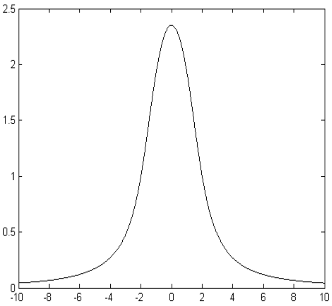
14. Implicit Differentiation

Example 14.1 Consider the equation

$$y^3 + 3x^2 y = 13$$

whose graph is given below. Clearly, this equation defines a function $y = g(x)$ such that $[g(x)]^3 + 3x^2[g(x)] = 13$ for all x.

Qn:



How do we find $g'(x)$?

One might attempt to write y in terms of x and then carry out the differentiation. In this example, however, this is not possible. An alternative approach is to differentiate both sides of the equation with respect to x and solve for $g'(x)$. This technique is called *implicit differentiation*. Keeping in mind that y is a function of x, we differentiate both sides of the equation:

$$y^3 + 3x^2 y = 13 \quad \Rightarrow \quad (y^3 + 3x^2 y)' = (13)'$$

$$\Rightarrow \quad 3y^2 y' + 6xy + 3x^2 y' = 0$$

$$\Rightarrow \quad y'[3y^2 + 3x^2] = -6xy$$

$$\Rightarrow \quad y' = \frac{-2xy}{y^2 + x^2}.$$

Example 14.2 Suppose $x^2 y = 1$. Find y'. Differentiating both sides, we have

$$x^2 y' + 2xy = 0$$

so that $y' = -2y/x$. Noting that $y = 1/x^2$, we have $y' = -2/x^3$.

I did the substitution of $y = 1/x^2$ at the end to obtain y' in terms of x. In fact, because I could solve the equation for y in terms of x, I could have carried out the differentiation directly, without implicit differentiation. The power of the technique lies in situations where we are not able to solve for y in terms of x. In these cases, of course, one may have to leave the derivative y' as an expression involving both x and y (as in example 14.1, and the next example).

Example 14.3 Suppose $y^5 - xy = 24$. Find y'.

Differentiating both sides gives $5y^4 y' - xy' - y = 0$, and solving gives

$$y' = \frac{y}{5y^4 - x}.$$

Implicit differentiation is also useful when the function to be differentiated is defined by an equation involving general functional forms.

Example 14.4 Consumption Tax

Suppose $D = f(P+t)$ and $S = g(P)$ where D is quantity demanded, S is quantity supplied, P is price, and t is a consumption tax. Suppose the functions f and g satisfy $f' < 0$ and $g' > 0$. At equilibrium, $D = S$, i.e.,

$$f(P+t) = g(P) .$$

Suppose this equation defines a function $P(t)$. What happens when the level of tax is altered? That is, find $P'(t)$. Implicitly differentiating $f(P+t) = g(P)$ we get

$$f'(P+t)\left(\frac{dP}{dt}+1\right) = g'(P)\left(\frac{dP}{dt}\right)$$

Solving, we have $\dfrac{dP}{dt} = \dfrac{f'(P+t)}{g'(P) - f'(P+t)}$ which is negative.

Example 14.5 A firm has production function

$$F(K), K > 0.$$

It faces unit price $p > 0$ and has cost function $C(K)$ with $C'' > 0$. Then, profit is

$$\pi(K) = pF(K) - C(K), \ K > 0$$

Qn: What is the optimal quantity of K ?

We will need to make assumptions about $F(K)$ and $C(K)$. We assume $F(K)$ is differentiable, $F'(K) > 0$, $F''(K) < 0$, and $C''(K) > 0$. Also, because $F(K)$ and $C(K)$ are not specified completely, we cannot find K^* explicitly. However, we can characterize the solution. In particular, the solution must satisfy

[FOC] $\pi'(K^*) = PF'(K^*) - C'(K^*) = 0$, i.e., $pF'(K^*) = C'(K^*)$.

The solution $K^*(p)$ is *implicitly defined* by the FOC.

Differentiating $\pi'(K)$ a second time, we have

[SOC] $$\pi''(K) = pF''(K) - C''(K)$$

which is negative because $F''(K) < 0$ and $C''(K) > 0$. That is, the profit function is a concave function, and therefore K^* gives the maximum point.

Although we cannot generate an explicit solution here, we are in fact already saying a lot. In particular, the first order condition

$$pF'(K^*) = C'(K^*)$$

gives a useful general principle: at the optimum, the addition revenue obtained from adding more input K is equal to the additional costs incurred. Frequently, it will be general principals like this that you will be interested in, not solutions involving specific functions.

We can also carry out comparative statics to answer the question: how does K^* change as p changes? Differentiating the first-order condition (and remembering that K^* is a function of p), we have

$$\frac{d}{dp} pF'(K^*) = \frac{d}{dp} C'(K^*)$$

$$pF''(K^*)\frac{dK^*}{dp} + F'(K^*) = C''(K^*)\frac{dK^*}{dp}$$

$$\frac{dK^*}{dp} = \frac{F'(K^*)}{C''(K^*) - pY''(K^*)} > 0$$

We can also find out how π^*, the optimal profit level changes with K^*. The optimal profit level is $\pi^* = \pi(K^*) = pY(K^*) - C(K^*)$. Again, differentiation implicitly gives

$$\begin{aligned} \frac{d\pi^*}{dp} &= pF'(K^*)\frac{dK^*}{dp} + F(K^*) - C'(K^*)\frac{dK^*}{dp} \\ &= \left[pY'(K^*) - C'(K^*)\right]\frac{dK^*}{dp} + F(K^*) \\ &= F(K^*) > 0 \end{aligned}$$

As p increases, the firm hires more capital, and its profit level rises, as intuition would expect.

We will revisit implicit differentiation after studying partial derivatives, when we will develop a neat and handy formula for computing derivatives implicitly.

You can repeat the implicit differential process to obtain higher derivatives.

Example 14.6 Suppose $x^2 y = 1$. Find y''.

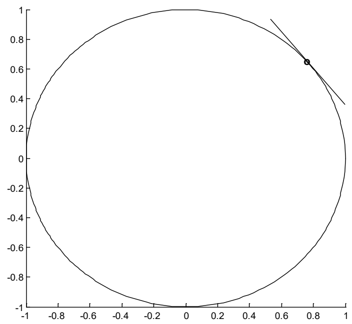
Earlier, we found that $y' = -2y/x$. Then differentiating both sides of the second equation gives

$$y'' = (-2y'x + 2y)/x^2 = (4y + 2y)/x^2 = 6y/x^2.$$

Again making the substitution $y = 1/x^2$, we have $6/x^4$.

Local Solutions The main caveat in our discussion so far is that we have *assumed* that the given equation defines a function that is differentiable. In many cases this assumption is clearly not appropriate.

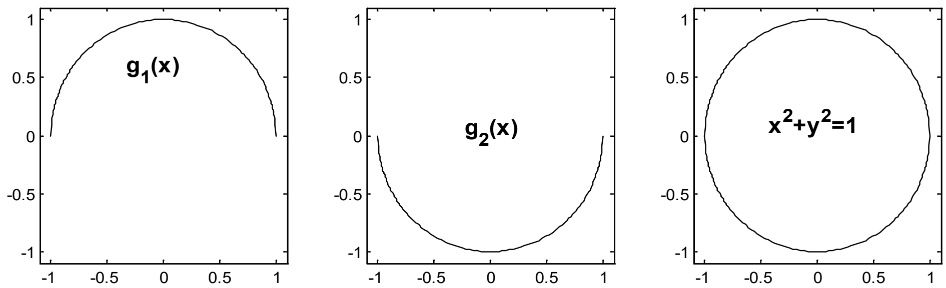
Example 14.7 It is not possible to find a function $y = g(x)$ whose graph coincides exactly and completely with the graph of the equation $x^2 + y^2 = 1$. Yet, it is a sensible question to ask what is the slope of the curve at the certain point (x_0, y_0)? And, what is it that implicit differentiation gives us in these situations?



While it is not possible to find a function $y = g(x)$ that accounts for every point on the graph of the equation, it is often possible to find valid functions $g_1(x)$, $g_2(x)$, …, that each satisfy *part* of the graph. For instance, the functions

$$g_1(x) = \sqrt{1-x^2} \quad \text{and} \quad g_2(x) = -\sqrt{1-x^2}, x \in (-1,1)$$

each satisfy part of the graph of $x^2 + y^2 = 1$.



A local solution to an equation is a function $g: I \to \mathbb{R}$ that satisfies the equation for each x in some interval I.

If there is a unique differentiable function $g: I \to \mathbb{R}$ passing through the point (x_0, y_0) on the graph of an equation, then the expression obtained from implicit differentiation gives us the derivative of the local solution at x_0, i.e., $g'(x_0)$. If there is no such function, or if there are too many of them passing through (x_0, y_0), then the expression obtained from implicit differentiation will not be defined at that point.

Example 14.7 continued   If we implicitly differentiate $x^2 + y^2 = 1$ w.r.t. x we get

$$2x + 2y\,y' = 0, \text{ i.e.,}$$

$$y' = -x/y\ .$$

Evaluated at any point (x_0, y_0) on the graph of $x^2 + y^2 = 1$, except the points $(x_1, y_1) = (-1, 0)$ and $(x_2, y_2) = (1, 0)$, we obtain the slope of the curve at that point. At (x_1, y_1) and (x_2, y_2), $y' = -x/y$ is not defined. The reason is that there is no valid function that passes through these points (i.e., no local solution).

Differentiating Systems of Equations   One application of implicit differentiation is in "differentiating systems of equations". First, we clarify what this means. Suppose we have a system of two equations in two unknowns:

(A)
$$\begin{aligned} 4x + \ \ y &= 3 \\ x + 2y &= 1 \end{aligned}$$

Solving this system means finding the values of x and y that satisfy both equations. You can easily find this out to be $(x, y) = (5/7, 1/7)$. The solution is simply a point.

Now consider the system

(B)
$$\begin{aligned} 4x + \ \ y &= 3t \\ x + 2y &= t^2 \end{aligned}$$

for some given t. If we solve for x and y, holding t to be fixed, we get the solution to be

$$x = (6t - t^2)/7 \qquad \text{and} \qquad y = (4t^2 - 3t)/7\ .$$

The solutions are now functions of t. When t changes, the solution values of x and y will also change. We can ask: how does the solution values for x and y change when t changes? To answer this, simply compute dx/dt and dy/dt. Here we have

$$dx/dt = (6 - 2t)/7 \text{ and } dy/dt = (8t - 3)/7\ .$$

This procedure of solving for x and y first, then differentiating is feasible in simple linear systems. But in nonlinear systems, this might be difficult or impossible.

Example 14.8   Consider the following system of equations

(C)
$$u^2 + v = x$$
$$uv = 1 - x^2$$

which we will treat as two equations in two unknowns, u and v, to be solved for some given x. The solution will be $u =$ some function of x, and $v =$ some function of x. This system is tough (perhaps impossible?) to solve explicitly. Nonetheless, we can compute the derivatives du/dx and dv/dx quite easily. Implicitly differentiating both sides of both equations wrt x, we have

$$u^2 + v = x \quad \Rightarrow \quad 2u\frac{du}{dx} + \frac{dv}{dx} = 1$$

$$uv = 1 - x^2 \quad \Rightarrow \quad v\frac{du}{dx} + u\frac{dv}{dx} = -2x.$$

Note that from two nonlinear equations in u and v we now have two linear equations in two unknowns (du/dx and dv/dx), and we can solve for these in terms of u, v, and x in the usual way:

Multiply the first eq by u to get $2u^2\frac{du}{dx} + u\frac{dv}{dx} = u$.

Subtracting the second eq from this gives $(2u^2 - v)\frac{du}{dx} = u + 2x$, so $\frac{du}{dx} = \frac{u + 2x}{2u^2 - v}$.

Substituting this into the first eq gives $\frac{dv}{dx} = 1 - 2u\left(\frac{u + 2x}{2u^2 - v}\right) = \frac{v + 4ux}{v - 2u^2}$.

Of course, we can also apply this technique to linear systems.

Example 14.9   Returning to system (B), we can implicitly differentiate the two equations to get:

$$\begin{aligned} 4x + \ y &= 3t \\ x + 2y &= t^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4\frac{dx}{dt} + \frac{dy}{dt} &= 3 \\ \frac{dx}{dt} + 2\frac{dy}{dt} &= 2t \end{aligned}.$$

This is a system of two equations in two unknowns (dx/dt and dy/dt) which we can solve, in the usual fashion. Doing so gives us $dx/dt = (6-2t)/7$ and $dy/dt = (8t-3)/7$, as before.

Example 14.10 Consider the macroeconomic model

$$C = f(Y)$$
$$Y = C + G$$

where C and Y are endogenous, and G is exogenous. Find $\frac{dC}{dG}$ and $\frac{dY}{dG}$?

We cannot solve for C and Y in terms of G because the form of $f(.)$ is not known. Here the functions $C = C(G)$ and $Y = Y(G)$ are only defined implicitly by a system of equations.

Differentiating both sides of both equations, we have

$$\frac{dC}{dG} = f'(Y)\frac{dY}{dG} \quad \text{and} \quad \frac{dY}{dG} = \frac{dC}{dG} + 1 .$$

We can solve these two equations for dC/dG and dY/dG: substituting $dC/dG = f'(Y)dY/dG$ into the second equation gives

$$\frac{dY}{dG} = \frac{dC}{dG} + 1 = f'(Y)\frac{dY}{dG} + 1$$

$$\Rightarrow \left(\frac{dY}{dG} - f'(Y)\frac{dY}{dG}\right) = 1$$

$$\Rightarrow \frac{dY}{dG} = \frac{1}{1 - f'(Y)}$$

Substituting back into the first equation gives

$$\frac{dC}{dG} = f'(Y)\frac{dY}{dG} = f'(Y)\left(\frac{1}{1 - f'(Y)}\right) = \frac{f'(Y)}{1 - f'(Y)} .$$

[If $0 < f' < 1$ is assumed, as it usually is, then dY / dG and dC / dG are both positive.]

Exercises

1. For the following equations,

   a. Find y' by implicit differentiation,

   b. Solve the original equation for y,

   c. Verify your answer in (a) by directly differentiating the equation in part (b),

   (i) $x^2e^y = 1$ (ii) $x^2 + y\ln x = 3$ (iii) $x^3 + xy - 2x = 1$

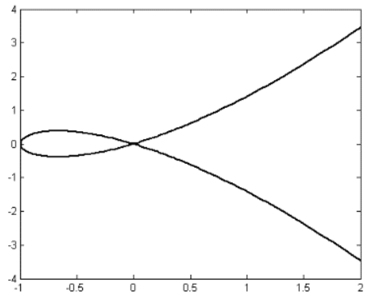
   (iv) $x^2 = \dfrac{x+y}{x-y}$ (v) $yx + y + 1 = x$ (vi) $e^{2y} = x^3$

2. Let $xy = 1$. Find y' by implicit differentiation. As part of your answer, you should have obtained the equation $xy' + y = 0$. Implicitly differentiate this equation a second time, and solve for y''. Verify your answers by differentiating $y = 1/x$ twice.

3. Find dy/dx and d^2y/dx^2 by implicit differentiation

   (i) $x^2y + 3xy^3 - x = 3$ (ii) $x^3y^2 - 5x^2y + x = 1$ (iii) $x^3 - y^3 = 6xy$

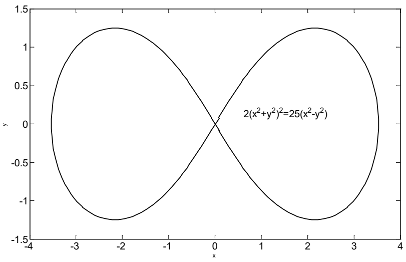
4. The following is the graph of $y^2 - x^3 - x^2 = 0$. Find y' by implicit differentiation. At what points are y' not defined?



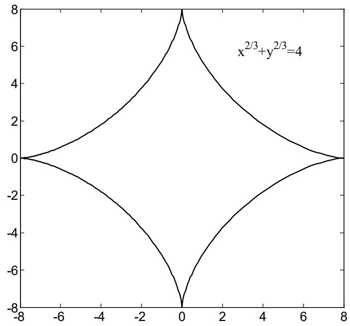
5. Suppose x and y satisfies the equation

$$2(x^2 + y^2)^2 = 25(x^2 - y^2).$$

Find y'. Compute y' at the point $(x, y) = (3,1)$. Show that the derivative does not exist at the point $(0,0)$. Check that your answers are consistent with the following graph.



6. Use implicit differentiation to find the slope of the tangent line to $x^{2/3} + y^{2/3} = 4$ at the point $(-1, 3\sqrt{3})$. Check that your answer is consistent with the following graph. At what points does dy/dx not exist? Check that your answers are consistent with the following graph.



7. Find the values of a and b for the curve $x^2 y + ay^2 = b$ such that the point $(1,1)$ lies on its graph, and the tangent line at $(1,1)$ has the equation $4x + 3y = 7$.

8. Suppose $x^3 y^2 + y = 3$ and assume that x and y are both functions of t. Use implicit differentiation to find dy/dt in terms of x, y, and dx/dt.

9. In Example 14.8, we have $u^2 + v = x$ and $uv = 1 - x^2$. Write the first equation as $v = x - u^2$ and substitute into the second equation. Use this expression to find du/dx using implicit differentiation. Reconcile your answer with the result obtained in the text.

10. Draw the system of equations in Example 14.3.1 on the u-v axis (put v on the vertical axis) for the case when $x = 0$. How will your diagram change if x were to increase by a small amount? What are the values of u and v that solve the system at $x = 0$? Find the values of of u' and v' at $x = 0$.