Mathematics for Economics **Anthony Tay**

5. Functions of One Variable

You have already worked with equations such as

$$y = ax^2 + bx + c$$

that relate a 'dependent variable' y to an 'independent variable' x. In such cases, we often say 'y is a function of x' and write something like $y = f(x)$, where in this example, $f(x) = ax^2 + bx + c$. The main objective of this section is to give you a deeper understanding of a function as a mathematical concept, to get familiarized with the related terminology and the $f(.)$ notation, and to review the properties of some important functions.

I will use the symbol $\mathbb{R}$ to represent the set of all real numbers. To represent the set of all pairs of real numbers, I will use the symbol $\mathbb{R}^2$. The symbol $\mathbb{R}^3$ will represent the set of all triplets of real numbers. The symbol "(a,b)" where a and b are real numbers, will sometimes be used to represent the set of all real numbers between a and b (noninclusive). Thus $(a,b) \subset \mathbb{R}$. To say that $x > 0$, we may write $x \in (0,\infty)$. However, sometimes the symbol "(a,b)" is *also* used to represent the pair of numbers a and b. In this context, $(a,b) \in \mathbb{R}^2$. Which interpretation is used depends on context. Please be aware of this.

5.1 Definition and Notation for Functions of One Variable

If x and y are variables that take numerical values, and if the value of y depends on what the value of x is, then we say that "y is a function of x", and we may write $y = f(x)$ where $f(.)$ indicates the form of the association. Often, these can be represented by an equation, e.g., if the association between

$y = a + bx$ for some constants a and b,

then $y = f(x) = a + bx$. The notation $f(x)$ in this example represents the expression $a + bx$. Occasionally you will find authors who use the variable name and the function name synonymously, and write something like $y = y(x)$.

There is more that must be said to fully describe the relationship between y and x. For what values of x are we applying this association? If x represents something that cannot take negative values, then we must specify $y = a + bx$, $x \geq 0$. If no such restriction needs to applied, we might simply say $y = a + bx$, $x \in \mathbb{R}$ (that is, "for all real numbers x").

Example 5.1.1 If A represents the area of a circle, and r its radius, then $A = \pi r^2$, $r > 0$.

Note that we could have defined $A = \pi r^2$ for all $r \in \mathbb{R}$. Mathematically this is fine. We could even have defined $A = \pi r^2$ for all $r \in \mathbb{C}$, i.e., for complex numbers. But for the given application of the function – to relate the radius of a circle to its area – it makes sense to restrict r to positive real numbers.

Definition The values of x over which a function $f(x)$ is to be defined is called the domain of the function. The values taken by the function over its domain is its range.

Example 5.1.1 continued The range of the function $A = \pi r^2$, $r > 0$ is $(0, \infty)$.

Remember, the domain of a function is an important part of the definition of a function. The appropriate domain to use depends on a number of things such as the regions over which x and y make sense given what the function is being used for. Sometimes the domain of a function also has to be limited because the function cannot be defined mathematically over certain regions:

Example 5.1.2 $f(x) = (x+1)/(x-1)$, then $f(x)$ cannot be defined for $x = 1$ because we cannot divide by zero. The largest possible domain (limiting ourselves to real numbers) is the set of all real numbers $x \neq 1$. We will simply say

$$f(x) = (x+1)/(x-1), \ x \neq 1$$

If, in addition, our application requires that $f(x) \geq 0$, then the domain must be further restricted to $x \in (-\infty, -1) \cup (1, \infty)$. If negative values of x have no meaning, then we should specify $x \in (1, \infty)$.

Example 5.1.3 What is the largest possible domain of $f(x) = \sqrt{x}$? If we require $f(x)$ to take real values only, then this is $x \geq 0$. However, if we allow $f(x)$ to take complex values, then this restriction is unnecessary, e.g., $\sqrt{-1} = i$ is perfectly valid.

In this course, we will restrict both the domain and range to real numbers.

The full, general, definition of a function is:

Definition. A function f from a set A to another set B is a rule that associates each element of A with an element of the set B.

We write $f: A \to B$. "$f(x)$" denotes that element of B which is associated with the element $x \in A$. The set A is called the domain of the function f.

<u>Example 5.1.4</u> Sequences are merely functions whose domain are the natural numbers $f: \mathbb{N} \to \mathbb{R}$.

In the notation $f: A \to B$, the set B is not necessarily the range, any set encompassing the range will be fine. Once again, in this course, our functions are restricted to be real-valued, i.e., $B \subset \mathbb{R}$. For the first part of this course, we will work with 'real-valued functions of one real variable', That is, $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$. We will simply call these "functions of one variable".

Later in the course, we will consider relationships between one dependent variable y, and multiple independent variables. For instance, the value of y might depend on two variables w and x. In this case, the function is a mapping of pairs of values (w, x) to y. Then A will be a subset of all pairs of real numbers. We write $A \subset \mathbb{R}^2$.

Note that the function definition can be applied to non-numerical sets of objects.

<u>Example 5.1.6</u> Let A be the set {product 1, product 2, product 3} and B = {1, 2, 3, …, 10}. The following rule $f: A \to B$ is a function

f(prod. 1) = 2 ; f(prod. 2) = 2 ; f(prod. 3) = 9.

The set {2, 9} is called the range of the function f; it is the set of all values of the function. This function might represent the results of a survey where ten people are asked if they would purchase each of the three products. If the variable "x" represents "product", we might write this function as "$f(x)$" defined by the association described above. In more advanced courses, we can talk about functions whose domains are sets of functions, matrices, and other objects! For now, we will deal strictly with numerical functions.

An important feature of the definition might not be obvious at first.

(i) <u>Every</u> element of A must be mapped to some element of B.

(ii) Each element of A is mapped to <u>exactly one</u> element of B. An element of A cannot be associated with two or more elements of B. Symbolically, we can say that if f is a function, then

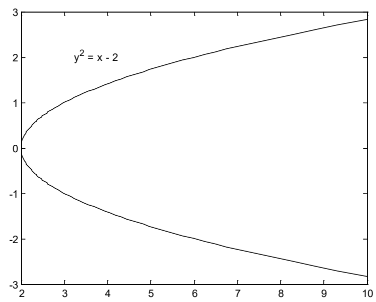
$$x_1 = x_2 \Rightarrow f(x_1) = f(x_2) \,.$$

Any rule that satisfies these two conditions is a function, any rule that doesn't, isn't. Note that nothing in the definition says that every element of B must be associated with an element of A.

<u>Example 5.1.7</u> The following rule g is not a function from the set {product 1, product 2, product 3} to the set $B = \{1, 2, 3, \ldots, 10\}$

g (prod. 1) = 2 ; g (prod. 1) = 4 ; g (prod. 2) = 4.

because prod 3 wasn't mapped, and because prod 1 was mapped to two numbers.

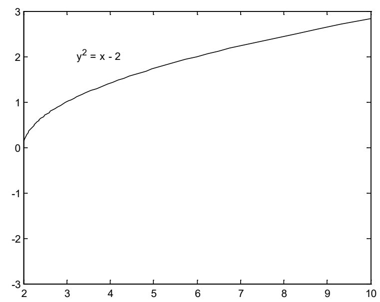


Although we will typically use equations to represent functions, <u>they are not the same thing</u>. Take the equation

$$y^2 = x - 2$$

The graph of this equation is plotted on the right.

The equation (without further restriction) does not represent y as a function of x because some values of x are mapped to more than one value of y.



The function $f(x) = \sqrt{x-2}$ is *defined* to be only the top portion of this graph, as illustrated on the right.

To further emphasize that each member of the domain cannot be given more than one value by the function, consider the following example.

Example 5.1.8 What is the square root of four?

The term 'square root' is used ambiguously. By 'y is the square root of x' do we mean the *solution* to the equation $y^2 = x$, or do we mean the value of the *function* $y = \sqrt{x}$? Most people treat the phrase 'square root' as the former, thus they would say the square roots of 4 are 2 and -2, the first called the 'principal square root', and the second the 'negative square root'. In this case, the term 'square root' is not being used in a function sense. However, there are others that treat the term 'square root' in the function sense, so they always mean the positive square root.

Note that the symbols "$\sqrt{\ }$" and $(\ )^{1/2}$ are unambiguous. These represent functions with positive values. Thus $\sqrt{4} = 2$, never $\sqrt{4} = -2$, and $4^{1/2} = 2$, never $4^{1/2} = -2$.

Other remarks

1. Functions stand separately from whatever interpretation you might associate with it. Suppose cost C depends on quantity produced q by the equation $C = \pi q^2$, $q \in (0,\infty)$, then this cost function and the earlier area of a circle function are identical as mathematical objects. We can write $C = f(q)$ and $A = f(r)$, where $f(x) = \pi x^2$, $x \in (0,\infty)$.
2. Notice that when using the notation $f(.)$ to describe a function, what appears inside the bracket is merely a placeholder. If $f(x) = \pi x^2$, then

   $f(z) = \pi z^2$ (whatever z is supposed to represent, it doesn't matter)

   $f(x^2) = \pi (x^2)^2 = \pi x^4$

   $f(1/x) = \pi (1/x)^2 = \pi / x^2$

   If $g(x)$ is another function, then $f(g(x)) = \pi g(x)^2$

   (we call $h(x) = f(g(x))$ a composite function. A number of the examples just given are also examples of composite functions. It is often convenient to "break down" complicated functions into compositions of simpler functions.)

   On the other hand, if $f(x^2) = \pi x^2$, then $f(z) = \pi z$.
3. The domain of a function is not always stated explicitly, because it is impractical to always do so, but remember that the domain is an essential part of a function's definition. For instance, $A = \pi r^2$,

$r \in (0,\infty)$, and $A = \pi r^2$, $r \in \mathbb{R}$, are both valid functions, but strictly speaking they are different functions, because their domains differ.

5.2 Onto, One-to-One Functions; Inverse Functions

Consider the function $f : A \to B$. If every element of *B* is associated with at least one member of *A*, then f is a function from *A* onto *B*. Such a function is also called a "surjective function".

Remember that two or more elements of *A* may be mapped onto the same element of *B*. If no element of *B* is associated with two or more elements of *A*, then the function is called one-to-one, or injective. If a function is one-to-one and onto, it is also called "bijective", or a "one-to-one correspondence from A to B ".

Example 5.2.1 The relationship mapping the radius *r* of a circle to its area is a one-to-one function.

Example 5.2.2 The function $f(x) = x^2$, $x \in \mathbb{R}$, is not a one-to-one function.

Example 5.2.3 The function $f(x) = x^2, x \in [0,\infty)$, is a one-to-one function.

Given a one-to-one function *f* from A to B, we can always define a related function, called the inverse function f^{-1} from $f(A)$ to A. We have $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

Example 5.2.4 Suppose f is a function from $A = \{1,2,3,4\}$ to $\mathbb{R}$ defined by

$$f(1) = -2,\ f(2) = -19,\quad f(3) = 3.1,\qquad f(4) = 8.$$

then its inverse is

$$f^{-1}(-19) = 2,\ f^{-1}(-2) = 1,\quad f^{-1}(3.1) = 3,\quad f^{-1}(8) = 4.$$

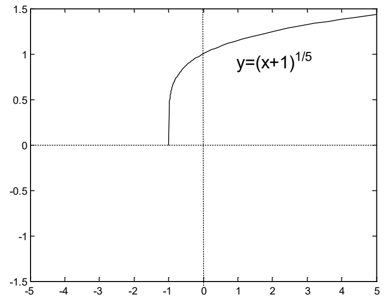
The inverse is a function defined on the set $\{-19,-2,3.1,8\}$.

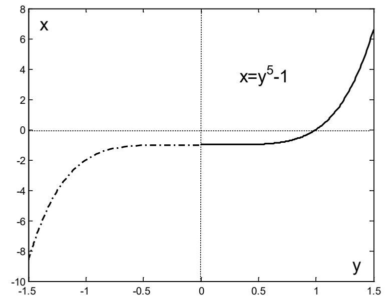
Example 5.2.5 Suppose $y = \sqrt[5]{x+1}$, a function from $(-1,\infty)$ to $[0,\infty)$. Its inverse function is $x = y^5 - 1$, a function from $[0,\infty)$ to $(-1, \infty)$. We get the expression $x = y^5 - 1$ by "solving" $y = \sqrt[5]{x+1}$ for x.

Using the f and f^{-1} notation, we would to write the inverse of the function $f(x) = \sqrt[5]{x+1}$ as $f^{-1}(x) = x^5 - 1$, $x \ge 0$, instead of $f^{-1}(y) = y^5 - 1$.

It may be a little confusing because we use x in the bracket for both $f(.)$ and $f^{-1}(.)$. Remember that what appears inside the bracket in the function notation is merely a placeholder.

Note also that I was careful to note that the domain of $f^{-1}(x)$ is $x \geq 0$, even though $x^5 - 1$ can be defined for all x. However, $g(x) = x^5 - 1$, $x \in \mathbb{R}$, is NOT the inverse of $f(x) = \sqrt[5]{x+1}$.





5.3 Important Functions

Here is a list of functions you must be very familiar with.

Linear Functions: $f(x) = a + bx$, or, $y = a + bx$

Quadratic Functions: $f(x) = ax^2 + bx + c$, where a, b, and c are any real numbers ($a \neq 0$). This function can be defined on the entire real line $\mathbb{R}$.

Polynomials

Cubic Function $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

Quartic Function $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$

…

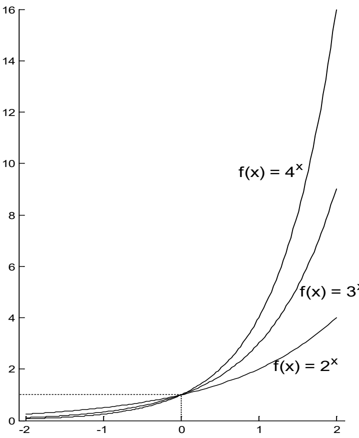
Polynomial of degree n :

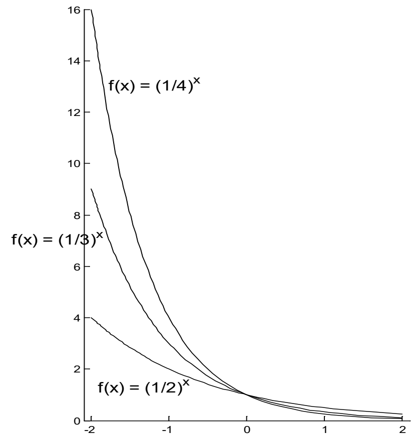
$$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots a_2x^2 + a_1x + a_0$$

Power Function $f(x) = ax^r$, for some $r \in \mathbb{R}$.

Exponential and Logarithmic Functions These are extremely important. Here I will emphasize the relationship of one as an inverse function of the other.

General Exponential Function $f(x) = Aa^x$, $x \in \mathbb{R}$ where $a > 0$. For now, take $A > 0$. It is obvious how the discussions would change if A is negative. The (general) exponential function is one-to-one. If $A > 0$, the range is $(0, \infty)$.





This function is especially useful when x is interpreted as 'time' (in this case we often use t instead of x). The exponential function can be used to describe constant growth (or depreciation) rates: when x increases by one unit, $f(t+1) = Aa^{t+1} = af(t)$, so $f(t)$ grows/depreciates by a constant factor of

$$\frac{f(t+1) - f(t)}{f(t)} = a - 1, \text{ i.e., } f(t) \text{ grows by } 100(a-1)\%$$

In other words, a is the gross rate of change for every unit change in t. The exponential function arises very naturally in applications such as population growth and compound interest calculations.

We call $f(x) = e^x$, where $e = 2.718281828459\ldots$ the "natural exponential function." Sometimes e^x is written $\exp(x)$. This function, and the number e, has a special place in mathematics.

Logarithmic Function Define

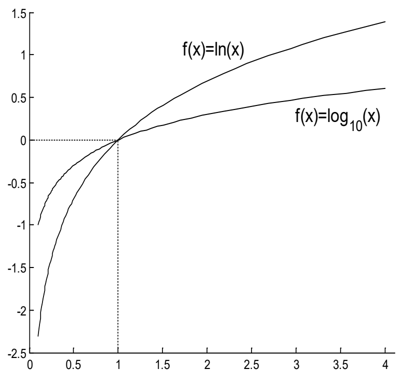
$$f(x) = \log_a(x) \text{ where } y = \log_a(x) \text{ if } x = a^y \text{ (usually } a > 1).$$

That is, $y = \log_a(x)$ is *defined* as the inverse of the exponential function $x = a^y$:

$$a^{\log_a(x)} = x \text{ and } \log_a(a^x) = x$$

We call $\log_a(x)$ the logarithm of x to the base a. Logs to the base 10 are called common logarithms.

[Qn: What is the domain of $f(x) = \log_a(x)$?]



Rules for manipulating logs:

$\log_a 1 = 0$ and $\log_a a = 1$ proof: Because $a^0 = 1$ and $a^1 = a$.

$\log_a x^r = r \log_a x$ proof: Let $y = \log_a x$, i.e., $x = a^y$. Then

$$(a^y)^r = x^r \Leftrightarrow ry = \log_a x^r \Leftrightarrow r\log_a x = \log_a x^r$$

This implies $\log_a a^x = x$.

$a^{\log_a x} = x$	proof:	Let $\log_a x = y$, i.e., $x = a^y$.
		Then substitution of y gives $a^{\log_a x} = x$
$\log_a(xy) = \log_a x + \log_a y$	proof:	$a^{\log_a xy} = xy = a^{\log_a x} a^{\log_a y} = a^{\log_a x + \log_a y}$
$\log_a(x/y) = \log_a x - \log_a y$	proof:	Similar to previous case.
$\log_a x = \log_b x / \log_b a$	proof:	Let $\log_a x = y$, i.e., $a^y = x$. Then
		$\log_b x = \log_b a^y = y\log_b a = \log_a x \log_b a$

This implies, in particular, that $\log_a b = \dfrac{1}{\log_b a}$.

The function $\log_e x$ is called the natural logarithmic function. We use the symbol $\ln x$ to represent $\log_e$ x.

<u>Note!</u> Because in economics (and in many other disciplines) we use natural logs much more frequently than we use logs of any other base, many authors (and software writers) will use ‘log’ to represent ‘ln’ without saying so (you may perhaps be more familiar with the practice of using ‘log’ to mean ‘$\log_{10}$’). From this point, we will use ‘log’ and ‘ln’ interchangeably to mean natural logs. Logs to other bases will be indicated explicitly. If in doubt, check.

Exercises

1. Restricting the domain and value of the function to real numbers, what is the largest possible domain for each of the following functions?

   (a) $f(x)=\sqrt{x^2-3}$ (b) $g(x)=3+\sqrt{x}$ (c) $y=\sqrt{3-x}$

   (d) $f(x)=1/(3-\sqrt{x})$ (e) $y=\dfrac{2x-1}{x^2-x}$ (f) $y=\sqrt{\dfrac{x-1}{(x-2)(x+3)}}$

2. If the following functions are to be used in an application where y is a variable that must always be non-negative (for instance, if y is the volume of some object), then what is the largest reasonable domain for x?

   (a) $y=\dfrac{2x-1}{x^2-x+1}$ (b) $y=x^3+1$ (c) $y=(x^2-1)/(x+1)$

3. (a) Find all values of x that satisfies the equation $x+2=\sqrt{4-x}$.

   (b) Find all (x,y) values that solve the (nonlinear) system of equations

   $$\text{eq1: } y=\frac{x}{\sqrt{x+1}-1} \quad \text{and eq2: } y=5-3\sqrt{x+1}\,.$$

4. Restricting both the domain and range to real numbers, what is the largest possible domain for the function

   $$y=f(x)=\log\left(\frac{2x+4}{x^2-x}\right)$$

   If this function is to be used in an application where y is a variable that must always be non-negative (for instance, if y is the volume of water in some container), then what is the largest appropriate domain for x?

5. (a) If $f(x)=x^2+\sqrt{x}$ and $g(x)=\log(x^2+1)$, find $h(x)=f(f(x))f(g(x))$,

   (b) Write $g(x)=x^8$ in the form $f(f(f(x)))$.

   (c) Let $g(x)=\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+x}}}$. Find $f(x)$ such that $g(x)=f(f(f(x)))$. Evaluate $g(1)$.

6. Describe the differences between the two functions. Plot them

   (a) $f(x)=2+3x$ (b) $g(x)=\dfrac{3x^2-x-2}{x-1},\quad x\neq 1$.

7. If $f(x)=x^2+\sqrt{x}$, compute

   (a) $f(4)$ (b) $f(2)$ (c) $f(\pi)$ (d) $f(z)$ (e) $f(z^2)$

   (f) $f(x^2)$ (g) $f(x+2)$ (h) $f(g(x))$ where $g(x)=\sqrt{x+2}$

8. Let the set $A=\{1,2,3\}$ and $B=\{a,b,c,d,e,f\}$. Which of the following "rules" $f(.)$ qualifies as a function from set A to set B, (in notation, this is written $f:A\to B$). Explain why or why not.

(i) $f(1)=a, f(2)=b, f(3)=c$ (ii) $f(1)=a, f(2)=b$ and $f(3)$ not defined.

(iii) $f(1)=a, f(2)=b$ or c, $f(3)=d$ (iv) $f(1)=a, f(2)=a, f(3)=b$

9. Let $A=\{0,1,2,3,4,5,6,7,8,9,10\}$ and $B=\{0,1,2,3,...,100\}$

(i) Let $f:A\to B$ be defined by $f(x)=x^2$. Is this function one-to-one? What is the range of this function? Is it possible to define a function from A to B that is one-to-one and also onto?

(ii) Let $f:A\to B$ be defined by $f(x)=(x-5)^2$. Is this function one-to-one? What is the range of this function?

10. Let A and B both be the set of all positive integers. Let $f:A\to B$ be defined by $f(x)=x^2$.

(i) What is the range of this function? Is this function one-to-one? Is it onto? Is it possible to define a one-to-one function from A to B?

(ii) Can you see that sequences can be thought of as a function $f:A\to B$ where A is the set of all positive integers. E.g. the given function results in the sequence $1,4,9,16,25,...$ corresponding to $x=1,2,3,4,5,...$

11. Convince yourself that the set of points (x,y) satisfying the following equations all lie on a straight line. In each case, compute the slope and intercept of the line.

(i) $y = a+bx$ (ii) $y-y_0 = m(x-x_0)$. Show that the line passes through (x_0,y_0)

(iii) $y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$. Show that the line passes through (x_1,y_1) and (x_2,y_2).

(iv) $ay+bx+c=0$

For part (iv) describe the line when $a=0$. Explain why the equation does not represent y as a function of x unless we restrict $a\neq 0$.

12.(a) Let $f(x)=x^2$. For each of the following, graph $g(x)$ and describe the relationship between $g(x)$ and $f(x)$

(i) $g(x)=f(x-a)$ (ii) $g(x)=f(x+a)$ (iii) $g(x)=f(x-a)+b$

(b) Show that $f(x)=ax^2+bx+c$ can be written as

$$f(x)=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right].$$

Show, <u>without using calculus</u>, that

(i) the minimum of $f(x)$ occurs at $x=-b/2a$, (ii) the min. value of $f(x)$ is $(4ac-b^2)/4a$,

(iii) $f(x)$ has one root (i.e. one zero value) if $b^2=4ac$, and two if $b^2>4ac$

Find an expression for the roots of $f(x)$.