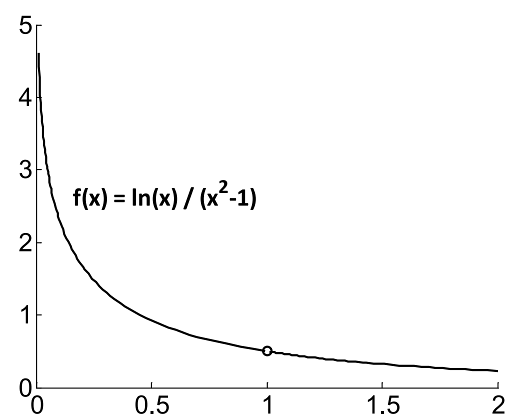
Mathematics for Economics **Anthony Tay**

7. Limits of a function

The concept of a limit of a function is one of the most important in mathematics. Many key concepts are defined in terms of limits (e.g., derivatives and continuity). The primary objective of this section is to help you acquire a firm intuitive understanding of the concept.

Roughly speaking, the limit concept is concerned with the behavior of a function f *around* a certain point (say a) rather than with the value $f(a)$ of the function at that point. The question is: what happens to the value of $f(x)$ when x gets closer and closer to a (without ever reaching a)?

Example 7.1 Take the function $f(x) = \dfrac{\ln x}{x^2 - 1}$.

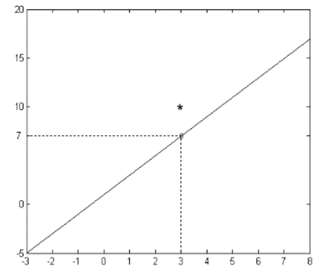


This function is not defined at the point $x = 1$, because the denominator at $x = 1$ is zero. However, as x 'tends' to 1, the value of the function 'tends' to $\frac{1}{2}$. We say that "the limit of the function $f(x)$ is $\frac{1}{2}$ as x approaches 1, or

$$\lim_{x \to 1} \frac{\ln x}{x^2 - 1} = \frac{1}{2}$$

It is important to be clear: **the limit of a function and the value of a function are two completely different concepts**. In example 7.1, for instance, the value of $f(x)$ at $x = 1$ does not even exist; the function is undefined there. However, the limit as $x \to 1$ does exist: the value of the function $f(x)$ does tend to something (in this case: $\frac{1}{2}$) as x gets closer and closer to 1. To emphasize this point, consider the following example:

Example 7.2 Let $f(x) = \begin{cases} 10, & x = 3 \\ 2x+1, & x \neq 3 \end{cases}$

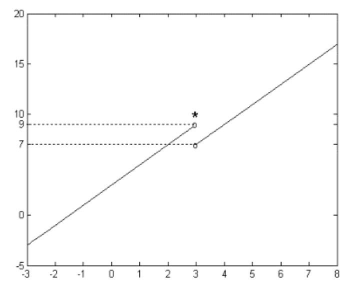


This function is defined at $x = 3$; we have $f(3) = 10$. However, the limit of $f(x)$ as $x \to 3$ is not 10. If we imagine any sequence of x's approaching 3, (but never reaching 3), then it should be clear that the value of the function $f(x) = 2x+1$ approaches 7:

$$\lim_{x \to 3} f(x) = 7$$

The next example again emphasizes the point that the limit of a function and the value of a function are two different concepts. The example also illustrates the point that limits do not always exist.

Example 7.3 Let $f(x) = \begin{cases} 2x+3, & x < 3 \\ 10, & x = 3 \\ 2x+1, & x > 3 \end{cases}$



The value of the function at $x = 3$ is 10: $f(3) = 10$. What is its limit as $x \to 3$?

If we restrict ourselves to sequence of x's approaching 3 from the left, the value of the function approaches 9. However, for sequence of x's approaching 3 from the right, the function approaches 7. It appears that the limit of the function depends on how the approach towards $x = 3$ was made.

In this case, we say that "the left limit of the function $f(x)$ is 9", and "the right limit of the function $f(x)$ is 7". We write

$$\lim_{n \to 3^-} f(x) = 9 \text{ and } \lim_{n \to 3^+} f(x) = 7.$$

But we say that the limit of this function does not exist.

If the limit of a function exists, then their right- and left-limits must also exist, and must be equal. In fact, to find limits (or existence of limits) of 'piece-wise' functions such as examples 7.2 and 7.3, it is often a fruitful strategy to find their left- and right-limits, and see whether they coincide.

Why it is useful to have a concept that focuses on what a function tends to as $x \to a$, as opposed to the value of the function at $x = a$? As a quick application that highlights this distinction, consider the concept of continuous functions. How might we define continuity? One way to do this is to say that a function is continuous at a point a if

$$\lim_{x \to a} f(x) = f(a)$$

A function is continuous at a point a if the limit of the function as $x \to a$ and the value $f(a)$ of the function at a coincides. A function is not continuous at $x = a$ if either (i) the limit doesn't exist as $x \to a$, (ii) the function doesn't exist at a, or (iii) both the limit and the value at $x = a$ exist, but are not equal to each other. The function in example 7.1 is not continuous at $x = 1$. The functions in examples 7.2, and 7.3 are not continuous at $x = 3$. They are continuous at all other points in their domains.

In more advanced courses, you will come across a more formal definition of limits. This definition is given in an appendix to this set of notes. For simple functions it is easy to use this definition to formally prove a certain limit result (such as $\lim_{x \to a} 3x = 3a$), but we will rely on informal arguments at this point. For more complicated expressions, the following rules are helpful:

Rules for finding limits If $\lim_{x \to x_0} f(x) = a$ and $\lim_{x \to x_0} g(x) = b$, then

(i) $\lim_{x \to x_0} [f(x) + g(x)] = a + b$;

(ii) $\lim_{x \to x_0} [f(x)g(x)] = ab$;

(iii) $\lim_{x \to x_0} [f(x)/g(x)] = a/b$, provided $b \neq 0$;

(iv) $\lim_{x \to x_0} [f(x)]^{p/q} = a^{p/q}$;

(v) if $g(x)$ is continuous, then $\lim_{x \to x_0} g(f(x)) = g(f(x_0))$.

We will not prove these rules, but merely illustrate their use with a few examples.

Example 7.4 Find $\lim_{x \to -1} \dfrac{3 + 3x}{x - 1}$.

It is clear from the limit laws that $\lim_{x \to -1}(3 + 3x) = 0$ (the full argument is

$$\lim_{x \to -1}(3 + 3x) = \lim_{x \to -1} 3 + \lim_{x \to -1} 3x = \lim_{x \to -1} 3 + \left(\lim_{x \to -1} 3\right)\left(\lim_{x \to -1} x\right) = 3 + 3(-1) = 0$$

but this can be easily done in one step.) Similarly, $\lim_{x\to -1}(x-1) = -2$. Therefore

$$\lim_{x\to -1}\frac{3+3x}{x-1} = \underbrace{\frac{\lim_{x\to -1}(3+3x)}{\lim_{x\to -1}(x-1)}}_{\text{valid because } \lim_{x\to -1}(x-1)\neq 0} = \frac{0}{-2} = 0\,.$$

Note It is tempting to simply substitute $x = -1$ into $(3+3x)/(x-1)$ to find this limit. In this case you do get the correct answer (because the function happens to be continuous at $x = -1$). However, the *argument* would be incorrect.

Note that if the limit of the denominator had been zero, we would not have been able to use the rules for calculating limits. Yet these are very important cases. Here are some examples to help us understand what can happen in such situations.

Example 7.5 $\quad f(x) = \dfrac{1}{(x-a)^2}$, $x \neq a$.

Clearly $f(x)$ increases without bound for any sequence of x approaching a (whether from the right or from the left) so the limit does not exist. We write

$$\text{“}\lim_{x\to a^+} f(x) = \infty\text{” and “}\lim_{x\to a^-} f(x) = \infty\text{”},$$

as shorthand for the statements "$f(x)$ increases without bound as x approaches a from the left", and "$f(x)$ increases without bound as x approaches a from the right". We can also write

$$\lim_{x\to a} f(x) = \infty$$

for this example (*but strictly speaking, the limit does not exist!*)

Example 7.6 $\quad f(x) = \dfrac{1}{(x-a)}$, $x \neq a$.

Clearly $f(x)$ decreases without bound for any sequence of x approaching a from the left, and increases without bound for any sequence of x approaching a from the right. We write

$$\lim_{x\to a^+} f(x) = \infty \text{ and } \lim_{x\to a^-} f(x) = -\infty.$$

The limit itself, $\lim_{x\to a} f(x)$, does not exist.

Example 7.7 $\quad$ Find $\lim_{x\to 2}\dfrac{x+1}{x-2}$.

The numerator limit is 3 (specifically, not 0), but the denominator limit is 0. As $x \to 2$ from the left, $x-2$ is negative and close to zero whereas the numerator is close to three, the ratio is thus a large negative number and gets larger in absolute value, without bound, as $x \to 2$. Thus, $\lim_{x\to 2^-}(x+1)/(x-2) = -\infty$. Likewise, $x \to 2$ from the right, $x-2$ is positive and close to zero whereas the numerator is again close to three, so the ratio is a large positive number and gets larger without bound as $x \to 2$. Thus, $\lim_{x\to 2^+}(x+1)/(x-2) = \infty$.

<u>Example 7.8</u> Using an argument similar to example 7.7, it should be straightforward to show that

$$\lim_{x\to 2}(x+1)/(x-2)^2 = \infty .$$

Much more interesting are examples where both the numerator and denominator are zero at a certain point, as in the example at the very start of this section. The function is therefore not defined at that point, but does the limit exist?

<u>Example 7.9</u> Let $f(x) = \dfrac{x^2 - x - 2}{x-2}$, $x \neq 2$.

This function does not exist at $x = 2$; we cannot talk about the value of the function at $x = 2$. Does it have a limit at $x = 2$? As $x \to 2$, both the numerator and the denominator goes to zero. But what about the ratio of the numerator and denominator? It turns out that the ratio converges to 3 as $x \to 2$. The easiest way to see this is to factor the numerator to get

$$f(x) = \frac{x^2 - x - 2}{x-2} = \frac{(x-2)(x+1)}{x-2} = x+1, \ x \neq 2$$

In other words, this function is actually simply a straight line, but with a ‘hole’ at $x = 2$. However, the ‘hole’ doesn’t matter as far as the limit is concerned. Thus,

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2 - x - 2}{x-2} = \lim_{x\to 2}(x+1) = 3$$

<u>Example 7.10</u> Let $f(x) = \dfrac{x^2 - x - 2}{(x-2)^2}$, $x \neq 2$.

We have

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2 - x - 2}{(x-2)^2} = \lim_{x\to 2}\frac{(x-2)(x+1)}{(x-2)^2} = \lim_{x\to 2}\frac{(x+1)}{(x-2)} = dne$$

In particular, the left limit is $-\infty$ and the right limit is ∞.

Example 7.11 Let $f(x) = \dfrac{(x-2)^2}{x^2-x-2}$, $x \neq 2$.

Employing the same trick as in Ex 7.3.4, we have

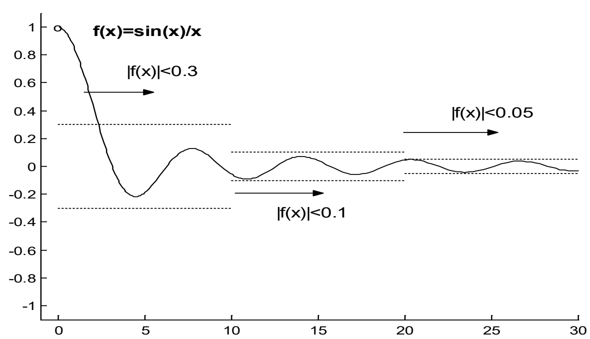
$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{(x-2)^2}{x^2-x-2} = \lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+1)} = \lim_{x\to 2}\frac{(x-2)}{(x+1)} = 0.$$

So there are many possibilities in the "$0/0$" case (this is an example of what is known as an "indeterminate form"). Later, we will look more closely at a technique that helps with solving such problems.

Limits at Infinity In addition to limits at a point, it is also often useful, if f is a function defined over arbitrarily large positive real numbers, to discuss "limits at infinity". That is, to describe the behavior of the functions "as x grows towards infinity". (This is similar to limits of sequences, which are, after all simply functions over the integers.)

We write $\lim_{x\to\infty} f(x) = A$ if $f(x)$ can be made arbitrarily close to A if we take values of x that are large enough.

Example 7.12 Consider the function $f(x) = \sin(x)/x$.



Clearly, $\lim_{x\to\infty} f(x) = 0$. Given any small number $\varepsilon > 0$, the absolute difference between $f(x)$ and 0 is less than ε for all values of x beyond some number.

The following are obvious, and easy to show:

<u>Example 7.13</u> $\lim_{x\to\infty} c = c$; $\lim_{x\to\infty} 1/x = 0$;

$\lim_{x\to\infty} x^2 = \infty$; $\lim_{x\to-\infty} x = -\infty$.

A useful limit which we have already seen in terms of the limit of a sequence is:

<u>Example 7.14</u> $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$.

Referring back to Ex. 7.13, what $\lim_{x\to\infty} x^2 = \infty$ means exactly, is that the function increases without bound. If you set a bound of M, no matter how large, the function x^2 will exceed that value for x large enough. Also, like sequences, some functions simply oscillate as $x\to\infty$. An obvious example is $\sin(x)$, which is bounded yet never converges to any number.

The usual limit laws can be used to compute limits at infinity.

<u>Example 7.15</u>

$$\lim_{m\to\infty} A\left(1+\frac{r}{m}\right)^{mt} = \lim_{m\to\infty} A\left(1+\frac{1}{m/r}\right)^{(m/r)rt} = A\left[\lim_{m\to\infty}\left(1+\frac{1}{m/r}\right)^{(m/r)}\right]^{rt} = Ae^{rt}$$

For rational functions, it is often useful to first divide both the numerator and the denominator by the highest power of x in the <u>denominator</u>.

<u>Example 7.16</u>

$$\lim_{x\to\infty}\frac{3x^2+x-1}{x^2+1} = \lim_{x\to\infty}\frac{\left(3x^2+x-1\right)/x^2}{(x^2+1)/x^2} = \frac{\lim_{x\to\infty}\left(3+1/x-1/x^2\right)}{\lim_{x\to\infty}\left(1+1/x^2\right)} = 3$$

$$\lim_{x\to-\infty}\frac{3x^2+x-1}{x^2+1} = \lim_{x\to-\infty}\frac{\left(3x^2+x-1\right)/x^2}{(x^2+1)/x^2} = \frac{\lim_{x\to-\infty}\left(3+1/x-1/x^2\right)}{\lim_{x\to-\infty}\left(1+1/x^2\right)} = 3$$

The problem here is that both the numerator and denominator $\to\infty$ as $x\to\infty$, and we have to be vary careful with things like "$\pm\infty/\pm\infty$". Such functions can converge, increase/decrease without bound, or even oscillate.

We also have to be careful with "$\infty-\infty$" and "$\infty.0$" situations:

Example 7.17

$$\lim_{x\to\infty}(x^2 - x) = \infty\,; \qquad \lim_{x\to\infty}(x - x^2) = -\infty\,;$$

$$\lim_{x\to\infty}(x^2)(1/x) = \infty\,; \qquad \lim_{x\to\infty} x(1/x^2) = 0\,.$$

The issue in situations like these is not so much the convergence to infinity of the various components of the functions, but how quickly (or slowly) those components are converging to infinity. Another example:

Example 7.18

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^2-3x}\; - \;x &= \lim_{x\to\infty}\left(\sqrt{x^2-3x}-x\right)\frac{\sqrt{x^2-3x}+x}{\sqrt{x^2-3x}+x} \\ &= \lim_{x\to\infty}\frac{x^2-3x-x^2}{\sqrt{x^2-3x}+x} \\ &= \lim_{x\to\infty}\frac{-3x}{x\left(\sqrt{1-3/x}+1\right)} \quad = \quad -3/2\end{aligned}$$

Example 7.18 shows that a "$\infty - \infty$" situation might well converge to a constant (oscillation is possible too!). The example also illustrates a technique very useful in situations involving radicals.

Asymptotes Functions that converge to a constant are sometimes said to converge to a horizontal asymptote. For functions that increase or decrease without bound, we can also define an (non-horizontal) asymptote.

Example 7.19

$$\lim_{x\to\infty}\frac{1-x^5}{x^4+x+1} \;=\; \lim_{x\to\infty}\frac{\frac{1}{x^4}-\frac{x^5}{x^4}}{\frac{x^4}{x^4}+\frac{x}{x^4}+\frac{1}{x^4}} \;=\; \lim_{x\to\infty}\frac{\frac{1}{x^4}-x}{1+\frac{1}{x^3}+\frac{1}{x^4}} \;=\; -\infty$$

This function decreases without bound, but we may be able to say something about *how* it decreases without bound. In this case, we have

$$f(x) \;=\; \frac{1-x^5}{x^4+x+1} \;=\; -x+\frac{x^2+x+1}{x^4+x+1}$$

so that $f(x)-h(x) = \dfrac{x^2+x+1}{x^4+x+1}$ and clearly $\lim_{x\to\infty}\dfrac{x^2+x+1}{x^4+x+1} = 0$.

The function thus decreases without bound, but looks more and more like the straight line $h(x) = -x$ as x increases. We say that $h(x) = -x$ is an <u>asymptote</u> of $f(x)$.

One misunderstanding concerning asymptotes is that the function cannot 'cross' its asymptote. This is not correct. In Example 7.12, the function $h(x)=0$ is a (horizontal) asymptote of $f(x)=\sin(x)/x$, but $f(x)$ crosses $h(x)$ infinitely many times.

We can also generalize asymptotes to non-linear curves:

Example 7.20 $\qquad f(x)=\dfrac{x^3-x^2+2}{x-1}$

This function increases without bound as $x\to\infty$. But because

$$f(x)=\frac{x^3-x^2+2}{x-1} \;=\; x^2+\frac{2}{x-1},$$

we have

$$\lim_{x\to\infty}(f(x)-x^2) \;=\; \lim_{x\to\infty} 2/(x-1) \;=\; 0,$$

so $f(x)$ looks more and more like x^2 as $x\to\infty$. We say that $h(x)=x^2$ and $f(x)$ are asymptotic to each other, as $x\to\infty$.

Exercises

1. Find the following limits using the "rules for computing limits"

   a. $\lim\limits_{y\to 2}\dfrac{(y-1)(y-2)}{(y+1)}$ $\qquad$ b. $\lim\limits_{x\to 3}\sqrt{\dfrac{x^2-2x}{x+1}}$

2. Find the following limits if they exist. If they don't exist, find the left and right limits. (For this exercise, do not use L'Hospital's rule, even where it applies.)

   a. $\lim_{x\to 2}\dfrac{3x^2+3x-18}{x-2}$ $\qquad$ b. $\lim_{x\to 0}\dfrac{x^3-2x-1}{x^5-x^2-1}$ $\qquad$ c. $\lim_{x\to 0}\dfrac{x^3+3x^2-2x}{x}$

   d. $\lim_{x\to 1}\dfrac{x^2-4x+3}{x^2-2x+1}$ $\qquad$ e. $\lim_{x\to 3}\dfrac{x}{x-3}$ $\qquad$ f. $\lim_{x\to 2}\dfrac{1}{|2-x|}$

   g. $\lim_{x\to 9}\dfrac{x-9}{\sqrt{x}-3}$ $\qquad$ h. $\lim_{x\to 0} f(x)$ where $f(x)=\begin{cases} t^2 & t\ge 0\\ t-2 & t<0\end{cases}$

   i. $\lim_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$ $\qquad$ j. $\lim_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x}$

   Plot the graph of the expression in (c).

3. Prove that polynomials are continuous everywhere.

4. For functions (a) to (g), determine if there are points of discontinuity. If it is continuous everywhere, prove it (you may use any result given in the notes). If it has a point of discontinuity, state why it is discontinuous there (e.g., because not defined at that point, or perhaps the limit disagrees with the value at that point.)

a. $f(x)=\begin{cases} 2x-1, & x\geq 1 \\ 3x^2+1, & x<1 \end{cases}$ b. $f(x)=\begin{cases} 5x-1, & x\geq 1 \\ 3x^2+1, & x<1 \end{cases}$ c. $f(x)=\dfrac{x^2-4}{x^3-8}$

d. $g(x)=1/\sqrt{x^4+7x^2+1}$ e. $f(x)=\begin{cases} 2x-3, & x\leq 2 \\ x^2, & x>2 \end{cases}$ f. $f(x)=|x|/x$

5. a. Evaluate the following limits

i. $\lim_{x\to-\infty}\sqrt{\dfrac{2+3x}{x-1}}$ ii. $\lim_{x\to-\infty}\sqrt{x^2+3}-x$ iii. $\lim_{x\to-\infty}\sqrt{x^2+3x}-x$

iv. $\lim_{x\to-\infty}\dfrac{1-x^5}{x^4+x+1}$ v. $\lim_{x\to\infty}\dfrac{2x^3-3x^2+3x-6}{x^2+1}$

Find the asymptote for the expressions in (iv) and (v) as $x\to\pm\infty$.

b. Find the asymptote to the function $f(x)=\sqrt{x^2+x/3}$ as $x\to\infty$.

Hint: use part (a.iii)

6. For each function, find a simpler function to which it is asymptotic as $x\to\infty$.

a. $f(x)=\dfrac{x^5-x^3+3}{x^2-1}$ b. $f(x)=\sqrt{\dfrac{x^3-x^2+2}{x-1}}$ c. $f(x)=\sqrt{\dfrac{x^2-2}{x-2}}$

7. (a) Find $\lim_{x\to 1}\dfrac{x^3+x^2-5x+3}{x^3-3x+2}$ without using L'Hospital's rule (even though it applies).

(b) Find $\lim_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$. If the limit doesn't exist, find the left and right limits.

(c) Show that $\lim\limits_{x\to\infty}\sqrt{2x^2+x/3}-x=\infty$, and find its (linear) asymptote.

Hint: write $f(x)=\sqrt{2x^2+x/3}-x$ *as*

$$f(x)=\left(\sqrt{2x^2+x/3}-\sqrt{2}x\right)+\left(\sqrt{2}x-x\right)$$

and analyze the two parenthesized terms separately.

Appendix (Optional)

Definition The limit of f at x_0 is q if for any $\varepsilon > 0$ (no matter how small), there is a $\delta > 0$ such that $q - \varepsilon < f(x) < q + \varepsilon$ for all x satisfying $x_0 - \delta < x < x_0 + \delta$ and $x \neq x_0$.

With this definition it is very easy to prove, for instance, that $\lim_{x \to 3} 2x + 1 = 7$. Suppose I pick a small $\varepsilon > 0$. To show $\lim_{x \to 3} 2x + 1 = 7$, I need to find δ such that $7 - \varepsilon < 2x + 1 < 7 + \varepsilon$ whenever $3 - \delta < x < 3 + \delta$. (I don't have to use a specific value of ε. In fact it will not be enough to use specific values of ε because I need to show that result for all ε. Instead, I'll find δ in terms of ε, so the argument applies for all ε). This is easy to do, because

$$7 - \varepsilon < 2x + 1 < 7 + \varepsilon \quad \Leftrightarrow \quad \tfrac{6-\varepsilon}{2} < x < \tfrac{6+\varepsilon}{2}, \text{ or } 3 - \tfrac{\varepsilon}{2} < x < 3 + \tfrac{\varepsilon}{2}.$$

In other words, I can make $2x + 1$ within a $\pm\varepsilon$ 'tolerance' of 7 by taking x within a range of $\pm\frac{\varepsilon}{2}$ of 3, no matter how small ε is. This is <u>exactly</u> what we mean by "$2x + 1$ approaches 7 as x approaches 3".

You can use the formal definition of the limit to prove these rules. E.g., to prove that the limit of a sum is the sum of a limit, we would show that for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that $|f(x) + g(x) - a - b| < \varepsilon$ for all x satisfying $x_0 - \delta < x < x_0 + \delta$, $x \neq x_0$. Because $\lim_{x \to x_0} f(x) = a$, for any given $\varepsilon_1 > 0$, we can find $\delta_1 > 0$ such that $|f(a) - a| < \varepsilon_1$ for all x satisfying $x_0 - \delta_1 < x < x_0 + \delta_1$, $x \neq x_0$. Because $\lim_{x \to x_0} g(x) = a$, for any given $\varepsilon_2 > 0$, we can find $\delta_2 > 0$ such that $|g(x) - b| < \varepsilon_2$ for all x satisfying $x_0 - \delta_2 < x < x_0 + \delta_2$ and $x \neq x_0$. For any $\varepsilon > 0$, choose ε_1 and ε_2 such that $\varepsilon_1 + \varepsilon_2 = \varepsilon$ (for instance, choose $\varepsilon_1 = \varepsilon_2 = \varepsilon / 2$). Then, choosing $\delta = \min(\delta_1, \delta_2)$ gives us

$$|f(x) + g(x) - a - b| \leq |f(x) - a| + |g(x) - b| < \varepsilon_1 + \varepsilon_2 = \varepsilon$$

for all x such that $x_0 - \delta < x < x_0 + \delta$, $x \neq x_0$. (We used the fact that $|x + y| \leq |x| + |y|$).

<u>Example 7.21</u> Most of the examples you will come across are situations akin to the simple examples in the main part of this section. However, there are examples that can really test intuition. Consider for example,

$$f(x) = \sin(\pi / x).$$

This function is not defined at $x = 0$, but what is its behavior near $x = 0$? Consider the sequence $x_n = 1/2n$, $n = 1, 2, 3, \ldots$.. Then $f(x_n) = \sin(2n\pi) = 0$ for all n. So does $f(x)$ approach 0 as x approaches 0? But now consider the sequence $x_n = 2/(4n+1)$ which is another sequence going to zero. Then $f(x_n) = \sin((4n+1)\pi/2) = \sin(2n\pi + \pi/2) = 1$ for all n, so it now appears that $f(x)$ approach 1 as x approaches 0. In fact for any $a \in [-1, 1]$, I can find a sequence x going to zero such that $f(x)$ approaches a. It turns out this function behaves very irregularly as x approaches zero. In fact, its limit at $x = 0$ does not exist.