2. Elementary Set Theory and Methods of Proof

This section reviews set theory, the notation and language used in logical arguments, and some elementary tools for proving results. Another purpose is to highlight to you the deeper features of real numbers. We begin by listing some important 'logic' notation. Throughout, let x represent real numbers, and N represent an integer.

Quantifiers	$\forall$	"for all" or "for each"	$\forall x,\ x^2 + 1 > 0$
	$\exists$	"there exists"	$\exists x,\ x^2 - 1 = 0$

Read the second line as "there exists (at least one) x such that $x^2 - 1 = 0$."

Another example:

$$\forall x, \exists N, N > x .$$

That is, for any real number x, we can find an integer N such that $N > x$. This fact seems obvious, but is important enough to have its own name: the "archimedean property of real numbers". It basically says that there are arbitrarily large integers. We will explore some implications of this fact a little later on. Be clear about the fact that the order of the quantifiers is important: "$\forall x, \exists N, N > x$" and "$\exists N, \forall x, N > x$" are two very different statements (the latter is obviously false).

$\Rightarrow$	implies	
$\equiv$	equivalent to	Sometimes we write $\Leftrightarrow$
$\neg$	not	Thus, $\neg(x > y) \Rightarrow x \le y$. Note that $\neg(\neg A) \equiv A$.

The statement $A \Rightarrow B$ says that if A is true, then B is true, or A implies B.

$A \Rightarrow B$: "The fruit is a ripe lemon implies that that it is yellow", or

"If the fruit is a ripe lemon, then it is yellow".

We say A is a sufficient condition for B (i.e. it guarantees B). That a fruit is a ripe lemon guarantees that it is yellow. If $A \Rightarrow B$, we can also say that B is a necessary condition for A. For a fruit to be a ripe lemon, it must be yellow. Another way of saying this is "A is true only if B is true".

Example: $ab \neq 0 \;\Rightarrow\; a \neq 0$ and $b \neq 0$.

That is, if $ab \neq 0$, then $a \neq 0$ and $b \neq 0$. We say

"$ab \neq 0$" is a sufficient condition for "$a \neq 0$ and $b \neq 0$".

"$ab \neq 0$" only if "$a \neq 0$ and $b \neq 0$"

"$a \neq 0$ and $b \neq 0$" is necessary for "$ab \neq 0$".

Note that $A \Rightarrow B$ does not say anything about B in situations where A does not hold. (If a fruit is not a ripe lemon, can it be yellow in color? *Well, it depends*…). In particular,

$A \Rightarrow B$ does not imply that $\neg A \;\Rightarrow\; \neg B$.

Note that $A \Rightarrow B$ does not imply that $B \Rightarrow A$. A ripe lemon is yellow, but a yellow fruit might be a banana. The statement "$A \Rightarrow B$" only says we cannot have, at the same time, A holding and B not holding. That is,

$A \Rightarrow B$:	A	B	
	false	false	$A \Rightarrow B$ is silent about this situation
	false	true	$A \Rightarrow B$ is silent about this situation
	~~true~~	~~false~~	$A \Rightarrow B$ only says this cannot hold,
	true	true	$A \Rightarrow B$ says if A is true, then B must be true

It **<u>is</u>** true that

$A \Rightarrow B \;\equiv\; \neg B \Rightarrow \neg A$ A ripe lemon is yellow. If it's not yellow, it's not a ripe lemon.

Can you see this from the 'truth table' above? The statement "$\neg B \Rightarrow \neg A$" is the called the **contrapositive** of "$A \Rightarrow B$".

If both $A \Rightarrow B$ and $B \Rightarrow A$ hold, then A and B are equivalent statements. Thus,

$x = 0 \;\Rightarrow\; x^2 = 0$ $x^2 = 0$ if $x = 0$

and $x^2 = 0 \;\Rightarrow\; x = 0$ $x^2 = 0$ only if $x = 0$

We say "$x^2 = 0$ is equivalent to $x = 0$"

"$x^2 = 0$ if and only if $x = 0$"

"$x^2 = 0$ iff $x = 0$" 'iff' is shorthand for "if and only if"

"$x^2 = 0 \;\equiv\; x = 0$"

"$x^2 = 0 \Leftrightarrow x = 0$"

To prove that two statements A and B are equivalent, you need to show that $A \Rightarrow B$ and $B \Rightarrow A$. It is often easier to show that $A \Rightarrow B$ and $\neg A \Rightarrow \neg B$.

Example x is even if and only if x^2 is even

Proof

(*Sufficiency: x is even $\Rightarrow$ x^2 is even*)

x is even $\Rightarrow x = 2m$ for some integer m $\Rightarrow x^2 = 2(2m^2)$ $\Rightarrow x^2$ is even.

('Necessity: x is even $\Leftarrow$ x^2 is even; we will prove the contrapositive)

x is not even $\Rightarrow$ x is odd $\Rightarrow$ $x = 2m+1$ $\Rightarrow$ $x^2 = 2(2m^2 + 2m) + 1$ $\Rightarrow$ x^2 is not even.

More on logic symbols

& and

$\vee$ or Sometimes $\wedge$ is used for 'and'.

Note that 'or' in mathematics is always inclusive. It is always A or B or both. This is not always true in everyday English. If I say 'swim or sink' I probably mean it in an exclusive sense. When I say 'study or fail', do I mean it in an inclusive sense, or exclusive? Sometimes we need to say 'exclusive or' (A or B but not both). In this case, we have the special "logic" word "xor". We might say A xor B.

Example $\neg(A \,\&\, B) \Leftrightarrow \neg A \vee \neg B$

That is, if it is not the case that both A and B are true, then either A is false or B is false (or both). If either A is false or B is false (or both), then it is certainly not true that both A and B are true. Using truth tables to represent this:

A	B	$A \,\&\, B$	$\neg(A \,\&\, B)$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

We use '1' for 'True' and '0' for 'False'

Remember that "or" is inclusive.

Exercise Prove $\neg(A \vee B) \Leftrightarrow \neg A \& \neg B$.

Exercise Prove that $A \& (B \vee C) = (A \& B) \vee (A \& C)$.

Complete the statement: $A \vee (B \& C) =$ ______ .

Exercise Prove that $A \Rightarrow B$ is equivalent to $\neg A \vee B$.

Proof $A \Rightarrow B$ says that $\neg(A \& (\neg B))$, which is equivalent to $\neg A \vee B$.

Exercise: Show that $\neg(A \Rightarrow B)$ is equivalent to $A \& \neg B$.

Exercise: Show that $A \Rightarrow (B \Rightarrow C) \Rightarrow (A \& B) \Rightarrow C$.

Negation of statements

Suppose someone said to you that there is a (small) positive number ε such that $\frac{1}{N} \geq \varepsilon$ for all N. Using the notation in this section, we write this as

$$\exists \varepsilon > 0, \ \forall N, \ \tfrac{1}{N} \geq \varepsilon$$

This statement is, of course, false. I can always choose N large enough such that $\frac{1}{N}$ is less than ε, no matter how small ε is . What is $\neg\left(\exists \varepsilon > 0, \ \forall N, \ \frac{1}{N} \geq \varepsilon\right)$? It is

$$\forall \varepsilon > 0, \ \exists N, \ \frac{1}{N} < \varepsilon$$

Change the $\forall$ to $\exists$, the $\exists$ to $\forall$, and negate the final statement.

Proving things Here are two examples illustrating the use of two different techniques for proving results: proof by contradiction, and mathematical induction.

Proposition $\sqrt{2}$ is not a rational number.

Proof Suppose $\sqrt{2}$ is a rational number, i.e., $\sqrt{2} = m/n$ for some integers m and n, $n \neq 0$. If both m and n are even, then the ratio can be simplified. Suppose therefore that we are dealing with the simplest form, and that either m and n are odd. But this leads to a contradiction: $\sqrt{2} = m/n$ implies $m^2 = 2n^2$, so m^2 is even, and therefore m is even. But if m is divisible by 2, it must be that m^2 is divisible by 4, and it follows that n^2 is divisible by 2. So n is even also. Assuming that $\sqrt{2} = m/n$ therefore leads to a contradiction ♦

Exercise Prove that $\sqrt{12}$ is not a rational number.

Proposition For all natural numbers $n = 1,2,3,\ldots$, we have $2^{n-1} \le n!$

(The notation $n!$ means $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$.) We prove the proposition by "mathematical induction". First note that the result holds for $n = 1$, since $2^{n-1} = 1 = 1!$. Now we show that if $2^{m-1} \le m!$ for some natural number m, then the statement must hold for $m+1$, i.e., $2^m \le (m+1)!$. We have

$$2^m = 2(2^{m-1}) \le 2m! \le (m+1)m! = (m+1)! \ \blacklozenge$$

The last inequality holds because, of course, $2 \le m+1$ holds for all natural numbers. Because we have proven the result for $n = 1$, the induction step says the result must also hold for $n = 2$, and therefore $n = 3$, and so on.

Exercise Prove that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

by induction. One disadvantage of induction as a strategy for writing proofs is that you must already know the result. Can you prove the equality without using induction?

Exercise Prove, using induction, that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Can you prove it without using induction?

Naive Set Theory

A set is a collection of objects, called elements.

$x \in A \quad \equiv \quad x$ is an element (or member) of the set A

$x \notin A \quad \equiv \quad x$ is not an element $A \quad \equiv \quad \neg(x \in A)$

Important sets

$\mathbb{N}$	The set of all "natural numbers" $1,2,3,...$ We write $\mathbb{N} = \{1,2,3,...\}$
$\mathbb{Z}$	The set of all integers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\varnothing$	The empty set, which contains no elements.

Examples $-2 \in \mathbb{Z}$; $\sqrt{2} \notin \mathbb{Q}$. The set $\{x: x \in \mathbb{R} \ \& \ x^2 = -1\}$, read as "the set of all x such that x is real and x^2 is minus one" is an empty set. Note that we can have a set of sets, e.g., $S = \{\{1\},\{1,2\},\{2,3\}\}$ is a set with three elements, each of which is a set. Qn: is there a difference between the sets $\{1\}$ and $\{\{1\}\}$?

If every element of A is also an element of B, we say "A is a subset of B", and write $A \subset B$. Note that we do not limit our usage of "$\subset$" to proper subsets, e.g. we would say $\{1,2,3\} \subset \{1,2,3\}$. Some authors reserve "$\subset$" for proper subsets, and use the symbol "$\subseteq$" for subsets, but we will not adopt this convention. Two sets A and B are equal if $A \subset B$ and $B \subset A$. To say that A is a proper subset of B, we will write $A \subset B$ and $A \neq B$.

$\{1,2\}$ is a proper subset of $\{1,2,3\}$ $\{1,2\} \subset \{1,2,3\}$ but $\{1,2\} \neq \{1,2,3\}$

Intersection $x \in A \cap B \equiv x \in A \ \& \ x \in B$

Sometime we write $A \cap B = \{x \mid x \in A \ \& \ x \in B\}$.

Examples $\{1,2,3\} \cap \{2,3,4\} = \{2,3\}$

$$\bigcap_{n=1}^{\infty}[0,\ 1/n) = [0,1) \cap [0,\tfrac{1}{2}) \cap [0,\tfrac{1}{3}) \cap ... = \{0\}$$

Incidentally, a set containing a single element is called a singleton

The expression "$\bigcap_{n=1}^{\infty}$" may look unusual. There is nothing preventing us from taking the intersection of an infinite number of sets, and there are indeed an infinite number of set of the form $[0,1)$, $[0,\tfrac{1}{2})$, $[0,\tfrac{1}{3})$, …. The number "0" is the only number that appears in all of these sets, therefore we have our result above.

Union $x \in A \cup B \equiv x \in A \vee x \in B$

Examples $\{1,2,3\} \cup \{2,3,4\} = \{1,2,3,4\}$

$$\bigcup_{n=1}^{\infty}(-1+\tfrac{1}{n}, 1-\tfrac{1}{n}) = \varnothing \cup (-\tfrac{1}{2}, \tfrac{1}{2}) \cup (-\tfrac{2}{3}, \tfrac{2}{3}) \cup \ldots = (-1,1)$$

Subtraction $x \in A - B \Rightarrow x \in A \ \& \ x \notin B$

Example $\{1,2,3,4\} - \{2,3,6\} = \{1,4\}$

We have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Why do we call this "Naive Set Theory"? Because we did not think very hard about what can be considered a set. Suppose I said something like, "let S be the set of all sets that do not include itself", that is:

$$R = \{x \mid x \notin x\}$$

Does the set R include itself?

If $R \in R$, then by definition, $R \notin R$.

If $R \notin R$, then R satisfies the condition for inclusion in itself, i.e., $R \in R$.

This is "Russell's Paradox", and its discovery led to the 'axiomatization' of set theory – a careful description of what can and cannot be considered a set. This fuller theory is called "Axiomatic Set Theory". Treatments that proceed without worrying about such issues are 'naive'. We will get around this problem by avoiding sets that are 'too large'; we limit the analysis always to some fixed, clearly defined, "Universal Set" X, such as the set of all real numbers, the set of all continuous functions defined on [0,1], and so on.

Complements $A^c = X - A$ E.g. if the universal set is $\mathbb{R}$, then $[0,1)^c = (-\infty,0) \cup [1,\infty)$.

We have $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

Exercise Prove that $(A \cup B)^c = A^c \cap B^c$

$$x \in (A \cup B)^c \Rightarrow \neg(x \in A \vee x \in B) \Rightarrow \neg(x \in A) \ \& \ \neg(x \in B) \Rightarrow x \in A^c \ \& \ x \in B^c \Rightarrow x \in A^c \cap B^c$$

Exercise Prove that $A - B = A \cap B^c$.

Cartesian Products

If $A = \{1,2,3\}$, and $B = \{3,4\}$, then $A \times B = \{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$

If $A = [0,1]$ and $B = [0,1]$, then $A \times B = \{ (x,y) : \ 0 \le x \le 1, 0 \le y \le 1 \}$, i.e. all ordered pairs of numbers (x,y) such that $0 \le x \le 1$ and $0 \le y \le 1$. 'Ordered' here means that the position of the numbers are important.

The set of all pairs of real numbers (x,y) is $\mathbb{R} \times \mathbb{R}$, often denoted $\mathbb{R}^2$.

Note the potential for confusion: is "$(0,1)$" the set $\{ x \in \mathbb{R} : \ 0 < x < 1 \}$, or is it the pair $(0,1)$? Depends on context, of course. Some authors use $]0,1[$ to denote the interval $(0,1)$, but sadly, this is not common usage.

Slightly more advanced material regarding the set of real numbers

Sometimes you will hear the real numbers described as an "ordered field". What does that mean? A field is a set F, with two operators '+' and '$\times$' defined on it, such that

(a1) $\forall x, y \in F$, $x + y \in F$

(a2) $\forall x, y \in F$, $x + y = y + x$

(a3) $\forall x, y, z \in F$, $(x + y) + z = x + (y + z)$

(a4) there is a member of F, which we call '0', such that $x + 0 = x$ for all $x \in F$

(a5) $\forall x \in F$, there is a member of F, called "$-x$" such that $x + (-x) = 0$

(m1) $\forall x, y \in F$, $xy \in F$

(m2) $\forall x, y \in F$, $xy = yx$

(m3) $\forall x, y, z \in F$, $(xy)z = x(yz)$

(m4) there is a member of F, which we call '1', such that $1x = x$ for all $x \in F$

(m5) $\forall x \in F$ such that $x \ne 0$, there is a member of F, called "$1/x$" such that $x(1/x) = 1$

(d1) $\forall x, y, z$, $x(y+z) = xy + xz$. 'd' stands for distribution. What does 'a' and 'm' stand for?

In other words, you can do all the usual arithmetic on real numbers that you have been learning since kindergarten. Are there other fields? Yes, the set of complex numbers, with addition and multiplication defined in a certain way, is also a field.

A super short primer on complex numbers

The imaginary number i is defined as $i=\sqrt{-1}$ (so that $i^2=-1$). Complex numbers are defined as $z=a+bi$ where a and b are real numbers. Sometimes this is written as

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)i \, .$$

Or $$z = a + bi \, .$$

Addition and multiplication are *defined* for complex numbers as

$$\begin{aligned}(a_1+b_1i)+(a_2+b_2i) &= (a_1+a_2)+(b_1+b_2)i \\ (a_1+b_1i)(a_2+b_2i) &= a_1a_2+a_1b_2i+a_2b_1i+b_1b_2i^2 \\ &= (a_1a_2-b_1b_2)+(a_2b_1+a_1b_2)i\end{aligned}$$

These definitions for addition and multiplication turn out to follow all the essential properties for addition and multiplication that are satisfied by real numbers, as defined by properties a1 to a5, m1 to m5, and d1.

We ofter write complex numbers z as as a pair of real numbers $z=(x,y)$, where x is the 'real part' and y is the 'imaginary' part, and define

addition $$(x_1,y_1)+(x_2,y_2)=(x_1+x_2, y_1+y_2)$$

multiplication $$(x_1,y_1)(x_2,y_2)=(x_1x_2-y_1y_2, y_1x_2+x_1y_2)$$

You can show that the set of all complex numbers $\mathbb{C}$ is a field. These definitions for addition and multiplication satisfy all the axioms (a1-a5; m1-m5; d1) above. We can do the usual arithmetic with complex numbers, manipulating them the same way we manipulate real numbers.

Exercise Show that $\mathbb{C}$ is a field.

Non-trivial imaginary and complex numbers cannot be related to the usual 'quantity' concept. I cannot say I have 1+3*i* liters of water in my water bottle. Nobody will speak with you if you talk like this. Complex numbers are nonetheless extremely useful. For example, quadratic equations

$f(x)=ax^2+bx+c$, where *a*, *b*, and *c* are real numbers

do not always have roots if the roots are restricted to the real numbers (roots are values of x such that $f(x)=0$. However, quadratic equations will always have roots if complex roots are allowed

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \, .$$

Applications go well beyond ensuring that the roots of quadratic expressions always exist. Without complex numbers, a lot of the mathematics used in the modern engineering might not have been possible.

Real numbers are a special case of complex numbers with $b = 0$, that is, we can identify each real number x with the complex number $(x,0)$. In this sense, $\mathbb{R}$ is a "subfield" of $\mathbb{C}$.

Exercise Show that $(x_1,0)+(x_2,0)=(x_1+x_2,0)$, $(x_1,0)(x_2,0)=(x_1x_2,0)$

Exercise Show that $(0,1)(0,1)=(-1,0)$

(That is: $i=(0,1)$)

Complex numbers can therefore be viewed as a generalization of real numbers.

What's the difference between $\mathbb{C}$ and $\mathbb{R}^2$? The use of the 'addition' and 'multiplication' operators for complex numbers. It is these two operators that turn $\mathbb{R}^2$ into $\mathbb{C}$.

What about the 'ordered' part of "ordered field"? An ordered set is a set S, together with a relation "$<$" satisfying

(i) $\forall x, y \in S$, exactly one of the following hold: $x < y$, $x = y$, $x > y$.

(ii) $\forall x, y, z \in S$, $x < y$ & $y < z \Rightarrow x < z$.

An ordered field is a field which is also an ordered set, such that

(i) $\forall x < y,\ x + z < y + z$

(ii) $\forall x > 0, y > 0$, we have $xy > 0$.

The set of real numbers $\mathbb{R}$ is an ordered field. The set of complex numbers $\mathbb{C}$ is a field, but not an ordered one.

Don't get too hung up on this. All this says is that you can do the usual arithmetic with real numbers and complex numbers (with the appropriate definition of addition and multiplication). You can do the usual manipulations with inequalities with real numbers, but you cannot do so with complex numbers).

We end with two "remarks about real numbers. For the rest of this section, (a,b) will refer to the set x of real numbers such that $a < x < b$, not the complex number $a+bi$, and $[a,b]$ refers to the set $\{x \in \mathbb{R}:\ a \le x \le b\}$. The set (a,b) is called an "open interval", whereas $[a,b]$ is called a "closed interval".

What is the largest number in the set $[0,1]$? (The answer is 1)

What is the largest number in the set $[0,1)$? (The answer is not $0.\dot{9}$, which is equal to 1)

> $0.\dot{9}$ (that is 0.999999…) is <u>equal</u> to 1? Yes.
>
> Let $x = 0.\dot{9}$. Then $10x = 9.\dot{9}$. We have $9x = 10x - x = 9.\dot{9} - 0.\dot{9} = 9$. Therefore $x = 1$. This is not a trick. Consider this: we know that $1/3 = 0.\dot{3}$. What is 1/3+1/3+1/3? What is $0.\dot{3} + 0.\dot{3} + 0.\dot{3}$?

There is no largest number in the set $[0,1)$. We can only speak of upper bounds. The number 2 is an upper bound of $[0,1)$. Any number greater or equal to 1 is an upper bound of the set. The "least upper bound" of $[0,1)$ is 1. We say "$\sup[0,1) = 1$".

Real numbers have an extremely important property. **Every non-empty set of real numbers that is bounded above will always have a least upper bound**. We say that $\mathbb{R}$ has the "least upper bound property". This is not the case for maximums. The set $[0,1)$ is a non-empty bounded set of real numbers, but it does not have a maximum. It does have a sup (which is 1).

Cardinality Finally, we look at a very interesting aspect of set theory and of real and rational numbers. This is cardinality: how many elements are there in a set?

$\{2,4,6,8,10\}$	five elements
$\{1,2,3,4,5,......\}$	"infinity"?
$\{2,4,6,8,10,...\}$	"infinity"?

Which set has more elements? $\{1,2,3,4,5,...\}$ or $\{2,4,6,8,10,...\}$? The odd numbers are not in the latter set, so it is tempting to say the latter set is larger. Yet I can match every number in the second set with a number in the first: match 1 with 2, 2 with 4, 3 with 6, 4 with 8, etc.

We define the concept of "the number of elements in a set" in the following manner:

Definition Let $J_n = \{1,2,3,...,n\}$. If I can put every element of a set A into a one-to-one mapping with J_n, then we say that A is a finite set with n elements. If I can put every element of A into a one-to-one mapping with $\mathbb{N}$, then we say that A is “countably infinite”.

Two countably infinite sets are considered to have the same “cardinality” (the same number of elements). Therefore $\{1,2,3,4,5,...\}$ and $\{2,4,6,8,10,...\}$ have the same number of elements. It doesn’t matter that one is a proper subset of the other.

Exercise What is the cardinality of the set of squares $\{1,4,9,16,25,...\}$?

Exercise What is the cardinality of the set of rational numbers?

(Answer: there are as many rational numbers as there are natural numbers. Can you put them into a one-to-one mapping with the set of natural numbers?)

What is the cardinality of the set of the interval $(0,1)$? That is, how many real numbers are there between 0 and 1? There are certainly an infinite number of them. However, it turns out that we cannot put all of these numbers into a one-to-one mapping with the natural numbers. Every number in $(0,1)$ can be written as $0.u_1u_2u_3u_4u_5u_6....$ where the u’s are digits from 0 to 9. Suppose we can put all numbers in $(0,1)$ into a one-to-one correspondence with the natural numbers. Then

1 $\leftrightarrow$ $0.u_{11}u_{12}u_{13}u_{14}u_{15}u_{16}....$

2 $\leftrightarrow$ $0.u_{21}u_{22}u_{23}u_{24}u_{25}u_{26}....$

3 $\leftrightarrow$ $0.u_{31}u_{32}u_{33}u_{34}u_{35}u_{36}....$

etc.

Now I am going to write down a number in $(0,1)$ that is not in this list. This will show that there are more numbers in $(0,1)$ than in $\mathbb{N}$. Let this number be...

$$0.w_1w_2w_3w_4w_5w_{6.}....$$

where $w_1 \neq u_{11}$, $w_2 \neq u_{22}$, $w_3 \neq u_{33}$, and so on. By construction, this number is different from any of the numbers in the list.

The set of real numbers in the interval (0,1) therefore cannot be put into a one-to-one mapping with the set of integers. There is always at least one left over if such a mapping is attempted (in fact, there will be many *many* real numbers that are left over). The point is: there are more real numbers in the interval (0,1) than there are integers.

The set $(0,1)$ has infinite cardinality (an infinite number of elements) but it is a 'larger' infinity than that of countable infinity. We say that $(0,1)$ is "uncountably infinite" or simply "uncountable". (It is rather like being able to count grains of sand, but not being able to count water).

A set can be finite, countably infinite, or uncountably infinite.

Exercise Does $(0,1)$ and $\mathbb{R}$ have the same "cardinality"? Does $\mathbb{R}$ and $\mathbb{R}^2$ have the same cardinality?