9. Elasticities

Economic analysis frequently focuses on elasticities – the percentage change in one variable that results from a certain percentage change in another variable: price elasticities of demand, income elasticities of demand, elasticity of substitution, etc. The elasticity is just another way of expressing a rate of change.

<u>*Average Elasticity*</u> Suppose $y = f(x)$. The **Average Elasticity** of x in the interval $[x, x+\Delta x]$ is defined as

$$\frac{[f(x+\Delta x) - f(x)]/f(x)}{\Delta x / x}$$

where "Δx" represent a change in x. With some rearrangement, we can write this as

$$\frac{x}{f(x)}\frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The **Elasticity** of $f(x)$ with respect to x is defined as

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x).$$

This is just the limiting version of the average elasticity:

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)]/f(x)}{\Delta x / x} &= \lim_{\Delta x \to 0} \frac{x}{f(x)} \frac{[f(x+\Delta x) - f(x)]}{\Delta x} \\ &= \frac{x}{f(x)} \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)]}{\Delta x}. \end{aligned}$$

The elasticity of $f(x)$ w.r.t. x is therefore just the instantaneous *proportional* rate of change of $f(x)$ wrt x, just as the derivative is the instantaneous rate of change. Note that the elasticity of $f(x)$ w.r.t. x is itself a function of x.

<u>Example 9.1</u> If $f(x) = x^2$, then $f'(x) = 2x$, and therefore

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x) = \frac{x}{x^2} 2x = 2.$$

Example 9.2 If $f(x) = \left(\frac{x+1}{x-1}\right)^{1/3}$, then

$$f'(x) = (\text{...grind grind...}) = -\frac{2}{3}(x+1)^{-2/3}(x-1)^{-4/3}.$$

Thus,

$$\begin{aligned} \mathrm{El}_x f(x) &= \frac{x}{f(x)} f'(x) \\ &= x(x+1)^{-1/3}(x-1)^{1/3}\left(-\frac{2}{3}(x+1)^{-2/3}(x-1)^{-4/3}\right) \\ &= -\frac{2}{3}\frac{x}{(x+1)(x-1)} \\ &= -\frac{2}{3}\frac{x}{x^2-1} \end{aligned}$$

It is worthwhile "translating" the rules for differentiation into rules for elasticities.

Example 9.3 If $f(x) = g(x)h(x)$, then $\mathrm{El}_x(gh) = \mathrm{El}_x g + \mathrm{El}_x h$.

We have $f' = g'h + gh'$. Therefore

$$\mathrm{El}_x f = \frac{x}{f} g'h + gh' = \frac{x(g'h + gh')}{gh} = \frac{xg'}{g} + \frac{xh'}{h} = \mathrm{El}_x g + \mathrm{El}_x h$$

You can easily (and should!) derive "elasticity rules" corresponding to the other differentiation rules.

Often, the easiest way is the straightforward application of the definition

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x).$$

On occasion, it might be easier to apply the following trick:

$$\frac{d}{dx}\ln f(x)$$

and multiply the result by x. This is because

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)},$$

so that

$$\left(\frac{d}{dx}\ln f(x)\right)x = \frac{f'(x)}{f(x)}x = \mathrm{El}_x f(x).$$

<u>Example 9.4</u> Suppose

$$f(x) = \left(\frac{x+1}{x-1}\right)^{1/3}, \; x \in (-\infty,-1) \cup (1,\infty)$$

then $\ln f(x) = \frac{1}{3}\left[\ln\left(x+1\right) - \ln(x-1)\right]$ so that $\frac{f'(x)}{f(x)} = -\frac{2}{3}\left[\frac{1}{x^2-1}\right]$, and

$$\mathrm{El}_x f(x) = \frac{f'(x)}{f(x)} x = -\frac{2}{3}\left[\frac{x}{x^2-1}\right].$$

Exercises

1. Find

(a) $El_x y$ when $y = \frac{1+x}{1-x}$

(b) $El_x f(x)$ when $f(x) = e^{ax}$

(c) $El_x g(x)$ when $g(x) = [f(x)]^2$

(d) $El_x\, g(x) f(x)$ in terms of $El_x g(x)$ and $El_x f(x)$.

(e) $El_x\, x^r$

(f) $El_x \log x$