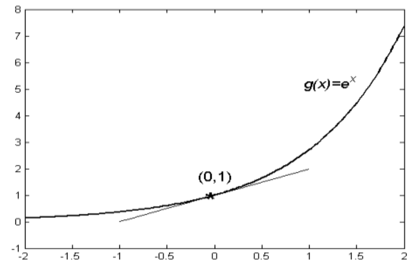
8. Derivatives

It is a fundamental task in many mathematical applications to find the rate of change of a function $f(x)$ at some given point a.

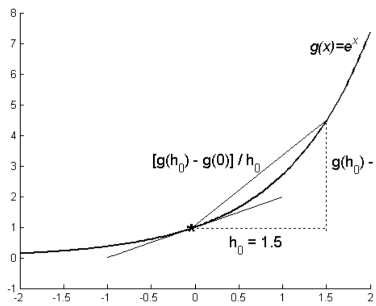


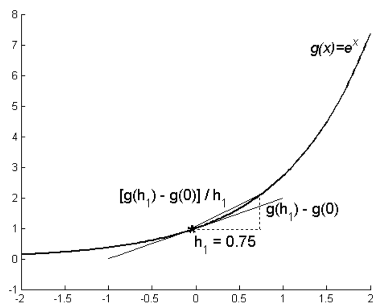
In an earlier example, we were interested in finding out the equation of the tangent to the function $g(x) = e^x$ at the point $x = 0$.

What we did was to take the ratio

$$\frac{g(0+h)-g(0)}{h} = \frac{e^h - 1}{h}$$

and ask what happens when $h \to 0$. Even though both numerator and denominator goes to zero, we discovered that $\lim_{h\to 0}(e^h - 1)/h$ exists (and is equal to one), and so this strategy for computing the slope of $g(x)$ at $x = 0$ is feasible.





Differentiation is the application of this strategy for determining the slope of a function. When we apply this technique to all values of x, we obtain the slope of the function $g(x)$ at each of these values. In other words, we obtain another function, which we will denote by $g'(x)$, that tells us the slope of the function at any given value of x. The function $g'(x)$ is called the derivative of $g(x)$.

I assume you already know the elementary techniques for finding the derivatives of functions (*perhaps additional practice might be needed*). You may, however, be less familiar with the reasoning behind these differentiation techniques, or may have learnt simplified accounts of these arguments. To go further, you will need a proper understanding of the foundations underlying derivatives, and it is the objective of this section to help you acquire this background.

8.1 *Formal definition of the derivative*

The derivative of a function $f(x)$ is defined as the function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

It gives us the slope of the tangent to the graph of $f(x)$ at any point x in its domain. The process of computing the derivative of a function is called *differentiation*. I will apply this definition to obtain, from 'first principles', the derivatives of certain functions. Later, we develop and apply general rules.

Example 8.1.1 If $f(x) = x^2$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}(2x+h) = 2x$$

Example 8.1.2 If $f(x) = a+bx$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a+b(x+h)-(a+bx)}{h} = \lim_{h\to 0} \frac{b\cancel{h}}{\cancel{h}} = b$$

There are many ways of denoting the derivative, including:

(i) $f'(x)$ (ii) $\frac{d}{dx}f(x)$ (iii) $\frac{df(x)}{dx}$ (iv) $\frac{dy}{dx}$ (v) y' (vi) $\dot{y}$

We use whichever is more convenient. Typically we use the first three when the function is represented in $f(x)$ form. The forms (iv) and (v) are used when the function is represented as $y = (\text{expression involving } x)$. The last is used exclusively when y is a function of time t.

Thus, all of the following say exactly the same thing:

"If $y = x^2$, then $\frac{dy}{dx} = 2x$" "If $y = x^2$, then $y' = 2x$"

"If $f(x) = x^2$, then $f'(x) = 2x$" "If $f(x) = x^2$, then $\frac{df(x)}{dx} = 2x$"

"If $f(x) = x^2$, then $\frac{d}{dx}f(x) = 2x$" "$\frac{d(x^2)}{dx} = 2x$" "$(x^2)' = 2x$"

It is important to emphasize that the derivative is a function. The derivative of the function $g(x) = x^2$ is the function $g'(x) = 2x$. As emphasized by the form (ii), a derivative is not a ratio of the two quantities "dy" and "dx". At this point, we have not given any rationale for writing

$$\text{“}\frac{dy}{dx} = 2x \text{, therefore } dy = 2x\,dx\text{”}$$

(We will make such a statement later, when discussing 'differentials', and it is important to understand properly the rationale for doing so). For now, develop the correct perspective that "$\frac{dy}{dx}$" is a symbol representing a certain function, 'derived' from the original. It is also useful to think of "$\frac{d}{dx}$" as an operator that when applied to the function $y = f(x)$ produces another function, the derivative. This perspective is emphasized by the notational form (ii).

8.2 Techniques for Differentiation

We have shown from the definition of the derivative that

$$f(x) = x^2 \Rightarrow f'(x) = 2x .$$

For simple functions like this, and even functions like $f(x) = 1/x$ ($f'(x) = -1/x^2$), it is easy to compute the derivative from definition. For more complicated functions such as

$$f(x) = (2x^7 - x^2)\left(\frac{x-1}{x+1}\right),$$

working directly from the definition is not really feasible. The power of the derivative idea lies in the ability to generate easy-to-apply '*rules*' that make finding derivatives of very complicated functions a simple task. The most commonly used techniques are:

If A is some constant value, then

Constant Rule $f(x) = A \;\Rightarrow\; f'(x) = 0$ (*This follows directly from 8.1.2.*)

Constant Multiple Rule $g(x) = Af(x) \;\Rightarrow\; g'(x) = Af'(x)$

Example 8.2.1 $f(x) = 6x^2 \;\Rightarrow\; f'(x) = 6(2x) \;=\; 12x$.

Summation Rule $k(x) = g(x) + f(x) \;\Rightarrow\; k'(x) = g'(x) + f'(x)$

Example 8.2.2

$$f(x) = 6x^2 + 1/x \;\Rightarrow\; f'(x) = 12x - 1/x^2 .$$

The Product Rule $k(x) = g(x)f(x) \;\Rightarrow\; k'(x) = g'(x)f(x) + g(x)f'(x)$

(This is often written "$y = uv \;\Rightarrow\; y' = u'v + uv'$" if y, u, v are all functions of x.)

Example 8.2.3 If $f(x) = (3x+10)(6x^2 - 7x)$. Then

$$\begin{aligned} f'(x) &= (3x+10)\left(\frac{d}{dx}(6x^2 - 7x)\right) + \left(\frac{d}{dx}(3x+10)\right)(6x^2 - 7x) \\ &= (3x+10)(12x-7) + 3(6x^2 - 7x) \\ &= 54x^2 + 78x - 70 \end{aligned}$$

Chain Rule $k(x) = f(g(x)) \Rightarrow k'(x) = f'(g(x))g'(x)$

(*perhaps more familiar as* " $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$ *where* y *is a fn of* z, *and* z *is a fn of* x.)

Example 8.2.4 Let $f(x) = \frac{1}{(6x^2 - 7x)}$.

Let $f(x) = g(h(x))$ where $g(x) = 1/x$, $h(x) = 6x^2 - 7x$. Then $g'(x) = -1/x^2$ and $h'(x) = 12x - 7$, therefore

$$f'(x) = g'(h(x))h'(x) = -\frac{1}{(6x^2 - 7x)^2}(12x - 7).$$

Quotient Rule $k(x) = f(x)/g(x) \Rightarrow k'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Example 8.2.5 Let $f(x) = \frac{(3x + 10)}{(6x^2 - 7x)}$. Then

$$\begin{aligned} f'(x) &= \frac{(3x+10)'(6x^2 - 7x) - (3x+10)(6x^2 - 7x)'}{(6x^2 - 7x)^2} \\ &= \frac{3(6x^2 - 7x) - (3x+10)(12x - 7)}{(6x^2 - 7x)^2} \\ &= \frac{70 - 120x - 18x^2}{(6x^2 - 7x)^2} \end{aligned}$$

Logarithm Rules $f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a}$, $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

Exponential Rules $f(x) = a^x \Rightarrow f'(x) = a^x \ln a$, $f(x) = e^x \Rightarrow f'(x) = e^x$

The exponential and logarithmic rules are especially useful in conjunction with the other differentiation rules:

Example 8.2.6 $f(x) = \ln(x^2 + 7) \Rightarrow f'(x) = \frac{1}{x^2 + 7}(2x)$

Example 8.2.7 $f(x) = e^{1-9x^2} \Rightarrow f'(x) = (1 - 9x^2)' e^{1-9x^2} = -18x\, e^{1-9x^2}$.

Example 8.2.8 $f(x) = a^{1-9x^2} \Rightarrow f'(x) = -18x(a^{1-9x^2})(\ln a)$

Power rule $f(x) = x^r \Rightarrow f'(x) = rx^{r-1}$ for any real number r

Example 8.2.9 $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$

The proofs for all of these are collected in an appendix.

The following example illustrates a technique called 'logarithmic differentiation', which uses the fact that, by the logarithmic rule and the chain rule,

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$$

Example 8.2.10

Let $f(x) = x^x$, $x > 0$. Then

$$\ln f(x) = x\ln x \quad \Rightarrow \quad f'(x)/f(x) = x(1/x) + \ln x$$

$$\Rightarrow \quad f'(x) = (1+\ln x)f(x)$$

$$\Rightarrow \quad f'(x) = x^x(1+\ln x)$$

8.3 Some Features of Derivatives

The next two examples illustrate two very important features of derivatives.

Example 8.3.1 $\qquad f(x) = \begin{cases} 2x+1 & x \geq 1 \\ 2x & x < 1 \end{cases}$

For $x > 1$ and $x < 1$, we can apply the general result in example 8.2.2 and say that $f'(x) = 2$. It is not so clear if we can say the same thing at the point $x = 1$. The problem is that in the definition $f'(x) = [f(x+h) - f(x)]/h$, the form of the function $f(x)$ is different, depending on whether $h < 0$, in which case $x + h = 1 + h < 1$, or $h > 0$, in which case $x + h = 1 + h > 1$. We will have to take right-limits and left-limits, in computing the derivative.

We have at $x = 1$:

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-}\frac{[2(1+h)]-[2(1)+1]}{h} = \infty$$

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+}\frac{[2(1+h)+1]-[2(1)+1]}{h} = 2$$

Note that for both cases, $f(x) = 2x+1$ when $x = 1$. Because the left and right limits of the 'derivative' are not the same, the derivative of $f(x)$ does not exist at $x = 1$. We write $f'(x) = 2,\ x \neq 1$.

Example 8.3.2 $\qquad f(x) = |x|$. This can be written $f(x) = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$.

It is clear that $f'(x) = 1$ for $x > 0$, and $f'(x) = -1$ for $x < 0$. What happens at $x = 0$?

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^-} \frac{-(0+h) - (0)}{h} = -1$$

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{(0+h) - (0)}{h} = 1$$

The derivative $f'(x)$ does not exist at $x = 0$.

These examples illustrates the following facts about derivatives

– for the derivative of a function to exist at $x = a$, the function must be continuous at $x = a$

– the derivative of a function at $x = a$ need not exist even if the function is continuous at $x = a$. It also needs to be *smooth* enough.

Example 8.3.2 is sufficient proof of the second assertion. The first is easily seen from the definition of a derivative. We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Since the denominator goes to zero as $h \to 0$, in order for the limit to exist, the numerator must also go to zero as $h \to 0$. But this is exactly the definition of continuity at x. This also implies that the derivative at $x = a$ can exist only if the function is defined there. Thus, the domain of the derivative may be a strict subset of the domain of the function.

8.4 Using the derivative to describe a function

We often use the derivative to describe whether a function is increasing or decreasing over certain intervals. A function is non-decreasing if for all x_1 and x_2,

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2).$$

A function is strictly increasing if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

Note: (i) A strictly increasing function is obviously also non-decreasing.

(ii) Sometimes a non-decreasing function is also called an increasing function. This may seem odd to you, because a constant function $f(x) = c$ would then be an ‘increasing function’, and also a ‘decreasing function’.

We have $f'(x) \geq 0$ for all x in some interval I $\Leftrightarrow$ f is non-decreasing over I

$f'(x) \leq 0$ for all x in some interval I $\Leftrightarrow$ f is non-increasing over I

and $f'(x) > 0$ for all x in some interval I $\Rightarrow$ f is strictly increasing over I

$f'(x) < 0$ for all x in some interval I $\Rightarrow$ f is strictly decreasing over I

Note that a function can be strictly increasing over an interval I without $f'(x) > 0$ for all $x \in I$. Similarly, a function can be strictly decreasing over I without $f'(x) < 0$ for all $x \in I$ In particular, it is possible that $f'(x) = 0$ at some points

Example 8.4.1 x^3 is strictly increasing, but $f'(x) = 3x^2$ is not strictly positive for all x. In particular, $f'(0) = 0$.

Example 8.4.2 Consider the function $f(x) = 1/x$, defined over $(-\infty, 0) \cup (0, \infty)$. This function has derivative $-1/x^2$. This derivative is negative for all $x \in (-\infty, 0)$ and for all $x \in (0, \infty)$, and the function is strictly decreasing within these two intervals. However, over the full set $(-\infty, 0) \cup (0, \infty)$, the function is not strictly decreasing, because $-1 < 1$, but $f(-1) < f(1)$.

8.A.1 Appendix: Proofs of Rules for Differentiation

$f(x) = A$ $\qquad \frac{f(x+h)-f(x)}{h} = \frac{A-A}{h} = 0$, therefore $f'(x) = \lim_{h\to 0} 0 = 0$

$g(x) = Af(x)$ $\qquad \frac{g(x+h)-g(x)}{h} = \frac{Af(x+h)-Af(x)}{h} = A\left[\frac{f(x+h)-f(x)}{h}\right]$, therefore $g'(x) = \lim_{h\to 0} A\left[\frac{f(x+h)-f(x)}{h}\right] = Af'(x)$.

$k(x) = g(x)+f(x)$ $\qquad \frac{k(x+h)-k(x)}{h} = \frac{g(x+h)+f(x+h)-g(x)-f(x)}{h} = \frac{g(x+h)-g(x)}{h} + \frac{f(x+h)-f(x)}{h}$

therefore $\qquad k'(x) = \lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h} + \frac{f(x+h)-f(x)}{h}\right] = g'(x)+f'(x)$

$k(x) = g(x)f(x)$ $\qquad \frac{k(x+h)-k(x)}{h} = \frac{g(x+h)f(x+h)-g(x)f(x)}{h}$

$$= \frac{[(g(x+h)-g(x))+g(x)][(f(x+h)-f(x))+f(x)]-g(x)f(x)}{h}$$

$$= \frac{[g(x+h)-g(x)][f(x+h)-f(x)]+g(x)f(x)-g(x)f(x)}{h} + \frac{[g(x+h)-g(x)]f(x)+[f(x+h)-f(x)]g(x)}{h}$$

$$= \left[\frac{g(x+h)-g(x)}{h}\right]\left[\frac{f(x+h)-f(x)}{h}\right]h + \left[\frac{g(x+h)-g(x)}{h}\right]f(x) + \left[\frac{f(x+h)-f(x)}{h}\right]g(x)$$

therefore $\qquad k'(x) = \lim_{h\to 0} [\frac{g(x+h)-g(x)}{h}\frac{f(x+h)-f(x)}{h}h + \frac{g(x+h)-g(x)}{h}f(x) + \frac{f(x+h)-f(x)}{h}g(x)]$

$$= g'(x)\,f'(x)\,0 + g'(x)f(x) + f'(x)g(x) = g'(x)f(x) + f'(x)g(x).$$

$k(x) = f(g(x))$

$$\frac{k(x+h)-k(x)}{h} = \frac{f(g(x+h))-f(g(x))}{h}$$

$$= \frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} \frac{g(x+h)-g(x)}{h}$$

therefore, $k'(x) = \lim_{h\to 0} \frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} \frac{g(x+h)-g(x)}{h}$.

Writing $u = g(x+h)-g(x)$, or $g(x+h) = g(x)+u$, we have

$$k'(x) = \lim_{h\to 0} \frac{f(g(x)+u)-f(g(x))}{u} \frac{g(x+h)-g(x)}{h} = f'(g(x))g'(x)$$

$h(x) = \frac{f(x)}{g(x)}$

$$h(x) = \frac{f(x)}{g(x)} = f(x)\left[\frac{1}{g(x)}\right] = f(x)[g(x)]^{-1}$$

Therefore $h'(x) = f(x)\left[-\frac{g'(x)}{[g(x)]^2}\right] + f'(x)[g(x)]^{-1} = \frac{f'(x)g(x)}{[g(x)]^2} - \frac{g'(x)f(x)}{[g(x)]^2} = \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$

Recall $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828182845...$

$f(x) = \log_a x \qquad \frac{f(x+h)-f(x)}{h} = \frac{\log_a(x+h)-\log_a x}{h} = \frac{1}{h}\log_a\left(\frac{x+h}{x}\right) = \frac{1}{x}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right) = \frac{1}{x}\log_a\left(1+\frac{1}{x/h}\right)^{x/h}$

$$\text{Therefore } f'(x) = \lim_{h\to 0}\frac{1}{x}\log_a\left(1+\frac{1}{x/h}\right)^{x/h} = \frac{1}{x}\log_a\left[\lim_{h\to 0}\left(1+\frac{1}{x/h}\right)^{x/h}\right] = \frac{1}{x}\log_a\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right] = \frac{1}{x}\log_a e = \frac{1}{x}\frac{1}{\ln a}$$

Furthermore, if $f(x)=\ln x$ $(i.e., a=e)$, then $\quad f'(x) = \frac{1}{x}\ln e = \frac{1}{x}$.

The logarithm rule above and the chain rule implies that

$$\frac{d}{dx}\ln g(x) = \frac{1}{g(x)}\frac{dg(x)}{dx} = \frac{g'(x)}{g(x)}.$$

This can be extremely helpful in differentiation. We can also prove many of the rules using this technique.

$g(x) = a^x$ We have $\ln g(x) = x\ln a$, therefore $\frac{g'(x)}{g(x)} = \ln a$, which implies $g'(x) = g(x)\ln a = a^x \ln a$.

Furthermore, when $g(x)=e^x$, i.e., $a=e$, then $g'(x) = e^x \ln e = e^x$

$f(x)=x^r$, $r\in\mathbb{R}$ We have $\ln f(x) = r\ln x$, therefore $\frac{f'(x)}{f(x)} = \frac{r}{x}$, which implies that $f'(x) = f(x)\frac{r}{x} = x^r\frac{r}{x} = rx^{r-1}$.

Exercises

1. (a) For each of the following, find $\dfrac{dy}{dx}$.

   (i) $y = \sqrt{x} + \dfrac{1}{x}$ (ii) $y = \dfrac{1}{a}\left(x^2 + \dfrac{1}{b}x + c\right)$, a,b,c are constants

   (iii) $y = x^{\pi} + e^{x^2}$ (iv) $y = (2x^7 - x^2)\left(\dfrac{x-1}{x+1}\right)$

   (b) For each of the functions in (a), find $\left.\dfrac{dy}{dx}\right|_{x=1}$ the equation of the tangent line to the function at $x=1$.

   (c) For each of the following, find $f'(x)$.

   (i) $f(x) = \log\left(\dfrac{x}{x^2+1}\right)$ (ii) $f(x) = \sqrt{1+\ln^2 x}$

   (iii) $f(x) = x\left[\log_2(2x^2 - x)\right]^3$ (iv) $f(x) = e^x \ln(x^2+2)$

   (d) For each of the functions in (c), find $f'(1)$ and the equation of the tangent line to the function at $x=1$.

2. (a) Given $y = x^3 + 3x + 1$, find y', y'', and y''' and show that it satisfies the equation $y''' + xy'' - 2y' = 0$.

   (b) Find $\dfrac{d}{d\lambda}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2-\lambda_0}\right]$ where λ_0 is a constant; Find $\dfrac{d^2}{d\lambda^2}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2-\lambda_0}\right]$.

   (c) Find expressions for the first and second derivatives for each of the following

   (i) $\sqrt{x}\, f(x)$ (ii) $\dfrac{2x+1}{f(x)}$ (iii) $\dfrac{f(x)}{4+g(x)}$

   Find the value of the first and second derivatives at $x=1$ if

   $$f(1)=2,\ f'(1)=1,\ f''(1)=1/2,\ g(1)=1,\ g'(1)=3,\ g''(1)=1/3.$$

3. (a) Find $\dfrac{dy}{dx}$ when $y=\sqrt{u}$ and $u = x^3 - 2x + 5$

   (b) If $f(x) = 1/x$ and $g(x) = x^3 - 2x + 5$, find

   (i) $f'(g(x))$ (ii) $[f(g(x))]'$ (iii) $(f \circ g)'(x)$

   (c) Find $f'(x)$ when $f(x) = \sqrt{4+\sqrt{3x}}$

4. Find y' using logarithmic differentiation

   (i) $y = \dfrac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$ (ii) $y = (x^2+3)^{\ln x}$ (iii) $y = x\sqrt[3]{1+x^2}$

5. (a) Let f, g, and h be functions of x. The product rule says that

$$(f\,g)' = f'\,g + f\,g'.$$

Use the product rule twice to show that

$$(f\,g\,h)' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$

(b) Use the rule in (a) to compute $\frac{d}{dx}\left[(2x+1)\left(1+\frac{1}{x}\right)(x^{-3}+7)\right]$.

(c) Given the functions f and g, find expressions for

(i) $(f\,g)''$ (ii) $(f\,g)'''$ (iii) $(fg)'''$

Find an expression for $(fg)^{(n)}$ using the binomial coefficients.

6. Prove that $\frac{d}{dx}\log(|x|) = \frac{1}{x}$ for all $x \neq 1$. (Note that $|x| = \sqrt{x^2}$.) Plot the function $f(x) = \log(|x|)$ and its derivative.

7. (a) Suppose $g(x) = f(f(f(x)))$, show that

$$g'(x) = f'(f(f(x))f'(f(x))f'(x).$$

Verify this formula in the case where $f(x) = x^2$, i.e., $g(x) = x^8$, by (i) using the formula, (ii) differentiating $g(x)$ directly.

(b) Find the derivative of $g(x) = \exp(\exp(\exp(x)))$, i.e., of $g(x) = e^{e^{e^x}}$.

(c) Find the derivative of $g(x) = \sqrt{1+\sqrt{1+\sqrt{1+x}}}$.

8. Find out if each of the following functions are continuous and differentiable at $x = 1$:

(a) $f(x) = \begin{cases} x^2+1, & x \leq 1 \\ 2x, & x > 1 \end{cases}$ (b) $f(x) = \begin{cases} x^2+2, & x \leq 1 \\ x+2, & x > 1 \end{cases}$

Sketch the graphs as accurately as you can.