Mathematics for Economics **Anthony Tay**

13. L'Hopital's Rule

Derivatives are defined using limits. Here we have an application where derivatives are used to find limits.

L'Hopital's Rule for Limit at a Point Suppose we are interested in finding

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

but face the problem that $f(a) = g(a) = 0$, as for example, when we tried to find

$$\lim_{h \to 0} \frac{e^h - 1}{h}.$$

L'Hopital's Rule v1

If (i) $f(a) = g(a) = 0$,

(ii) $f'(x)$ and $g'(x)$ both exist at a, and

(iii) $g'(a) \neq 0$,

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The rationale for L'Hopital's rule is simple: if $f(a) = g(a) = 0$, $f'(x)$ and $g'(x)$ exists at a, and $g'(a)$ $\neq 0$, then

$$\frac{f(x)}{g(x)} = \frac{[f(x) - f(a)]/(x-a)}{[g(x) - g(a)]/(x-a)}.$$

Taking limits gives us the right hand side as $\dfrac{f'(a)}{g'(a)}$.

Example 13.1 The limit of $f(h) = \dfrac{e^h - 1}{h}$ as $h \to 0$ is

$$\lim_{h \to 0} \frac{e^h - 1}{h} = \text{“}\frac{0}{0}\text{”} = \lim_{h \to 0} \frac{e^h}{1} = 1.$$

Example 13.2 $$\lim_{x \to 1} \frac{x-1}{x^2 - 1} = \text{“}\frac{0}{0}\text{”} = \lim_{x \to 1} \frac{1}{2x} = \frac{1}{2}.$$

(*You didn't need to use L'Hopital's Rule here, but that doesn't mean you can't.*)

Example 13.3

$$\lim_{x\to 2}\frac{x^4-4x^3+6x^2-8x+8}{x^3-3x^2+4} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 2}\frac{4x^3-12x^2+12x-8}{3x^2-6x} = \text{“}\frac{0}{0}\text{”} =$$

$$\lim_{x\to 2}\frac{12x^2-24x+12}{6x-6} = \frac{48-48+12}{12-6} = 2.$$

(*Nothing to stop you using L'Hopital's Rule twice if it continues to apply!*)

Example 13.4 $\quad \lim_{x\to 1}\frac{x^2+3x-4}{2x^2-2x} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 1}\frac{2x+3}{4x-2}$.

Note that

$$\lim_{x\to 1}(2x+3)/(4x-2)$$

does not have the “ $0/0$ ” form, so you cannot apply L'Hopital's rule to this limit (why?). If you do you will get the wrong answer: differentiating numerator and denominator, you get $\lim_{x\to 1}(2/4)=1/2$. The correct answer is

$$\lim_{x\to 1}\frac{2x+3}{4x-2} = \frac{\lim_{x\to 1}(2x+3)}{\lim_{x\to 1}(4x-2)} = \frac{5}{2} .$$

L'Hopital's Rule for Limits at Infinity The rule also works for the case where the “0/0” form appears when taking the limit as $x\to\pm\infty$, or if both $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$:

L'Hopital's Rule v2

Suppose $f(x)$ and $g(x)$ are both differentiable for all $x\ge M$ for some real number M . Suppose $\lim_{x\to\infty} f(x)=\lim_{x\to\infty} g(x)=0$. Then if $g'(x)\ne 0$,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

Proof Let $x=1/t$ (note that $x\to\infty$ as $t\to 0^+$). Then

$$\begin{aligned}\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{t\to 0^+}\frac{f(1/t)}{g(1/t)}\\ &= \lim_{t\to 0^+}\frac{f'(1/t)[-1/t^2]}{g'(1/t)[-1/t^2]}\\ &= \lim_{t\to 0^+}\frac{f'(1/t)}{g'(1/t)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}\end{aligned}$$

Example 13.5 [SH pg 264 Ex 4] Find $\lim_{x\to\infty} \sqrt[5]{x^5 - x^4} - x$.

We have
$$\begin{aligned}\lim_{x\to\infty} \sqrt[5]{x^5 - x^4} - x &= \lim_{x\to\infty} x\sqrt[5]{1-1/x} - x \\ &= \lim_{x\to\infty} x(\sqrt[5]{1-1/x} - 1) \\ &= \lim_{x\to\infty} \frac{\sqrt[5]{1-1/x} - 1}{1/x}\end{aligned}$$

This has a "$\frac{0}{0}$" form. Using L'Hopital's rule, we have

$$\lim_{x\to\infty} \frac{\sqrt[5]{1-1/x} - 1}{1/x} = \lim_{x\to\infty} \frac{(1/5)(1-1/x)^{-4/5}(1/x^2)}{-1/x^2} = \lim_{x\to\infty} -(1/5)(1-1/x)^{-4/5} = \frac{1}{5}.$$

L'Hopital's Rule for Limits for Other Indeterminate Forms Example 13.5 is also an example of how to deal with indeterminate forms of the type "$0.\infty$", such as $\lim_{x\to\infty} x(\sqrt[5]{1-1/x} - 1)$. Basically, rewrite the limit as a "0/0" type (or as a "∞/∞" type as below).

L'Hopital's Rule v3

Suppose $f(x)$ and $g(x)$ are both differentiable, with $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$, and $g'(x) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof:

$$\begin{aligned}\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{1/g(x)}{1/f(x)} = \lim_{x\to a} \frac{-g'(x)/[g(x)]^2}{-f'(x)/[f(x)]^2} \\ &= \lim_{x\to a} \frac{g'(x)}{f'(x)}\left(\frac{[f(x)]}{[g(x)]}\right)^2 = \lim_{x\to a} \frac{g'(x)}{f'(x)}\left(\lim_{x\to a} \frac{[f(x)]}{[g(x)]}\right)^2\end{aligned}$$

This implies $$\frac{1}{\lim_{x\to a}[f(x)/g(x)]} = \lim_{x\to a}[g'(x)/f'(x)]$$

Therefore $\lim_{x\to a}[f(x)/g(x)] = \lim_{x\to a}[f'(x)/g'(x)]$.

Example 13.6   "Exponentials grow faster than powers"

What is the limit of x^p / a^x when $a > 1$ and $p > 0$? We have

$$\lim_{x\to\infty} \frac{x^p}{a^x} = \text{“}\frac{\infty}{\infty}\text{”}$$

Rather than apply L'Hopital's Rule to $\lim_{x\to\infty} \frac{x^p}{a^x}$, we rewrite this as

$$\lim_{x\to\infty} \frac{x^p}{a^x} = \lim_{x\to\infty} \left(\frac{x}{a^{x/p}} \right)^p = \left(\lim_{x\to\infty} \frac{x}{a^{x/p}} \right)^p$$

We have

$$\lim_{x\to\infty} \frac{x}{a^{x/p}} = \text{“}\frac{\infty}{\infty}\text{”} = \lim_{x\to\infty} \frac{1}{a^{x/p}(1/p)\ln a} = 0$$

Therefore

$$\lim_{x\to\infty} \frac{x^p}{a^x} = \left(\lim_{x\to\infty} \frac{x}{a^{x/p}} \right)^p = 0$$

In other words, a^x grows faster than x^p for any $a > 1$, $p > 0$.

L'Hopital's Rule can also be applied to indeterminate forms of the type "0^0", "∞^0", "1^∞". The trick is similar to logarithmic differentiation:

Example 13.7 $\lim_{x\to 0^+} (1+x)^{1/x} = e$

Let $y = (1+x)^{1/x}$. Taking logs on both sides gives

$$\ln y = \ln(1+x)^{1/x} = \frac{\ln(1+x)}{x}$$

Thus, $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \lim_{x\to 0^+} \frac{1/(1+x)}{1} = 1$.

Since $\ln(\lim_{x\to 0^+} y) = 1$, $\lim_{x\to 0^+} y = e^1 = e$.

Note that "0^∞" is not an indeterminate form. Also, sometimes it is useful to transform the problem using functions other than the logarithmic function.

Some Comments on Indeterminate Forms When someone writes $\infty + \infty = \infty$, there is only one way to interpret the statement: this person is taking the limit of the sum of two objects, both of which are going to infinity (increases without bound), and the sum as a result also goes to infinity, as in

$$\lim_{x\to 0}\left(\frac{1}{x^2}+\frac{1}{x^4}\right) = \infty + \infty = \infty .$$

While, strictly speaking, you cannot do arithmetic with infinities, you can think of the above as a shorthand way of saying "because $1/x^2$ and $1/x^4$ both increase without bound, their sum also increases without bound". Similarly, you might encounter a statement like

$$\lim_{x\to\infty}\frac{1-1/x}{x} = "\frac{1}{\infty}" = 0$$

where again, a 'shorthand' is used.

However, note that only some shorthands work. For example,

"$\infty + \infty = \infty$", "$-\infty - \infty = -\infty$", "$1/\infty = 0$", "$\infty/1 = \infty$",
"$0^\infty = 0$", "$\infty^\infty = \infty$", "$\infty/0 = \infty$", "$\infty.\infty = \infty$"

are all fine in the sense that you will never find an $f(x)\to\infty$ and $g(x)\to\infty$ such that $f(x)+g(x)$ does not also $\to\infty$, for instance. Likewise, you will never find functions $f(x)\to 0$ and $g(x)\to\infty$ such that $f(x)^{g(x)}$ does not go to zero.

However, the following statements are wrong:

"$\infty - \infty = 0$", "$\infty/\infty = 1$", "$0/0 = 1$", "$0/0 = \infty$"
"$0.\infty = 0$", "$\infty^0 = 1$", "$1^\infty = 1$", "$0^0 = 1$"

These are all "indeterminate forms" and it is easy to show counterexamples:

$\lim_{x\to\infty}(x - x^2) = "\infty - \infty"$ but $\lim_{x\to\infty}(x - x^2) = -\infty$

$\lim_{x\to\infty}(x^2 - x) = "\infty - \infty"$ but $\lim_{x\to\infty}(x^2 - x) = \infty$

$\lim_{x\to\infty}(x^2 - x^2) = "\infty - \infty"$ but $\lim_{x\to\infty}(x^2 - x^2) = 0$.

We saw in this section examples of a $"\infty - \infty"$ that converged to a non-zero finite number. Other examples:

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = "1^\infty" = e$, $\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = "1^\infty" = 1$,

$\lim_{x\to 0^+} x^x = "0^0" = 1$, $\lim_{x\to 0^+} x^{\left(\frac{\ln 2}{1+\ln x}\right)} = "0^0" = 2$.

Exercises

1. Find the limits

(i) $\lim_{x\to 1}\frac{\ln x}{1-x}$ (ii) $\lim_{x\to 3}\frac{x-3}{3x^2-13x+12}$ (iii) $\lim_{x\to 0}\frac{xe^x}{1-e^x}$

(iv) $\lim_{x\to 0^+}\frac{1-\ln x}{e^{1/x}}$ (v) $\lim_{x\to\infty}\frac{x^{100}}{e^x}$ (vi) $\lim_{x\to\infty} xe^{-x}$

(vii) $\lim_{x\to\infty}\frac{e^{3x}}{x^2}$ (viii) $\lim_{x\to\infty}\frac{\ln x}{e^x}$ (ix) $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$

(x) $\lim_{x\to\infty}\frac{\ln x}{x^n}$ for any positive n (xi) $\lim_{x\to\infty}\frac{\ln(\ln x)}{\sqrt{x}}$

2. Find the error in the following calculation, and find the correct solution:

$$\lim_{x\to 1}\frac{x^3-x^2+x-1}{x^3-x^2} = \lim_{x\to 1}\frac{3x^2-2x+1}{3x^2-2x} = \lim_{x\to 1}\frac{6x-2}{6x-2} = 1$$

3. Find the limits

(i) $\lim_{x\to\infty}(1-3/x)^x$ (ii) $\lim_{x\to 0}(e^x+x)^{1/x}$ (iii) $\lim_{x\to 0^+} x^x$