show that the two answers are the same.

10. Evaluate $\int \frac{dx}{(\frac{1}{3}x-8)^5}$ using an appropriate substitution.

11. Evaluate (a) $\int_0^1 \frac{1}{\sqrt{1-x}}\, dx$ (b) $\int_0^\infty e^{-x}dx$ (c) $\int_{-\infty}^{\infty} x^3\, dx$