Mathematics for Economics **Anthony Tay**

6. Sequences and Series

A sequence is a function defined over the integers, e.g., 1, 1/2, 1/3, … is a sequence which we can write as $1/n$, $n = 1,2,...$. Another example of a sequence is 2,4,8,16,… which we can write as 2^n, $n = 1,2,3,...$. With sequences, we are usually interested in its behavior as the index n gets larger and larger. For instance, it seems obvious that as the sequence 1, 1/2, 1/3, …, $1/n$, … evolves, the value of the sequence get closer and closer to zero. The sequence 2,4,8,16,… on the other hand, obviously gets larger and larger without bound. The concepts of convergence and divergence formalize these ideas.

Convergent Sequence Loosely speaking, a sequence x_n is said to converge to x if it gets arbitrarily close to x as n increases towards infinity. Formally, we say that

> x_n **converges** to x if for any arbitrarily small number $\varepsilon > 0$, we can find an integer N such that $|x - x_n| < \varepsilon$ for all $n > N$, i.e., $x - \varepsilon < x_n < x + \varepsilon$.

We write $\lim_{n\to\infty} x_n = x$, or $x_n \to x$ as $n \to \infty$.

Divergent Sequence: a sequence that does not converge is said to diverge.

<u>Example 6.1</u> To formally prove that the sequence $1/n$, $n = 1,2,...$ converges to 0, we have to find, for every small positive number ε, a number N such that $|1/n - 0| < \varepsilon$ whenever $n > N$. We do this by finding N in terms of ε. Such an N is easy to find: for any given ε, simply choose N to be the smallest integer larger than $1/\varepsilon$. Then for all $n > N$, we have

$$1/n \; < \; 1/N \; < \varepsilon$$

(which implies $|1/n - 0| < \varepsilon$, since $1/n$ is always positive).

Sometimes you'll hear the statement "the value of $1/n$ at infinity is 0". This statement is shorthand for the more precise statement that $\lim_{n\to\infty} 1/n = 0$. Note also that the sequence $x_n = 1/n$ never actually reaches 0; no matter how large n is, $1/n$ is strictly greater than zero.

Example 6.2 The sequence $-n$, $n = 1, 2, ...$ obviously does not converge.

Example 6.3 The sequence $1 - 1/n$, $n = 1, 2, ...$ converges to 1.

Example 6.4 The sequence n^2, $n = 1, 2, ...$ diverges.

Example 6.5 The sequence $\sin(n\pi/2)$, $n = 1, 2, ...$ diverges.

One basic fact regarding convergent sequences is that their limits are unique: no sequence can converge to two distinct limits.

The following is a formal proof of this statement. This proof makes use of an elementary fact of real numbers, that $|a+b| \le |a| + |b|$. Suppose $x_n \to x$ and $x_n \to x'$. This implies that given any $\varepsilon > 0$, no matter how small, we can find an N such that

$$|x_n - x| < \tfrac{\varepsilon}{2} \;\&\; |x_n - x'| < \tfrac{\varepsilon}{2}$$

for all $n \ge N$. But this means that

$$|x - x'| \le |x_n - x| + |x_n - x'| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

for all $n \ge N$. Because ε can be selected arbitrary small, this implies that $|x - x'| = 0$, i.e., $x = x'$. This justifies references to ***the*** limit of a convergent sequence.

With sequences, there are two basic questions: does the sequence converge? If it does, what does it converge to? It is often difficult to tell if a particular sequence converges, and mathematicians have developed a number of 'rules' to help them answer such questions. For instance, it can be proven that if a sequence is increasing and bounded above, then a limit must exist. A systematic study of such rules is beyond the scope of this course, but the following is an example of a series whose limit can be shown to exist using such means.

(Very Important) Example 6.7 The number *e*.

If interest at rate 1.00 is paid on every dollar at the end of the year, we get 2 dollars from an initial investment of 1 dollar.

If interest at rate 1/2 is paid on every dollar every half year (with reinvestment), then we get $(1+1/2)^2$ from an initial investment of one dollar (note that $(1+1/2)^2 = 2.25 > 2$)

If interest at rate 1/3 is paid on every dollar every 1/3 year (with reinvestment), then we get $(1+1/3)^3$ from an initial investment of one dollar (note that $(1+1/3)^3 = 2.37 > 2.25$)

If interest at rate $1/n$ is paid on every dollar at n different periods within the year (with reinvestment), then we get $(1+1/n)^n$ from the initial investment.

Is there a limit to this process of paying smaller rates more frequently throughout the year? In other words, does the sequence $(1+1/n)^n$ have a limit as $n \to \infty$, or does the sequence diverge? It turns out that the sequence $(1+1/n)^n$ does have a limit as $n \to \infty$ (the proof goes about by showing first, that this sequence is increasing, and second, that this sequence is bounded above, and therefore a limit exists). The limit turns out to be the irrational number

$$2.71828182845905\ldots$$

This number is be extremely important in all areas of science and mathematics. We give it a name "e". (For a fun and easy to read history of the discovery of this number, see Eli Maor's "e: The Story Of A Number" (Princeton University Press). Likewise, the function $f(x) = e^x$ is one of the most important in all mathematics. Note that this function is often written as $f(x) = \exp(x)$. These are just two different ways of writing the same thing.

The other question of actually finding limits of sequences is just as difficult, but the following facts allow us to break down complicated problems into smaller simpler ones.

Theorem Algebraic operators respect limits (stated here without proof)

Let x_n and y_n be sequences that converge to x and y. Then

(a) $x_n + y_n \to x + y$;

(b) $ax_n \to ax$

(c) $x_n y_n \to xy$;

(d) $1/x_n \to 1/x$ provided $x_n \neq 0$ and $y \neq 0$;

(e) $(x_n)^r \to x^r$

Example It is quite obvious that $\frac{n}{n+1} \to 1$. To show this using the rules above:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{\lim_{n\to\infty}(1)}{\lim_{n\to\infty}(1+1/n)} = 1$$

Theorem Squeeze theorem for sequences

If two x_n and z_n both converge to c, and $x_n \le y_n \le z_n$ for all n, then $y_n \to c$.

Proof If $x_n \to c$, $z_n \to c$, and $x_n \le y_n \le z_n$, then for any $\varepsilon > 0$, there is an N and an M such that

$$c - \varepsilon < x_n < c + \varepsilon \text{ for all } n \ge N, \text{ and } c - \varepsilon < z_n < c + \varepsilon \text{ for all } n \ge M.$$

It follows that

$$c - \varepsilon < x_n \le y_n \le z_n < c + \varepsilon \text{ for all } n \ge \max(M, N) \blacksquare$$

Example $p^{1/n} \to 1$ for all $p > 0$

The result is obvious for $p = 1$. For $p > 1$, let $x_n = p^{1/n} - 1$. Because $x_n > 0$, the binomial formula (check it out) implies that

$$(1 + nx_n) \le (1 + x_n)^n = p$$

so that $0 \le x_n \le (p-1)/n$. Because

$$\lim_{n\to\infty} \frac{(p-1)}{n} = (p-1)\lim_{n\to\infty} \frac{1}{n} = (p-1)0 = 0,$$

the squeeze theorem then implies that $x_n \to 0$, i.e., $p^{1/n} \to 1$. If $0 < p < 1$, let $x_n = (1/p)^{1/n}$ and proceed as in the case $p > 1$.

Application: Series

What is the infinite sum $\sum_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + ... + ar^{n-1} + ...$ when $|r| < 1$? For example, what is the infinite sum $1/2 + 1/4 + 1/8 + ...$?

Many of you know the answer to be

$$a + ar + ar^2 + ... + ar^{n-1} + ... = \frac{a}{1-r}.$$

But what does this mean, exactly? After all, no infinite sum can ever be computed literally. How do we interpret $a/(1-r)$ in this example?

One has to be very careful when working with infinite sums. Their behavior can be quite unintuitive. Here are examples to tease your mind…

What is the infinite sum $\sum_{i=1}^{\infty} (-1)^{i-1} = 1 - 1 + 1 - 1 + 1 - 1 + ...$?

Is it 0? $(1-1) + (1-1) + (1-1) + ...$

Is it 1? $1 + (-1+1) + (-1+1) + (-1+1) + ...$

What is the infinite sum $\sum_{i=1}^{\infty} 1/i = 1 + 1/2 + 1/3 + 1/4 + ...$?

In both instances, the infinite sum does not exist. You will see in a while in what sense it does not exist.

In this section, we will take a more careful look at infinite sums, or *series*. Among other things, we will use sequences to help us make sense of series.

Returning to the statement that

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r},$$

consider adding up the terms of the series one by one. This gives the sequence of "partial sums"

$$s_1 = a,\ s_2 = a + ar,\ s_3 = a + ar + ar^2,\ \ldots$$

which defines a sequence. If $|r| < 1$, then the sequence of partial sums has a limit in the sense of the definition given earlier: the formula for the n-partial sum (the nth term of the sequence above) is

$$s_n = \frac{a(1-r^n)}{(1-r)}$$

which we can get from the fact that

$$s_n = a + ar + ar^2 + ... + ar^{n-1}$$
$$rs_n = \quad ar + ar^2 + ... + ar^{n-1} + ar^n.$$

Subtracting the second equation from the first, gives $(1-r)s_n = a(1-r^n)$.

Because $|r|<1$, $r^n \to 0$ as $n \to \infty$, therefore

$$s_n \to \frac{a}{1-r} \text{ as } n \to \infty .$$

In other words, we are saying that we can make the sum

$$a + ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$$

as close to $a/(1-r)$ as we like by simply making n large enough (i.e., by adding up enough of these terms). We say that the geometric series $a + ar + ar^2 + ...$ converges to $a/(1-r)$.

We emphasize: the expressions

$$\text{“}a + ar + ar^2 + ...\text{”} \qquad \text{and} \qquad \text{“}\sum_{i=1}^{\infty} ar^{i-1}\text{”}$$

both mean exactly the same thing: they both refer to means $\lim_{n\to\infty} \sum_{i=1}^{n} ar^{i-1}$.

Example 6.6 A consol or perpetuity is a bond that pays a sum C annually, with no maturity date. At a fixed rate of interest R, the present value of the perpetuity is

$$\frac{C}{1+R} + \frac{C}{(1+R)^2} + \frac{C}{(1+R)^3} + ...$$

This is simply a geometric series with

$$a = \frac{C}{1+R} \text{ and } r = \frac{1}{1+R},$$

and so the present value of the perpetuity is

$$\frac{C}{1+R} \bigg/ \left(1 - \frac{1}{1+R}\right) = \frac{C}{R}.$$

For a series to converge, the individual terms in the series must obviously converge to zero. The sum $1+1+1+...$ obviously does not converge. However, there are some series whose terms converge to zero, yet the series itself diverges.

Example 6.7 The series $1+1/2+1/3+1/4+\ldots$ does not converge. It turns out that I can always *exceed* a value simply by adding together enough of the terms of the series.

One easy way to show that is to compare the sum

$$1+1/2+\underbrace{1/3+1/4}+\underbrace{1/5+1/6+1/7+1/8}+\ldots$$

with the sum

$$1+1/2+\underbrace{1/4+1/4}+\underbrace{1/8+1/8+1/8+1/8}+\ldots$$

(continuing the pattern as indicated.) The latter sum is obviously smaller than the former. But the latter is $1+1/2+1/2+1/2+\ldots$ which clearly does not converge. The sequence of partial sums of $1+1/2+1/2+1/2+\ldots$ obviously grows without bound, hence the partial sums of $1+1/2+1/3+1/4+\ldots$ must also grow without bound. Sometimes we write

$$\text{“}1+1/2+1/3+1/4+\ldots = \infty\text{”}$$

as a shorthand for this statement.

Proving convergence and divergence of series is a challenge. Finding the limit of convergent series is probably even more difficult. The easy ones include the geometric series, and certain special series such as telescoping series.

Example 6.8 The series

$$\sum_{i=1}^{\infty}\frac{1}{i(i+1)} = \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots = 1.$$

The ratio $\frac{1}{i(i+1)}$ can be written as $\frac{1}{i(i+1)} = \frac{1}{i}-\frac{1}{i+1}$. Considering only the partial sum, we have

$$\begin{aligned}\sum_{i=1}^{N}\frac{1}{i(i+1)} &= \sum_{i=1}^{N}\left(\frac{1}{i}-\frac{1}{i+1}\right)\\ &= 1-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\ldots+\cancel{\frac{1}{N-1}}-\cancel{\frac{1}{N}}+\cancel{\frac{1}{N}}-\frac{1}{N+1}\\ &= 1-\frac{1}{N+1}\end{aligned}$$

Hence $\sum_{i=1}^{\infty}\frac{1}{i(i+1)} = \lim_{N\to\infty}\sum_{i=1}^{N}\frac{1}{i(i+1)} = 1$

There are many interesting series that eluded all but the best mathematicians throughout history, e.g.

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \text{ and } \sum_{i=1}^{\infty} \frac{1}{i^4} = \frac{\pi^4}{90}$$

and many others remain unsolved. By far the most important series for you is the geometric series

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}.$$

You must know this well, and be able to apply it with some (mental) nimbleness. We will see one or two other important ones later in the course. Here is an example of a convergent series from statistics:

<u>Example 6.9</u> The Poisson distribution is often used to model the number of occurrences of an event in a fixed amount of time. [Simeon-Denis Poisson showed in the early 1800s that if events occur independently of the last occurrence, the number of occurrences of the event in a fixed amount of time will have this distribution.] The distribution is

$$\Pr[X = i] = e^{-\lambda}\lambda^i / i!, \; X = 0,1,2,3,....$$

Probabilities must add up to one in valid distributions. Perhaps by the end of this course you can show that

$$\sum_{i=0}^{\infty} e^{-\lambda}\lambda^i / i! = 1.$$

Exercises

1. Express the following in closed form (i.e., reduce to a simplified expression that does not use the summation notation). In the case of part (ii), also find the limit as $n \to +\infty$

   (i) $\sum_{k=1}^{n-1} \left(\frac{k^2}{n} \right)$ (ii) $\sum_{k=1}^{n} \left(\frac{5}{n} - \frac{2k}{n^2} \right)$

2. Find $\sum_{i=1}^{\infty} \frac{2}{i(i+2)}$.

3. (a) The geometric series $\sum_{i=1}^{\infty} 1/2^i$ converges to one. How many terms of the series are required for the partial sum $\sum_{i=1}^{n} 1/2^i$ to be within 0.00005 of 1?

   (b) The infinite sum $\sum_{i=1}^{\infty} ip(1-p)^i$, $0 < p < 1$, converges. Find its value in terms of p.

4. (*An early treatment of integration*) We wish to find the area between the x-axis, and the curve $y = x^2$ from $x = 0$ to $x = 1$. Consider the points

$$(x, y) = \left(\tfrac{1}{n}, \tfrac{1^2}{n^2}\right), \left(\tfrac{2}{n}, \tfrac{2^2}{n^2}\right), \left(\tfrac{3}{n}, \tfrac{3^2}{n^2}\right), \ldots, \left(\tfrac{n}{n}, \tfrac{n^2}{n^2}\right),$$

and approximate the desired area by the sum of the rectangles defined by these coordinates

$$A_n = \sum_{i=1}^{n} \left(\tfrac{1}{n}\right)\left(\tfrac{i^2}{n^2}\right).$$

Find a simplified expression for A_n, then find

$$\sum_{i=1}^{\infty} \left(\tfrac{1}{n}\right)\left(\tfrac{i^2}{n^2}\right) = \lim_{n\to\infty} A_n .$$

Find $\int_0^1 x^2 dx$ using the usual integration techniques.

5. Show that $p^{1/n} \to 1$ for $0 < p < 1$.

6. Show that $n^{1/n} \to 1$. Hint: let $x_n = n^{1/n} - 1$. The binomial formula then gives

$$0 \le \frac{n(n-1)}{2} x_n{}^2 \le (1 + x_n)^n = n$$

from which it follows that

$$0 \le x_n \le \sqrt{2/(n-1)} \quad \text{(for all } n \ge 2\text{)}.$$

7. Show that $\dfrac{\sin(1/n)}{n} \to 0$.

8. Use the fact that $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$ to show that $\lim_{n\to\infty}\left(1+\dfrac{r}{n}\right)^n = e^r$, and $\lim_{n\to\infty} A\left(1+\dfrac{r}{n}\right)^{nt} = Ae^{rt}$.

9. Examine the convergence of the sequences (i) $\dfrac{n^2-1}{n}$ and (ii) $\dfrac{3n}{\sqrt{2n^2-1}}$.

10. Let $x_n = \dfrac{3-n}{2n-1}$ and $y_n = \dfrac{n^2+2n-1}{3n^2-2}$. Find

(i) $\lim_{n\to\infty} x_n$ (ii) $\lim_{n\to\infty} y_n$ (iii) $\lim_{n\to\infty} x_n y_n$ (iv) $\lim_{n\to\infty} x_n / y_n$