Mathematics for Economics **Anthony Tay**

3. The Summation Notation

You will frequently deal with complicated expressions involving a large number of additions. Often, these expressions are simplified using the 'summation' notation. It seems to me that many students find difficulty in manipulating such expressions. The purpose of this section is to introduce the notation to you, and to get you comfortable with it.

1. Definition and Rules for Summations

The uppercase sigma "Σ" is used to denote summation. For an arbitrary set of numbers $\{x_1, x_2, ..., x_n\}$ define

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + ... + x_n .$$

Example 3.1.1 The average of a set of numbers $x_1, x_2, ..., x_n$ can be written

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i .$$

Example 3.1.2 Write the sum 4 + 8 + 12 + 16 + 20 + 24 in summation notation. Ans: $\sum_{i=1}^{6} 4i$.

Example 3.1.3 Suppose the following payments are to be made: a_1 in the first period, a_2 in the second period, and so on until a_n in the nth period. At a fixed interest rate of r per period, the **present value** of the payments is

$$\frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + ... + \frac{a_n}{(1+r)^n} = \sum_{i=1}^{n} \frac{a_i}{(1+r)^i} .$$

In Example 3.1.1, the "**index of summation**" i enter as subscripts, but otherwise do not enter into the computation of the terms of the summation. In Example 3.1.2, the index is actually part of the computation of the terms. In Example 3.1.3, the index is used both ways.

Example 3.1.4 Economists often use an aggregate price index to track the overall price level in an economy relative to some base year. This is usually done by tracking a weighted average of prices of a certain set of commodities. Let $i = 1, ..., n$ represent n commodities and

q_{0i} be the quantity of good i purchased in period 0 (the base year)

p_{0i} be the price of good i in period 0

q_{ti} be the quantity of good i purchased in period t

p_{ti} be the price of good i in period t

The Laspeyres Price Index is $\dfrac{\sum_{i=1}^{n} p_{ti}q_{0i}}{\sum_{i=1}^{n} p_{0i}q_{0i}}$

The Paasche Price Index is $\dfrac{\sum_{i=1}^{n} p_{ti}q_{ti}}{\sum_{i=1}^{n} p_{0i}q_{ti}}$

Expressions using summation notation are not unique; more than one expression can be used to represent a given sum.

Example 3.1.5 Write 1 – 1/3 + 1/5 – 1/7 + 1/9 – 1/11 in summation notation:

Ans: $\sum_{i=1}^{6}(-1)^{i-1}\dfrac{1}{2i-1}$. Alternate answer: $\sum_{i=0}^{5}(-1)^{i}\dfrac{1}{2i+1}$

3.2 Rules for Working with the Summation Notation

The summation notation greatly simplifies notation (once you get used to it), but this is only helpful then you know how to manipulate expressions written in it. There are only two rules to learn

(i) $\sum_{i=1}^{n}(a_i+b_i) = \sum_{i=1}^{n}a_i + \sum_{i=1}^{n}b_i$, (ii) $\sum_{i=1}^{n}ca_i = c\sum_{i=1}^{n}a_i$, where c is a constant.

Example 3.2.1 $\sum_{i=1}^{n}c = nc$

Example 3.2.2 $\sum_{i=1}^{n}(x_i-\overline{x}) = 0$, where $\overline{x} = \dfrac{1}{n}\sum_{i=1}^{n}x_i$.

Proof: $\sum_{i=1}^{n}(x_i-\overline{x}) = \sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\overline{x} = n\overline{x}-n\overline{x} = 0$

Example 3.2.3 $\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) = \sum_{i=1}^{n}(x_i-\overline{x})y_i = \sum_{i=1}^{n}(y_i-\overline{y})x_i$,

where $\overline{x} = \dfrac{1}{n}\sum_{i=1}^{n}x_i$, and $\overline{y} = \dfrac{1}{n}\sum_{i=1}^{n}y_i$

Proof We have $\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) = \sum_{i=1}^{n}(x_i-\overline{x})y_i - \sum_{i=1}^{n}(x_i-\overline{x})\overline{y}$

But the second term is zero: $\sum_{i=1}^{n}(x_i-\overline{x})\overline{y} = \overline{y}\sum_{i=1}^{n}(x_i-\overline{x}) = 0$,

which gives the desired result. The proof of $\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) = \sum_{i=1}^{n}(y_i-\overline{y})x_i$ is similar.

3.3 Some useful formulas involving summations

For every integer n,

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\sum_{i=1}^{n} i \right)^2$$

<u>Arithmetic Series</u>

$$\begin{aligned} \sum\nolimits_{i=0}^{n-1} (a + id) &= \sum\nolimits_{i=1}^{n} (a + (i-1)d) \\ &= a + (a+d) + (a+2d) + \ldots + (a + (n-1)d) \\ &= na + \frac{n(n-1)d}{2} \end{aligned}$$

where a and d are real numbers.

<u>Geometric Series</u>

$$\sum\nolimits_{i=0}^{n-1} ar^i = \sum\nolimits_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{(1-r^n)}{(1-r)},$$

where a and r are real numbers.

3.4 Double summations

Suppose we have a rectangular array of numbers

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix}$$

Let the total sum of these numbers be S. To get S we can first add up the rows and then add the results,

$$\text{i.e., } S = \sum_{j=1}^{n} a_{1\,j} + \sum_{j=1}^{n} a_{2\,j} + \cdots + \sum_{j=1}^{n} a_{m\,j} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right),$$

or first add up the columns and then add the results:

$$\text{i.e., } S = \sum_{i=1}^{m} a_{i\,1} + \sum_{i=1}^{m} a_{i\,2} + \cdots + \sum_{i=1}^{m} a_{i\,n} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right).$$

The parenthesis makes it clear which summation is to be done first, but it is conventional to leave out the parenthesis and write

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} \qquad \text{or} \qquad \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}$$

with the understanding that the summations are carried out from right to left, i.e., from the inner summation to the outer.

<u>Example 3.4.1</u> Expand $\sum_{i=1}^{m}\sum_{j=1}^{n} i\, j^2$.

One way to do this is $\sum_{i=1}^{m}\sum_{j=1}^{n} i\, j^2 = \sum_{i=1}^{m} i \sum_{j=1}^{n} j^2 = (1+2+\cdots m)(1^2+2^2+\cdots+n^2)$

A more explicit argument…

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i\, j^2 = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} i\, j^2\right) = \sum_{i=1}^{m}\left(i\sum_{j=1}^{n} j^2\right) = \left(\sum_{i=1}^{m} i\right)\left(\sum_{j=1}^{n} j^2\right)$$

$$= (1+2+\cdots+m)(1^2+2^2+\cdots+n^2)$$

In the examples so far, we could interchange the summation signs, i.e.,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij} .$$

We can do this only if the limits of the outer summation do not depend on the limit of any of the inner summations.

<u>Example 3.4.2</u> Suppose we have a triangular array of numbers to be expressed in summation notation.

$$\begin{array}{llll} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{array}$$

We can write this as

$$\textstyle\sum_{i=1}^{m}\sum_{j=1}^{i} a_{ij} .$$

However, we <u>cannot</u> interchange the summation symbols because the upper limit in the inner summation depends on the index of the outer summation.

Example 3.4.3 Write the sum of the following triangular array using summation notation.

$$\begin{array}{cccc} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{array}$$

where $n \le m$.

Solution: Let $a_{i,j}$ be the typical element in the sum. Then the first column has $j = 1$, and i running from 1 to m; the second column has $j = 2$, and i running from 2 to m. In general we have j running from 1 to n, for the jth column, i running from j to m. Thus the sum is

$$\sum_{j=1}^{n}\sum_{i=j}^{m} a_{ij} .$$

Exercises

1. Write out in full, then evaluate or simplify

a. $\sum_{i=1}^{4} 2i$ b. $\sum_{i=0}^{3} ix_i$ c. $\sum_{i=1}^{4} (i-1)x_{i-1}$ d. $\sum_{i=1}^{10} 2$

e. $\sum_{i=1}^{10} (2i-1)$ f. $\sum_{i=1}^{10} (-1)^i$ g. $\sum_{j=1}^{i} j$ h. $\sum_{j=1}^{i} i$

2. Write in summation notation

a. $1-3+5-7+9$

b. $a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \ldots + \binom{m}{m-1}ab^{m-1} + \binom{m}{m}b^m$

c. 2 + 3/2 + 4/3 + 5/4 + 6/5

d. $a_{i\,1}b_{1\,j} + a_{i\,2}b_{2\,j} + \ldots + a_{i\,n}b_{n\,j}$

3. Prove by writing out in full

a. $\sum_{k=1}^{10}[(k+1)^3 - k^3] = 11^3 - 1$ b. $\sum_{j=1}^{5} x_i x_j = x_i \sum_{j=1}^{5} x_j$

c. $\sum_{i=1}^{3}\sum_{j=1}^{2} x_i x_j = \sum_{j=1}^{2}\sum_{i=1}^{3} x_i x_j$ d. $\sum_{i=1}^{3}\sum_{j=1}^{2} x_i x_j = \left(\sum_{i=1}^{3} x_i\right)\left(\sum_{j=1}^{2} x_j\right)$

e. $\sum_{i=1}^{3} x_i \bar{x} = \bar{x}\sum_{i=1}^{3} x_i$ where $\bar{x} = \frac{1}{3}\sum_{i=1}^{3} x_i$

4. Prove using the rules of summation:

   a. $\sum_{i=1}^{n}\sum_{j=1}^{i} i = \sum_{i=1}^{n} i^2$ b. $\sum_{i=1}^{n}\sum_{j=i}^{n} j = \sum_{i=1}^{n} i^2$

5. Prove all the equality relations in the following:

$$\sum_{i=1}^{n}(x_i - \overline{x})^2 = \sum_{i=1}^{n}(x_i - \overline{x})x_i = \sum_{i=1}^{n} x_i^{\,2} - n\overline{x}^2$$

6. Show that $\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = \sum_{i=1}^{n} x_i y_i - n\,\overline{x}\,\overline{y}$.

7. Evaluate by first simplifying, then applying the appropriate formulas

   a. $\sum_{k=1}^{30} k(k-2)$ b. $\sum_{k=1}^{n} k(k-2)(k+2)$

8. Let $\{x_1, x_2, ..., x_{10}\} = \{1, 2, 6, \frac{1}{3}, \pi, e, 0, -1, 2, 4.2\}$. Verify using excel or otherwise (calculator, manual computation, mental computation, as you please) that

   a. $\sum_{i=1}^{10}(x_i - \overline{x}) = 0$

   b. $\sum_{i=1}^{10}(x_i - \overline{x})^2 = \sum_{i=1}^{10} x_i^2 - 10\overline{x}^2$

   c. $\sum_{i=1}^{10}(x_i - \overline{x})^2 = \sum_{i=1}^{10}(x_i - \overline{x})x_i$

   If in (i) you get an answer a little different from 0, explain why this occurs.

9. Express the following in the form $ax_1 + bx_2 + cx_3 + dx_4$:

   (i) $\sum_{i=1}^{4}\sum_{j=1}^{2} i\,x_j$ (ii) $\sum_{i=1}^{4}\sum_{j=i}^{4} i\,x_j$

   (i.e., you have to tell me what a, b, c, and d are in each case.)

10. Let $\{x_1, x_2, ..., x_n\}$ be an arbitrary set of n real numbers, and let

$$\overline{x} = \tfrac{1}{n}\sum_{i=1}^{n} x_i .$$

   Prove that

$$\sum_{i=1}^{n}(x_i - \overline{x})(x_i - 1) = \sum_{i=1}^{n}(x_i - \overline{x})(x_i - 10000).$$