Mathematics for Economics **Anthony Tay**

1. Introduction to mathematical economic models

The objective of this section is to introduce to you the elements of an economic model, and to provide a few examples highlighting the type of mathematics you are likely to encounter in your economics study.

Economists build "models" to explain various aspects of the economy. The models are in mathematical form, although we sometimes express or illustrate the mathematics graphically. It should not be surprising that economists choose to build mathematical models. Economics explains human behavior as a manifestation of choices made with the intention of achieving various objectives, all within the context of scarce resources that have alternate uses, and constraints. This is inherently a mathematical problem. Modeling economic problem mathematically also allows for the use of mathematical techniques to work out the logical conclusions and predictions of the model, and for the use of statistical techniques when taking the theories to data.

I will use a few examples of economic models

(i) as examples of the kind of models economists build;

(ii) to introduce the elements of mathematical economics models;

(iii) to motivate the mathematical topics included in this course.

The examples that follow may contain mathematics that you are unfamiliar with. They are not intended to be an indication of what you should already know, but to 'set the agenda', and to give you a look ahead to the type of mathematics we will be studying in this course.

<u>Example 1.1</u> Suppose a firm chooses how much to produce in order to maximizes profit (total revenue minus total costs). Let x denote the firm's output quantity choice. Suppose there is 'perfect competition', and the firm has to take price p as given. Then its revenue function is $R(x) = px$. Assume its total cost of producing x is given by $C(x) = x^2$. The firm's profit function is then

$$\pi(x) = R(x) - C(x) = px - x^2 .$$

This is sometimes called the firm's **objective function**.

What is the firm's optimal choice of x? (That is, how much should the firm produce so that its profits are maximized?) Let x^* denote the firm's optimal choice. You will learn, when we study optimization theory, that if the second derivative $\pi''(x)$ of the objective function has the property that $\pi''(x) < 0$ for all x, then the value x^* that maximizes profit $\pi(x)$ is that value such that $\pi'(x^*) = 0$, that is, such that the value of the first derivative is zero when evaluated at x^*. We have

$$\pi'(x^*) \;=\; R'(x^*) - C'(x^*) \;=\; p - 2x^* \;=\; 0$$

which gives $x^* = p/2$. Furthermore, $\pi''(x) \;=\; R''(x) - C''(x) \;=\; -2$, so $\pi''(x) < 0$ is satisfied for all values of x. The value $x^* = p/2$ is a profit maximizing level of production.

This model begins with a postulate: that the firm maximize profits. From this postulate, a relationship between two economic variables is derived, output x and price p. Price in this model is an **exogenous variable** (we don't try to explain it, but simply treat it as 'given'; we justified taking price as given by assuming perfect competition). The variable x is an **endogenous variable**, a variable whose behavior the model explains in relation to the exogenous variables. In this example, we derived $x^* = p/2$. More importantly,

$$\frac{dx^*}{dp} \;=\; \frac{1}{2} \;>\; 0,$$

so the theory says that when prices are higher, firms will choose to produce more.

The process of asking what happens (to the endogenous variable) when an exogenous variable or a parameter of the model changes is called **comparative statics**. In this case, the model generates a refutable proposition as a consequence of the assumption that firms maximize profits – that production increases when prices increase. This proposition can then be compared with ('tested against') actual observed behavior, a skill you will learn when you study econometrics.

A few remarks: all models are abstractions of the real world. It includes whatever is needed for providing an explanation for some observed behavior, and leaves out everything else. There is obviously a tremendous amount of abstraction in our example. Many firms produce many products, and even for the 'same' product there are often many 'varieties', but these have all been 'lumped' into one product type. Many variables one might expect to be related to the issue have also been omitted, for example, labor costs, rental costs, employment, variables related to available technology, etc. However, for the question that we asked, these are not relevant – in a sense, all of these factors have been summarized into the cost function.

One skill you will have to develop is the ability to work with general functions instead of specific functions. For instance, in dealing with the cost function, we would prefer to leave it as general as possible, and make only the minimal assumptions about it. For instance, we might assume only that the cost function is characterized by $C'(x) > 0$ (when x increases, $C(x)$ increases) and $C''(x) > 0$ (it increases at an increasing rate), instead of specifying a particular function such as $C(x) = x^2$. Example 1.1 is recast in this more general form in Example 1.2.

Example 1.2 Suppose a firm chooses how much to produce in order to maximize profits, which are computed as revenue minus costs. Let x denote the firm's output quantity choice. If the firm is a price taker, i.e., it has to take price p as given, then its revenue is $R(x) = px$. Denote its total costs by $C(x)$ where $C'(x) > 0$ and $C''(x) > 0$ for all x. The firm's profit function is then

$$\pi(x) \ = \ R(x) - C(x) \ = \ px - C(x) \ , \ C'(x) > 0 \text{ and } C''(x) > 0 \text{ for all } x$$

What is the firm's optimal choice of x? Again letting x^* denote the firm's optimal choice, the necessary and sufficient conditions for $x = x^*$ to maximize profit are

$$\pi'(x^*) \ = \ R'(x^*) - C'(x^*) \ = \ p - C'(x^*) \ = \ 0$$

and $$\pi''(x) \ = \ R''(x) - C''(x) \ = \ -C''(x) \ < \ 0 \text{ for all } x.$$

That is, x^* must satisfy

$$R'(x^*) \ = \ C'(x^*) \qquad \text{(“Marginal Revenue = Marginal Cost”)}$$

which in our specific example says that $C'(x^*) = p$, and

$$\pi''(x) = R''(x) - C''(x) < 0$$

The latter condition says that marginal costs rise increases faster than marginal revenue.

The cost function has been kept very general, with only two characteristics specified. This keeps the analysis as general as possible, and helps to identify the assumptions that drive the results. One can argue about the reasonableness of the assumptions made, but this a good thing about the analysis – we have based the argument on specific assumptions, and efforts to improve the analysis can focus on the validity of these assumptions. The analysis also highlights general principles rather than specific results. Here, it is that profit maximizing firms will operate where 'marginal cost equals marginal revenue, as long as the marginal revenue curve cuts the marginal cost curve from below'. Such a result should be familiar to those who have studied introductory economics; you may have already drawn the corresponding curves in an earlier course. If so, you have already carried out the analysis above! Graphs, however, are difficult to apply in more complicated modeling situations. It is easier to do so with mathematical analysis

One question students sometimes raise is whether we can reasonably assume that a firm is able to employ optimization theory to get to the maximum. Again, this is not an issue. We are usually not interested in *how* the firm gets *to* the maximum, but in how the firm behaves *at* the maximum. In our example, we want to *characterize* the firm's optimal choice as a function of the price given to the firm. The firm might get there in a variety of ways. We will get there by optimization theory.

Apart from characterizing behavior, we also want to know what happens to the endogenous variable when the exogenous variable changes. But how do we do this with general functions? In a later section you will learn how to compute comparative statics results using implicit differentiation with general functions. The argument for our example is given below:

Writing the optimal production level x^* as a function of p, $x^* = x^*(p)$, we have

$$p = C'(x^*(p)).$$

Implicit differentiation then gives

$$1 = C''(x^*(p))\frac{dx^*(p)}{dp}.$$

On the left hand side, the derivative of p with respect to itself is simply 1. On the right hand side, we have a composite function: C' is a function of x^*, which in turn is a function of p. Applying the chain rule – we differentiate C' wrt to x^*, and multiply the result by the derivative of x^* with respect to p, to give

$$\frac{d}{dp}\left[C'(x^*(p))\right] = C''(x^*(p))\frac{dx^*(p)}{dp}.$$

Rearranging $1 = C''(x^*(p))\frac{dx^*(p)}{dp}$ gives

$$\frac{dx^*(p)}{dp} = \frac{1}{C''(x^*(p))} > 0,$$

since $C''(x) > 0$ for all x, by assumption. Therefore, as price increases, the firm increases production (supply curve slopes upwards).

You may have done implicit differentiation before, but probably not with general functions. We will do so in this course. Note also that in working with general functions, you will have to be comfortable working with the function notation, and working with the function-prime notation for derivatives.

Another mathematical procedure we use a lot of is solving simultaneous equations.

<u>Example 1.3</u> *Demand and Supply* Suppose the supply and demand functions for a good are

(supply) $q^s = a_1 + a_2 p\,,\ a_2 > 0$

(demand) $q^d = a_3 + a_4 p + a_5 r\,,\ a_4 < 0,\ a_5 > 0$

where q^s is quantity supplied, q^d is quantity demanded, p is the price of the good, and r is the price of a substitute good. Suppose that the <u>equilibrium</u> price p^*, and quantity bought and sold q^*, is the level at which the market clears: $q^d = q^s (= q^*)$. Then p^* satisfies

$$a_1 + a_2 p^* = a_3 + a_4 p^* + a_5 r\,,$$

which gives

$$p^* = \frac{a_3 - a_1}{a_2 - a_4} + \frac{a_5}{a_2 - a_4} r\,.$$

Substituting into either the demand or supply equation (supply is easier) gives

$$q^* = a_1 + a_2\left(\frac{a_3 - a_1}{a_2 - a_4} + \frac{a_5}{a_2 - a_4} r\right) = \left(\frac{a_2 a_3 - a_1 a_4}{a_2 - a_4}\right) + \left(\frac{a_2 a_5}{a_2 - a_4}\right) r$$

In this model, we are explaining the behavior of the price and quantity sold of a good: p and q are the endogenous variables. The behavior of these two goods are explained in relation to the price of the substitute good, r, which we take as exogenous; we do not attempt to explain its behavior. An increase in r results in both the equilibrium price and the equilibrium quantity (since a_2 and a_5 are both positive).

The 'constants' a_1, a_2, a_3, a_4, a_5 are called parameters. We often also ask how equilibrium p and q change when one of the parameters change. For instance, we a_5 increases, then both p^* and q^* become more sensitive to changes in r.

Economists are also often interested in how consumers and producers are affected when the parameters of the model change. For instance, the area between the demand curve and the line $p = p^*$ from

$q=0$ to $q=q^*$ is a measure of *consumer surplus*, whereas the area between the supply curve and the line $p=p^*$ from $q=0$ to $q=q^*$ is *producer surplus*. Computing areas between curves involves computing integrals.

The following is another example of an analysis using simultaneous equations.

Example 1.4 Let C represent consumption and Y total output. Let G represent government expenditure, and assume $a>0$, and $0<b<1$. Suppose

$$C=a+bY$$

$$Y=C+G$$

For a fixed level of government expenditure, the equilibrium value of C and Y are the values where both equations are satisfied, which is the point at which they intersect. What are these values?

Here G is exogenous, and C and Y are endogenous. Subsituting $C=a+bY$ into $Y=C+G$, we get $Y=a+bY+G$, so

$$Y=\frac{a}{1-b}+\frac{G}{1-b}.$$

Substituting this into $C=a+bY$ gives $C=a+b\left(\frac{a}{1-b}+\frac{G}{1-b}\right)$, i.e.,

$$C = \frac{a}{1-b}+\frac{bG}{1-b}.$$

Models of this sort (but substantially more sophisticated) might be useful for policy analysis. This particular model says that a one unit increase in G increases Y by more than one unit, if $0<b<1$, since $dY/dG = 1/(1-b)$, and $0<b<1$.

A quick remark: in examples 1.3 and 1.4, the expressions may appear complicated, but the mathematical manipulations there are exactly the same as in the following simple problem: what values of x and y solve the two equations

$$2x+y=3$$

$$x+3y=1 \qquad\qquad \text{Ans:} \quad x=8/5 \text{ and } y=-1/5$$

Expressions in economics (and all applied mathematical sciences) tend to be more complicated than we are accustomed to in elementary mathematics classes. It is useful to bear in mind that the mathematical principles involved are often no deeper than what you already know.

The topics covered in this course are those that are used in the above examples and their extensions to multivariable situations, and in statistics/econometrics:

- differential and integral calculus (univariate and multivariate)
- optimization (univariate and multivariate, constrained and unconstrained)
- matrix algebra.

In covering the topics listed in course, there will be some emphasis on helping the student develop a deeper understanding of the mathematical concepts involved. This is harder than it sounds. All of you have studied quite a bit of calculus in terms of how to compute derivatives, but many of you have learnt how to do this in a mechanic fashion, or learnt about differentiation and derivatives at a very superficial "heuristic" level. Getting to the foundations of calculus will require 'unlearning', and then 're-learning', which for many of you will be challenging. It is important to go through the process, however. More advanced work in economics and econometrics use mathematics that go well beyond what we cover. To develop the ability to acquire more advanced material, on your own if necessary, will require a *proper foundational* understanding of calculus.

Exercises

1. Take example 1.1

(i) Plot the firm's cost function. Describe the important features of the function in your own words.

(ii) The firm's marginal revenue is the derivative of its revenue function, its marginal cost is the derivative of its cost function. Plot in a single diagram, the firm's marginal cost curve, and it's marginal revenue curve for different values of p (say, $p = 4, 8, 12$, 16). Explain intuitively why it is optimal for the firm to be producing at the intersection of the marginal revenue and the marginal cost curves. If p increases, will the firm produce more, or less?

(iii) Plot the firm's profit function for the same values of p as in part (ii). As p increases, does the firm's optimized profit levels increase, or decrease? Plot the firm's maximized profit levels against p;

2. In example 1.1, suppose the firm's cost function is

$$C(x) = \frac{x^3}{3} - 2x^2 + 4x + 2 .$$

(i) Plot the firm's marginal cost curve.

(ii) Show that for $0 < p < 4$, that the firm's MC = MR at two points, $x = 2 - \sqrt{p}$ and $x = 2 + \sqrt{p}$. Explain intuitively why the firm's profit is maximized at $x = 2 + \sqrt{p}$, but not at $x = 2 - \sqrt{p}$.

(iii) Show that the firm's MC curve cuts the MR curve from above at $x = 2 - \sqrt{p}$, whereas it cuts the MR curve from below at $x = 2 + \sqrt{p}$.

3. Plot the two equations $C = a + bY$ and $Y = C + G$ on the C - Y plane; use arbitrary values for $a > 0$ and $b \in (0,1)$, and plot three different versions of the second equation, corresponding to three different values of G.

4. In example 1.3, suppose the supply and demand curves are

(supply) $q^s = 3 + p$,

(demand) $q^d = 10 - 2p + r$,

Suppose $r = 2$. Plot the supply and demand functions with the horizontal axis representing q^s and q^d, and the vertical axis representing p.

(i) The equilibrium price is occurs when $q^s = q^d$. What is the equilibrium price?

(ii) What happens to the demand function when r increases from $r = 2$ to $r = 3$? From $r = 3$ to $r = 6$? Does anything happen to the supply curve? What happens to the equilibrium price?